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ON A FAMILY OF (p,q)-HYBRID POLYNOMIALS

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Abstract. In this paper, the class of (p,q)-Bessel-Appell polynomials is introduced. The generating function, series definition and determinant definition of this class are established. Certain members of (p,q)-Bessel-Appell polynomials are considered and some properties of these members are also derived. Further, the class of 2D (p,q)-Bessel-Appell polynomials is introduced by means of the generating function and series definition. In addition, the graphical representations of some members of (p,q)-Bessel-Appell polynomials and 2D (p,q)-Bessel-Appell polynomials are plotted with the help of Matlab.

1. Introduction

The quantum calculus (or called q-calculus) has been extensively studied and has applications in various fields of mathematics, physics and engineering. Further, motivated and inspired by these applications, many mathematicians and physicist have developed the theory of post quantum calculus (based on (p,q) numbers), an extension of the q-calculus and is denoted by (p,q)-calculus. The recent interest in the subject is due to the fact that the (p,q)-calculus has popped in such diverse areas as quantum algebra, number theory etc. [3–5, 12]. Recently, Duran et al. [5] defined (p,q)-analogues of Bernoulli, Euler and Genocchi polynomials and derived the (p,q)-analogues of some known earlier formulae. We now review briefly some definitions and notations of (p,q)-calculus taken from [3, 4, 12].

The (p,q)-numbers are defined as follows:

$$[\alpha]_{p,q} = p^{\alpha-1} + p^{\alpha-2}q + p^{\alpha-3}q^2 + \cdots + pq^{\alpha-2} + q^{\alpha-1} = \frac{p^\alpha - q^\alpha}{p-q}, \quad q < p \leq 1, \alpha \in \mathbb{N}.$$

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We note that $[\alpha]_{p,q} = p^{\alpha-1}[\alpha]_{q/p}$, where $[\alpha]_{q/p}$ is the q-number given by $[\alpha]_{q/p} = \frac{(q/p)^{\alpha}-1}{(q/p)-1}$. By appropriately using the relation $[\alpha]_{p,q} = p^{\alpha-1}[\alpha]_{q/p}$, most (if not all) of the (p,q)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved. In case of $p = 1$, (p,q)-numbers reduce to q-numbers [8, 9].

The (p,q)-factorial $[m]_{p,q}!$ is defined by

$$[m]_{p,q}! = \prod_{s=1}^{m}[s]_{p,q} = [1]_{p,q}[2]_{p,q}[3]_{p,q}\cdots[m]_{p,q}, \quad m \in \mathbb{N},\ [0]_{p,q}! = 1.$$

The (p,q)-binomial coefficient $\begin{bmatrix} m \\ s \end{bmatrix}_{p,q}$ is defined by

$$\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[s]_{p,q}!\ [m-s]_{p,q}!}, \quad s = 0, 1, 2, \ldots, m.$$

The (p,q)-analogue of $(x+y)^n$ is given by

$$(x+y)^m_{p,q} = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} p^{\binom{m-s}{2}} q^{\binom{s}{2}} x^s y^{m-s}, \quad (p,q)\text{-Gauss Binomial Formula}.$$

The (p,q)-analogue of the classical derivative Df of a function f with respect to t is defined by

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{pt - qt}, \quad t \neq 0.$$

Also, we note that

(i) $(D_{p,q}f)(0) = f^{'}(0)$, provided that f is differentiable at 0;

(ii) $D_{p,q}(a_1 f(t) + a_2\ g(t)) = a_1 D_{p,q}f(t) + a_2 D_{p,q}g(t)$;

(iii)

$$D_{p,q}(fg)(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t);$$

(iv)

$$D_{p,q}\left(\frac{f(t)}{g(t)}\right) = \frac{g(pt)D_{p,q}f(t) - f(pt)D_{p,q}g(t)}{g(pt)g(qt)} = \frac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(qt)}{g(pt)g(qt)}.$$

The (p,q)-exponential functions are given as:

$$e_{p,q}(t) = \sum_{m=0}^{\infty} p^{\binom{m}{2}} \frac{t^m}{[m]_{p,q}!}, \tag{1.1}$$

$$E_{p,q}(t) = \sum_{m=0}^{\infty} q^{\binom{m}{2}} \frac{t^m}{[m]_{p,q}!}, \tag{1.2}$$

which satisfy the following properties:

$$D_{p,q}e_{p,q}(t) = e_{p,q}(pt), \quad D_{p,q}E_{p,q}(t) = E_{p,q}(qt), \tag{1.3}$$

$$e_{p,q}(t)E_{p,q}(-t) = E_{p,q}(t)e_{p,q}(-t) = 1. \tag{1.4}$$

The class of Appell polynomials was introduced and characterized completely by Appell [2]. Further, Throne [16], Sheffer [15] and Varma [17] studied this class of polynomials from different points of views. Sharma and Chak [14] introduced a q-analogue for the class of Appell polynomials and called this sequence of polynomials as q-Harmonic. Later, Al-Salam [1] introduced the class of q-Appell polynomials $\{\mathcal{A}_{m,q}(x)\}_{m=0}^{\infty}$ and studied some of its properties. These polynomials arise in numerous problems of applied mathematics, theoretical physics, approximation theory and many other branches of mathematics. Recently, many researchers introduced and studied some hybrid special polynomials related to q-Appell polynomials (see for example [19]). The polynomials $\mathcal{A}_{m,q}(x)$ (of degree m) are called q-Appell provided that they satisfy the q-differential equation given by:

$$D_{q,x}\{\mathcal{A}_{m,q}(x)\} = [m]_q\mathcal{A}_{m-1,q}(x), \quad m = 0,1,2,3,\ldots, q \in \mathbb{C}, 0 < |q| < 1. \tag{1.5}$$

The (p,q)-Appell polynomials (pqAP) $\{\mathcal{A}_{m,p,q}(x)\}_{m=0}^{\infty}$ (see [11]) are defined by means of the followin generating functions

$$\mathcal{A}_{p,q}(t)\ e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}, \tag{1.6}$$

where

$$\mathcal{A}_{p,q}(t) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}\ \frac{t^m}{[m]_{p,q}!}, \quad \mathcal{A}_{p,q}(t) \neq 0, \mathcal{A}_{0,p,q} = 1 \tag{1.7}$$

and $\mathcal{A}_{m,p,q} := \mathcal{A}_{m,p,q}(0)$ denotes the (p,q)-Appell numbers.

The explicit form of the pqAP $\mathcal{A}_{m,p,q}(x)$ given as (see [11]):

$$\mathcal{A}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} p^{\binom{m-s}{2}} \mathcal{A}_{s,p,q} x^{m-s}. \tag{1.8}$$

The function $\mathcal{A}_{p,q}(t)$ may be called the determining function for the set $\mathcal{A}_{m,p,q}(x)$. Based on suitable selections for the function $\mathcal{A}_{p,q}(t)$, different members belonging to the family of (p,q)-Appell polynomial $\mathcal{A}_{m,p,q}(x)$ can be obtained. These members are mentioned in Table 1.

Table 1. Some known (p,q)-Appell polynomials

S. No.	$\mathcal{A}_{\mathbf{p,q}}(\mathbf{t})$	**Generating Functions**	**Polynomials**
I.	$\mathcal{A}_{p,q}(t) = \frac{t}{(e_{p,q}(t)-1)}$	$\frac{t}{(e_{p,q}(t)-1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathfrak{B}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$	The (p,q)-Bernoulli polynomials [6] (see also [11])
II.	$\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}}{(e_{p,q}(t)+1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$	The (p,q)-Euler polynomials [6]
III.	$\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{G}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$,	The (p,q)-Genocchi polynomials [6]

The Bessel polynomials form a set of orthogonal polynomials on the unit circle in the complex plane. They are important in certain problems of mathematical physics, for example, they arise in the study of electrical networks and when the wave equation is considered in spherical coordinates. Several important properties and applications of these polynomials can be found in [7].

The Bessel polynomials $\rho_m(x)$ [18] are defined by means of the following generating function

$$\sum_{m=0}^{\infty} \rho_m(x)\frac{t^m}{m!} = e^{x(1-\sqrt{1-2t})}.$$

This paper is organized as follows. In Section 2, the (p,q)-Bessel-Appell polynomials are introduced by means of the generating function and series definition. Also, the determinant definition and some properties for the (p,q)-Bessel-Appell polynomials are established. Further, some members of (p,q)-Bessel-Appell polynomials are considered. In Section 3, the 2D (p,q)-Bessel-Appell polynomials are introduced by means of the generating function and series definition. In Section 4, the graphical representations of some members belonging to (p,q)-Bessel-Appell and 2D (p,q)-Bessel-Appell families are plotted for suitable values of the indices.

2. (p,q)-Bessel-Appell Polynomials

In this section, we introduce the (p,q)-Bessel-Appell polynomials (pqBeAP) by means of generating function, series definition and determinant definition. First, we introduce the (p,q)-analogue of the Bessel polynomials denoted as (p,q)-Bessel polynomials $\rho_{m,p,q}(x)$.

Definition 2.1. The (p,q)-analogue of the Bessel polynomials $p_n(x)$ are defined by the following generating function:

$$\sum_{m=0}^{\infty} \rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!} = e_{p,q}(x(1-\sqrt{1-2t})) \tag{2.1}$$

and posses the following series expansion:

$$\rho_{m,p,q}(x) = \sum_{s=0}^{m-1} \frac{[m-1+s]_{p,q}!\ x^{m-s}}{[m-1-s]_{p,q}![s]_{p,q}!\ 2^s}.$$

In order to establish the generating function for the pqBeAP, the following result is proved.

Theorem 2.1. *The following generating function for the (p,q)-Bessel-Appell polynomials ${}_\rho\mathcal{A}_{m,p,q}(x)$ holds true:*

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} {}_\rho\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}. \tag{2.2}$$

Proof. By expanding the (p,q)-exponential function $e_{p,q}(xt)$ in the left hand side of the equation (1.6) and then replacing the powers of x, i.e., $x^0, x, x^2, \ldots, x^m$ by the corresponding polynomials $\rho_{0,p,q}(x), \rho_{1,p,q}(x), \rho_{2,p,q}(x), \ldots, \rho_{m,p,q}(x)$ in the left hand side and x by $\rho_{1,p,q}(x)$ in the right hand side of the resultant equation, we have

$$\mathcal{A}_{p,q}(t)\left(1+\rho_{1,p,q}(x)\frac{t}{[1]_{p,q}!}+\rho_{2,p,q}(x)\quad\frac{t^2}{[2]_{p,q}!}+\cdots+\rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!}+\cdots\right)$$

$$= \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(\rho_{1,p,q}(x))\ \frac{t^m}{[m]_{p,q}!}. \tag{2.3}$$

Further, summing up the series in left hand side and then using equation (2.1) in the resultant equation, we get

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(\rho_{1,p,q}(x))\ \frac{t^m}{[m]_{p,q}!}.$$

Finally, denoting the resultant pqBeAP in the right hand side of the above equation by ${}_{\rho}\mathcal{A}_{m,p,q}(x)$, that is

$$\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x)) = {}_{\rho}\mathcal{A}_{m,p,q}(x),$$

the assertion (2.2) is proved. □

Remark 2.1. It is remarked that for $p = 1$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to the q-Bessel-Appell polynomials (qBeAP) ${}_{\rho}\mathcal{A}_{m,q}(x)$ such that

$${}_{\rho}\mathcal{A}_{m,q}(x) := {}_{\rho}\mathcal{A}_{m,1,q}(x).$$

Thus, taking $p = 1$ in equation (2.2), we get

$$\mathcal{A}_{q}(t)e_{q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,q}(x)\ \frac{t^m}{[m]_{q}!},$$

which is the generating function for the q-Bessel-Appell polynomials.

Next, the series definition for the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ is derived by proving the following result.

Theorem 2.2. *The (p,q)-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ are defined by the following series definition:*

$${}_{\rho}\mathcal{A}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{A}_{s,p,q}\ \rho_{m-s,p,q}(x). \tag{2.4}$$

Proof. In view of equations (1.7) and (2.1), equation (2.2) can be written as:

$$\sum_{s=0}^{\infty} \mathcal{A}_{s,p,q}\ \frac{t^s}{[s]_{p,q}!} \sum_{m=0}^{\infty} \rho_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!},$$

which on using the Cauchy product rule gives

$$\sum_{m=0}^{\infty}\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{A}_{s,p,q}\ \rho_{m-s,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}.$$

Equating the coefficients of like powers of t in both sides of the above equation, we arrive at our assertion (2.4). □

Remark 2.2. For $p=1$, series definition (2.4) becomes

$$ {}_{\rho}\mathcal{A}_{m,q}(x)=\sum_{s=0}^{m}\begin{bmatrix}m\\ s\end{bmatrix}_{q}\mathcal{A}_{s,q}\ \rho_{m-s,q}(x), $$

which is the series definition for the q-Bessel-Appell polynomials.

Next, we establish the determinant definition for the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$.

Theorem 2.3. *The (p,q)-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ of degree m are defined by*

$$ {}_{\rho}\mathcal{A}_{0,p,q}(x)=\frac{1}{\mathcal{B}_{0,p,q}}, \tag{2.5} $$

$$ {}_{\rho}\mathcal{A}_{m,p,q}(x)=\frac{(-1)^m}{(\mathcal{B}_{0,p,q})^{m+1}} \times\begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x)\\ \mathcal{B}_{0,p,q} & \mathcal{B}_{1,p,q} & \mathcal{B}_{2,p,q} & \cdots & \mathcal{B}_{m-1,p,q} & \mathcal{B}_{m,p,q}\\ 0 & \mathcal{B}_{0,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \cdots & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q}\\ 0 & 0 & \mathcal{B}_{0,p,q} & \cdots & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mathcal{B}_{0,p,q} & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} \end{vmatrix}, \tag{2.6} $$

$$ \mathcal{B}_{m,p,q}=-\frac{1}{\mathcal{A}_{0,p,q}}\left(\sum_{s=1}^{m}\begin{bmatrix}m\\ s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\right),\quad m=1,2,3,\ldots, $$

where $\mathcal{B}_{0,p,q}\neq 0$, $\mathcal{B}_{0,p,q}=\frac{1}{\mathcal{A}_{0,p,q}}$ and $\rho_{m,p,q}(x)$, $m=0,1,2,\ldots,$ are the (p,q)-Bessel polynomials of degree m.

Proof. Consider ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ to be a sequence of the pqBeAP defined by equation (2.2) and $\mathcal{A}_{m,p,q}$, $\mathcal{B}_{m,p,q}$ be two numerical sequences (the coefficients of q-Taylor's series expansions of functions) such that

$$ \begin{aligned} \mathcal{A}_{p,q}(t)=&\mathcal{A}_{0,p,q}+\mathcal{A}_{1,p,q}\frac{t}{[1]_{p,q}!}+\mathcal{A}_{2,p,q}\frac{t^2}{[2]_{p,q}!}+\cdots+\mathcal{A}_{m,p,q}\frac{t^m}{[m]_{p,q}!}+\cdots,\\ &m=0,1,2,3,\ldots,\ \mathcal{A}_{0,p,q}\neq 0, \end{aligned} \tag{2.7} $$

$$ \begin{aligned} \hat{\mathcal{A}}_{p,q}(t)=&\mathcal{B}_{0,p,q}+\mathcal{B}_{1,p,q}\frac{t}{[1]_{p,q}!}+\mathcal{B}_{2,p,q}\frac{t^2}{[2]_{p,q}!}+\cdots+\mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!}+\cdots,\\ &m=0,1,2,3,\ldots,\ \mathcal{B}_{0,p,q}\neq 0, \end{aligned} \tag{2.8} $$

satisfying

$$ \mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t)=1. \tag{2.9} $$

On using Cauchy product rule for the two series production $\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t)$, we get

$$\begin{aligned}\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t) &= \sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\sum_{m=0}^{\infty}\mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\frac{t^m}{[m]_{p,q}!}.\end{aligned}$$

Consequently,

$$\sum_{s=0}^{m}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q} = \begin{cases}1, & \text{if } m=0,\\ 0, & \text{if } m>0.\end{cases} \tag{2.10}$$

That is

$$\begin{cases}\mathcal{B}_{0,p,q} = \dfrac{1}{\mathcal{A}_{0,p,q}},\\ \mathcal{B}_{m,p,q} = -\dfrac{1}{\mathcal{A}_{0,p,q}}\left(\displaystyle\sum_{s=1}^{m}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\right), \quad m=1,2,\ldots\end{cases} \tag{2.11}$$

Next, multiplying both sides of equation (2.2) by $\hat{\mathcal{A}}_{p,q}(t)$, we get

$$\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t})) = \hat{\mathcal{A}}_{p,q}(t)\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}.$$

Further, in view of equations (2.1), (2.8) and (2.9), the above equation becomes

$$\sum_{m=0}^{\infty}\rho_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty}\mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}. \tag{2.12}$$

Now, on using Cauchy product rule for the two series in the r.h.s of equation (2.12), we obtain the following infinite system for the unknowns ${}_{\rho}\mathcal{A}_{m,p,q}(x)$:

(2.13)
$$\begin{cases}\mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) = 1,\\ \mathcal{B}_{1,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) = \rho_{1,p,q}(x),\\ \mathcal{B}_{2,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]_{p,q}\mathcal{B}_{1,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{2,p,q}(x) = \rho_{2,p,q}(x),\\ \vdots\\ \mathcal{B}_{m-1,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix}m-1\\1\end{smallmatrix}\right]_{p,q}\mathcal{B}_{m-2,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \cdots + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{m-1,p,q}(x)\\ = \rho_{m-1,p,q}(x),\\ \mathcal{B}_{m,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix}m\\1\end{smallmatrix}\right]_{p,q}\mathcal{B}_{m-1,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \cdots + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{m,p,q}(x) = \rho_{m,p,q}(x),\\ \vdots\end{cases}$$

Obviously the first equation of system (2.13) leads to our first assertion (2.5). The coefficient matrix of system (2.13) is lower triangular, so, this helps us to obtain the unknowns ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ by applying Cramer rule to the first $m+1$ equations of system (2.13). According to this, we can obtain

(2.14)
$$
{}_{\rho}\mathcal{A}_{m,p,q}(x)=\frac{\begin{vmatrix}
\mathcal{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1\\
\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & 0 & \cdots & 0 & \rho_{1,p,q}(x)\\
\mathcal{B}_{2,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & \cdots & 0 & \rho_{2,p,q}(x)\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \cdots & \mathcal{B}_{0,p,q} & \rho_{m-1,p,q}(x)\\
\mathcal{B}_{m,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \cdots & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \rho_{m,p,q}(x)
\end{vmatrix}}{\begin{vmatrix}
\mathcal{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1\\
\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & 0 & \cdots & 0 & 0\\
\mathcal{B}_{2,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \cdots & \mathcal{B}_{0,p,q} & 0\\
\mathcal{B}_{m,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \cdots & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q}
\end{vmatrix}},
$$

where $m=1,2,3,\ldots,$ which on expanding the determinant in the denominator and taking the transpose of the determinant in the numerator, yields to

(2.15)
$$
\begin{aligned}
{}_{\rho}\mathcal{A}_{m,p,q}(x)=&\frac{1}{(\mathcal{B}_{0,p,q})^{m+1}}\\
&\times\begin{vmatrix}
\mathcal{B}_{0,p,q} & \mathcal{B}_{1,p,q} & \mathcal{B}_{2,p,q} & \cdots & \mathcal{B}_{m-1,p,q} & \mathcal{B}_{m,p,q}\\
0 & \mathcal{B}_{0,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \cdots & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q}\\
0 & 0 & \mathcal{B}_{0,p,q} & \cdots & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \mathcal{B}_{0,p,q} & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q}\\
1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x)
\end{vmatrix}.
\end{aligned}
$$

Finally, after m circular row exchanges, that is after moving the j^{th} row to the $(j+1)^{\text{th}}$ position for $j=1,2,3,\ldots,m-1$, we arrive at our assertion (2.6). $\square$

On taking $p=1$ in Theorem 2.3, we get the determinant definition for the q-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,q}(x)$.

Corollary 2.1. *The q-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,q}(x)$ of degree m are defined by*

(2.16)
$$
{}_{\rho}\mathcal{A}_{0,q}(x)=\frac{1}{\mathcal{B}_{0,q}},
$$

$$
(2.17)\qquad {}_\rho\mathcal{A}_{m,q}(x)=\frac{(-1)^m}{(\mathcal{B}_{0,q})^{m+1}}\begin{vmatrix} 1 & \rho_{1,q}(x) & \rho_{2,q}(x) & \cdots & \rho_{m-1,q}(x) & \rho_{m,q}(x)\\ \mathcal{B}_{0,q} & \mathcal{B}_{1,q} & \mathcal{B}_{2,q} & \cdots & \mathcal{B}_{m-1,q} & \mathcal{B}_{m,q}\\ 0 & \mathcal{B}_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\mathcal{B}_{1,q} & \cdots & \begin{bmatrix}m-1\\1\end{bmatrix}_q\mathcal{B}_{m-2,q} & \begin{bmatrix}m\\1\end{bmatrix}_q\mathcal{B}_{m-1,q}\\ 0 & 0 & \mathcal{B}_{0,q} & \cdots & \begin{bmatrix}m-1\\2\end{bmatrix}_q\mathcal{B}_{m-3,q} & \begin{bmatrix}m\\2\end{bmatrix}_q\mathcal{B}_{m-2,q}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mathcal{B}_{0,q} & \begin{bmatrix}m\\m-1\end{bmatrix}_q\mathcal{B}_{1,q}\end{vmatrix},
$$

$$
\mathcal{B}_{m,q}=-\frac{1}{\mathcal{A}_{0,q}}\left(\sum_{s=1}^{m}\begin{bmatrix}m\\s\end{bmatrix}_q\mathcal{A}_{s,q}\mathcal{B}_{m-s,q}\right),\quad m=1,2,3,\ldots
$$

Theorem 2.4. *The following identity for the pqBeAP ${}_\rho\mathcal{A}_{m,p,q}(x)$ holds true:*

$$
{}_\rho\mathcal{A}_{m,p,q}(x)=\frac{1}{\mathcal{B}_{0,p,q}}\left(\rho_{m,p,q}(x)-\sum_{s=0}^{m-1}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}\mathcal{B}_{m-s,p,q}\ {}_\rho\mathcal{A}_{s,p,q}(x)\right),\quad m=1,2,\ldots
$$

Proof. Expanding the determinant in equation (2.6) with respect to the $(m+1)^{\text{th}}$ row and using the same technique used in [10], we get the required result. □

On taking $p=1$ in Theorem 2.4, we get the following result for the q-Bessel-Appell polynomials ${}_\rho\mathcal{A}_{m,q}(x)$.

Corollary 2.2. *The following identity for the qBeAP ${}_\rho\mathcal{A}_{m,q}(x)$ holds true:*

$$
{}_\rho\mathcal{A}_{m,q}(x)=\frac{1}{\mathcal{B}_{0,q}}\left(\rho_{m,q}(x)-\sum_{s=0}^{m-1}\begin{bmatrix}m\\s\end{bmatrix}_{q}\mathcal{B}_{m-s,q}\ {}_\rho\mathcal{A}_{s,q}(x)\right),\quad m=1,2,\ldots
$$

2.1. **Certain Members of the (p,q)-Bessel-Appell Polynomials.** Recently, different members of the family of (p,q)-Appell polynomials are studied by many researchers (see for example [4, 5]). By making suitable selections for the function $\mathcal{A}_{p,q}(t)$, the members belonging to the family of the (p,q)-Bessel-Appell polynomials ${}_\rho\mathcal{A}_{m,p,q}(x)$ can be obtained. The (p,q)-Bernoulli polynomials (pqBP) $\mathfrak{B}_{m,p,q}(x)$, (p,q)-Euler polynomials (pqEP) $\mathcal{E}_{m,p,q}(x)$ and (p,q)-Genocchi polynomials (pqGP) $\mathcal{G}_{m,p,q}(x)$ are important members of the (p,q)-Appell family. In this subsection, we introduce the (p,q)-Bessel-Bernoulli polynomials (pqBeBP) ${}_\rho\mathfrak{B}_{m,p,q}(x)$, (p,q)-Bessel-Euler polynomials (pqBeEP) ${}_\rho\mathcal{E}_{m,p,q}(x)$ and (p,q)-Bessel-Genocchi polynomials (pqBeGP) ${}_\rho\mathcal{G}_{m,p,q}(x)$ by means of the generating functions, series definitions and determinant definitions.

2.1.1. (p,q)-*Bessel-Bernoulli polynomials.* Since, for $\mathcal{A}_{p,q}(t)=\frac{t}{e_{p,q}(t)-1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqBP $\mathfrak{B}_{m,p,q}(x)$ (Table 1 (I)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_\rho\mathcal{A}_{m,p,q}(x)$ reduce to pqBeBP ${}_\rho\mathfrak{B}_{m,p,q}(x)$, which are defined by means of following generating function:

$$
(2.18)\qquad \frac{t}{e_{p,q}(t)-1}\ e_{p,q}(x(1-\sqrt{1-2t}))=\sum_{m=0}^{\infty}{}_\rho\mathfrak{B}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}.
$$

The pqBeBP ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$ of degree m are defined by the series

$$ {}_{\rho}\mathfrak{B}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathfrak{B}_{s,p,q}\rho_{m-s,p,q}(x). $$

The following identity for the pqBeBP ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$ holds true:

$$ {}_{\rho}\mathfrak{B}_{m,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}} \left(\rho_{m,p,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{B}_{m-s,p,q}\, {}_{\rho}\mathfrak{B}_{s,p,q}(x) \right), \quad m = 1, 2, \ldots \tag{2.19} $$

Further, by taking $\mathcal{B}_{0,p,q} = 1$ and $\mathcal{B}_{j,p,q} = \frac{1}{[j+1]_{p,q}}$, $j = 1, 2, 3, \ldots$, in equations (2.5) and (2.6), we obtain the determinant definition of the pqBeBP ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$.

Definition 2.2. The (p, q)-Bessel-Bernoulli polynomials ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$ of degree m are defined by

$$ {}_{\rho}\mathfrak{B}_{0,p,q}(x) = 1, \tag{2.20} $$

$$ {}_{\rho}\mathfrak{B}_{m,p,q}(x) = (-1)^m \begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x) \\ 1 & \frac{1}{[2]_{p,q}} & \frac{1}{[3]_{p,q}} & \cdots & \frac{1}{[m]_{p,q}} & \frac{1}{[m+1]_{p,q}} \\ 0 & 1 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \frac{1}{[2]_{p,q}} & \cdots & \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{p,q} \frac{1}{[m-1]_{p,q}} & \begin{bmatrix} m \\ 1 \end{bmatrix}_{p,q} \frac{1}{[m]_{p,q}} \\ 0 & 0 & 1 & \cdots & \begin{bmatrix} m-1 \\ 2 \end{bmatrix}_{p,q} \frac{1}{[m-2]_{p,q}} & \begin{bmatrix} m \\ 2 \end{bmatrix}_{p,q} \frac{1}{[m-1]_{p,q}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix} m \\ m-1 \end{bmatrix}_{p,q} \frac{1}{[2]_{p,q}} \end{vmatrix}, \tag{2.21} $$

$m = 1, 2, 3, \ldots,$

where $\rho_{m,p,q}(x)$, $m = 0, 1, 2, 3, \ldots$, are the (p, q)-Bessel polynomials of degree m.

2.1.2. *(p, q)-Bessel-Euler polynomials.* Since, for $\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}}{e_{p,q}(t)+1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqEP $\mathcal{E}_{m,p,q}(x)$ (Table 1 (II)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to pqBeEP ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ which are defined by means of following generating function:

$$ \frac{[2]_{p,q}}{e_{p,q}(t) + 1}\, e_{p,q}(x(1 - \sqrt{1 - 2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{E}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}. \tag{2.22} $$

The pqBeEP ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ of degree m are defined by the series:

$$ {}_{\rho}\mathcal{E}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{E}_{s,p,q}\rho_{m-s,p,q}(x). $$

The following identity for the pqBeEP ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ holds true:

$$ {}_{\rho}\mathcal{E}_{m,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}} \left(\rho_{m,p,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{B}_{m-s,p,q}\, {}_{\rho}\mathcal{E}_{s,p,q}(x) \right), \quad m = 1, 2, \ldots $$

Further, by taking $\mathcal{B}_{0,p,q} = 1$ and $\mathcal{B}_{j,p,q} = \frac{1}{2}$, $j = 1, 2, 3, \ldots$, in equations (2.5) and (2.6), we obtain the determinant definition of the pqBeEP ${}_{\rho}\mathcal{E}_{m,p,q}(x)$.

Definition 2.3. The (p, q)-Bessel-Euler polynomials ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ of degree m are defined by

$$(2.23)\qquad {}_{\rho}\mathcal{E}_{0,p,q}(x) = 1,$$

$$(2.24)\qquad {}_{\rho}\mathcal{E}_{m,p,q}(x) = (-1)^m \begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \frac{1}{2} & \cdots & \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{p,q} \frac{1}{2} & \begin{bmatrix} m \\ 1 \end{bmatrix}_{p,q} \frac{1}{2} \\ 0 & 0 & 1 & \cdots & \begin{bmatrix} m-1 \\ 2 \end{bmatrix}_{p,q} \frac{1}{2} & \begin{bmatrix} m \\ 2 \end{bmatrix}_{p,q} \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix} m \\ m-1 \end{bmatrix}_{p,q} \frac{1}{2} \end{vmatrix},$$

$$m = 1, 2, 3, \ldots,$$

where $\rho_{m,p,q}(x)$, $m = 0, 1, 2, 3, \ldots$, are the (p, q)-Bessel polynomials of degree m.

2.1.3. *(p, q)-Bessel-Genocchi polynomials.* Since, for $\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}t}{e_{p,q}(t)+1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqGP $\mathcal{G}_{m,p,q}(x)$ (Table 1 (III)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ which are defined by means of following generating functions:

$$(2.25)\qquad \frac{[2]_{p,q}t}{e_{p,q}(t)+1}\; e_{p,q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{G}_{m,p,q}(x)\; \frac{t^m}{[m]_{p,q}!}.$$

The pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ of degree m are defined by the series:

$${}_{\rho}\mathcal{G}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{G}_{s,p,q}\rho_{m-s,p,q}(x).$$

The following identity for the pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ holds true:

$${}_{\rho}\mathcal{G}_{m,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}}\left(\rho_{m,p,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{B}_{m-s,p,q}\; {}_{\rho}\mathcal{G}_{s,p,q}(x)\right), \quad m = 1, 2, \ldots$$

3. 2D (p, q)-Bessel-Appell Polynomials

First, we introduce the (p, q)-analogue of the 2D Appell polynomials which are the 2-variable generalization of the (p, q)-Appell polynomials denoted as 2D (p, q)-Appell polynomials $\mathcal{A}_{m,p,q}(x, y)$.

Definition 3.1. The (p, q)-analogue of the 2D Appell polynomials $\mathcal{A}_{m,p,q}(x, y)$ are defined by the following generating function:

$$(3.1)\qquad \mathcal{A}_{p,q}(t)\; e_{p,q}(xt)E_{p,q}(yt) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(x, y)\; \frac{t^m}{[m]_{p,q}!}, \quad \mathcal{A}_{m,p,q} = \mathcal{A}_{m,p,q}(0, 0).$$

Table 2. Some members of 2D (p,q)-Appell polynomials

S. No.	$\mathcal{A}_{\mathbf{p,q}}(\mathbf{t})$	Generating Functions	Polynomials
I.	$\mathcal{A}_{p,q}(t)=\frac{t}{(e_{p,q}(t)-1)}$	$\frac{t}{(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}\mathfrak{B}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$	The 2D (p,q)-Bernoulli polynomials
II.	$\mathcal{A}_{p,q}(t)=\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}\mathcal{E}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$	The 2D (p,q)-Euler polynomials
III.	$\mathcal{A}_{p,q}(t)=\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}\mathcal{G}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$	The 2D (p,q)-Genocchi polynomials

Some members of the 2D (p,q)-Appell polynomials are listed in Table 2.

The approach used in previous section is further exploited to introduce the 2D (p,q)-Bessel-Appell polynomials (2DpqBeAP) and focus on deriving its generating functions and series definitions.

In order to establish the generating function for the 2DpqBeAP, the following result is proved.

Theorem 3.1. *The following generating function for the 2D (p,q)-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$ holds true:*

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x,y)\,\frac{t^m}{[m]_{p,q}!}. \tag{3.2}$$

Proof. By expanding the first (p,q)-exponential function $e_{p,q}(xt)$ in the left hand side of the equation (3.1) and then replacing the powers of x, i.e., $x^0,x,x^2,\dots,x^m$ by the corresponding polynomials $\rho_{0,p,q}(x),\rho_{1,p,q}(x),\rho_{2,p,q}(x),\dots,\rho_{m,p,q}(x)$ in the left hand side and x by $\rho_{1,p,q}(x)$ in the right hand side of the resultant equation, we have

$$\mathcal{A}_{p,q}(t)\left(1+\rho_{1,p,q}(x)\frac{t}{[1]_{p,q}!}+\rho_{2,p,q}(x)\frac{t^2}{[2]_{p,q}!}\quad+\cdots+\rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!}+\cdots\right)E_{p,q}(yt)$$
$$=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)\,\frac{t^m}{[m]_{p,q}!}.$$

Further, summing up the series in left hand side and then using equation (2.1) in the resultant equation, we get

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)\,\frac{t^m}{[m]_{p,q}!}.$$

Finally, denoting the resultant 2DpqBeAP in the right hand side of the above equation by ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$, that is

$$\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)={}_{\rho}\mathcal{A}_{m,p,q}(x,y),$$

the assertion (3.2) is proved. □

Remark 3.1. It is remarked that for $p = 1$, the 2DpqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x, y)$ reduce to the 2D q-Bessel-Appell polynomials (2DqBeAP) ${}_{\rho}\mathcal{A}_{m,q}(x, y)$ such that

$$ {}_{\rho}\mathcal{A}_{m,q}(x, y) := {}_{\rho}\mathcal{A}_{m,1,q}(x, y). $$

Thus, taking $p = 1$ in equation (3.2), we get

$$ \mathcal{A}_q(t)e_q(x(1-\sqrt{1-2t}))E_q(yt) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,q}(x, y)\ \frac{t^m}{[m]_q!}, $$

which is the generating function for the 2D q-Bessel-Appell polynomials.

Next, we give the series definition for the 2DpqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x, y)$, by proving the following result.

Theorem 3.2. *The 2D (p, q)-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x, y)$ are defined by the following series definition:*

$$ {}_{\rho}\mathcal{A}_{m,p,q}(x, y) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,p,q}(x). \tag{3.3} $$

Proof. In view of equations (1.2) and (2.2), equation (3.2) can be written as:

$$ \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} \sum_{s=0}^{\infty} q^{\binom{s}{2}}\ y^s \frac{t^s}{[s]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x, y)\ \frac{t^m}{[m]_{p,q}!}, $$

which on using the Cauchy product rule gives

$$ \sum_{m=0}^{\infty}\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x, y)\ \frac{t^m}{[m]_{p,q}!}. \tag{3.4} $$

Equating the coefficients of like powers of t in both sides of the above equation, we arrive at our assertion (3.3). ☐

Remark 3.2. For $p = 1$, series definition (3.3) becomes

$$ {}_{\rho}\mathcal{A}_{m,q}(x, y) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,q}(x), $$

which is the series definition for the 2D q-Bessel-Appell polynomials.

Certain members belonging to the 2D (p, q)-Appell family are given in Table 2. Since, corresponding to each member belonging to the 2D (p, q)-Appell family, there exists a new special polynomial belonging to the 2D (p, q)-Bessel-Appell family. Thus, by making suitable choices for the functions $\mathcal{A}_{p,q}(t)$ in equations (3.2) and (3.3), the generating functions and series definitions for the corresponding members belonging to the 2D (p, q)-Bessel-Appell family can be obtained. The resultant members of the 2D (p, q)-Bessel-Appell family along with their generating functions and series definitions are given in Table 3.

Table 3. Certain members belonging to the 2D (p,q)-Bessel-Appell polynomials

S. No.	$\mathcal{A}_{\mathbf{p,q}}(\mathbf{t})$	Generating Functions	Series Definition	Polynomials
I.	$\frac{t}{(e_{p,q}(t)-1)}$	$\frac{t}{(e_{p,q}(t)-1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty} {}_\rho\mathfrak{B}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$	${}_\rho\mathfrak{B}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s {}_\rho\mathfrak{B}_{m-s,p,q}(x)$	The 2D (p,q)-Bessel-Bernoulli polynomials
II.	$\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty} {}_\rho\mathcal{E}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$	${}_\rho\mathcal{E}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s {}_\rho\mathcal{E}_{m-s,p,q}(x)$	The 2D (p,q)-Bessel-Euler polynomials
III.	$\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}$	$\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty} {}_\rho\mathcal{G}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$,	${}_\rho\mathcal{G}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s {}_\rho\mathcal{G}_{m-s,p,q}(x)$	The 2D (p,q)-Bessel-Genocchi polynomials

4. Graphical Representation

In this section with the help of Matlab, we plot the graphs of (p,q)-Bessel-Bernoulli polynomials ${}_\rho\mathfrak{B}_{m,p,q}(x)$, (p,q)-Bessel-Euler polynomials ${}_\rho\mathcal{E}_{m,p,q}(x)$. To draw the graphs of these polynomials, we consider the values of the first four (p,q)-Bessel polynomials $\rho_{m,p,q}(x)$, the expressions of these polynomials are given in Table 4.

Table 4. Expressions of the first four $\rho_{m,p,q}(x)$.

m	0	1	2	3
$\rho_{m,p,q}(x)$	1	x	$x^2+\frac{[2]_{p,q}}{2}x$	$x^3+\frac{[3]_{p,q}[2]_{p,q}}{2}x^2+\frac{[4]_{p,q}[3]_{p,q}}{4}x$

Next, taking $p=\frac{1}{2}$, $q=\frac{1}{4}$ in the determinant definitions (2.21), (2.24) and using the expressions of the $\rho_{m,p,q}(x)$ from Table 4, we get the results mentioned in Table 5 for $m=0,1,2,3$.

Table 5. The first four expressions of ${}_\rho\mathfrak{B}_{m,\frac{1}{2},\frac{1}{4}}(x)$ and ${}_\rho\mathcal{E}_{m,\frac{1}{2},\frac{1}{4}}(x)$.

m	0	1	2	3
${}_\rho\mathfrak{B}_{m,\frac{1}{2},\frac{1}{4}}(x)$	1	$-\frac{4}{3}+x$	$x^2-\frac{5}{8}x-\frac{20}{21}$	$x^3-\frac{161}{384}x^2-\frac{7493}{12288}x-\frac{107}{45}$
${}_\rho\mathcal{E}_{m,\frac{1}{2},\frac{1}{4}}(x)$	1	$-\frac{1}{2}+x$	$x^2-\frac{5}{16}$	$x^3-\frac{7}{128}x^2-\frac{791}{4096}x-\frac{165}{512}$

Now, with the help of Matlab and using equations (2.20), (2.23) and the expressions of ${}_\rho\mathfrak{B}_{m,p,q}(x)$ and ${}_\rho\mathcal{E}_{m,p,q}(x)$ from Table 5, we get the graphs at Figure 1 and 2.

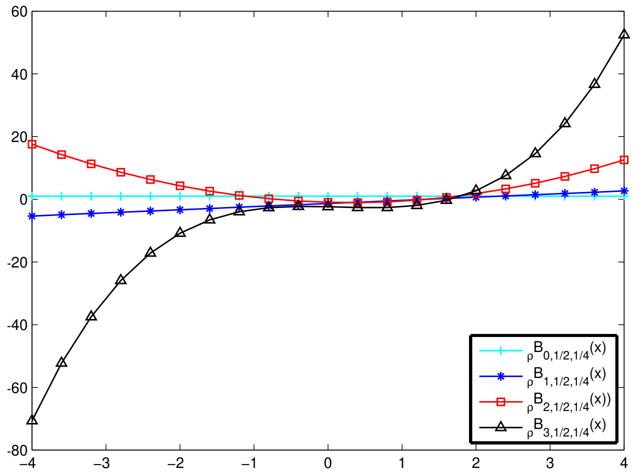

FIGURE 1. Graph of ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$

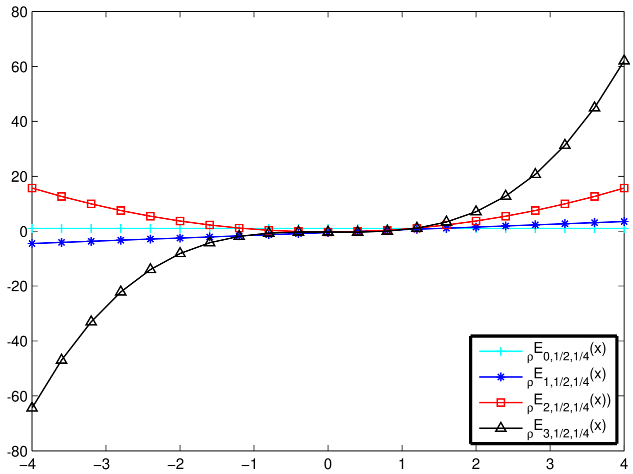

FIGURE 2. Graph of ${}_{\rho}\mathcal{E}_{m,p,q}(x)$

Further, setting $m = 3, p = \frac{1}{2}, q = \frac{1}{4}$ in the series definitions of ${}_{\rho}\mathfrak{B}_{m,p,q}(x,y)$, ${}_{\rho}\mathcal{E}_{m,p,q}(x,y)$ given in Table 3 and using the expressions of ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$, ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ from Table 5, we have

(4.1)
$$ {}_{\rho}\mathfrak{B}_{3,\frac{1}{2},\frac{1}{4}}(x,y) = x^3 - \frac{161}{384}x^2 - \frac{7493}{12288}x - \frac{107}{45} + \frac{7}{16}x^2y - \frac{35}{128}xy - \frac{5}{12}y - \frac{7}{48}y^2 + \frac{7}{64}xy^2 + \frac{1}{64}y^3, $$

(4.2)
$$ {}_{\rho}\mathcal{E}_{3,\frac{1}{2},\frac{1}{4}}(x,y) = x^3 - \frac{7}{128}x^2 - \frac{791}{4096}x - \frac{165}{512} + \frac{7}{16}x^2y - \frac{35}{256}y - \frac{7}{128}y^2 + \frac{7}{64}xy^2 + \frac{1}{64}y^3, $$

In view of equations (4.1)–(4.2), we get the surface plots at Figure 3 and 4.

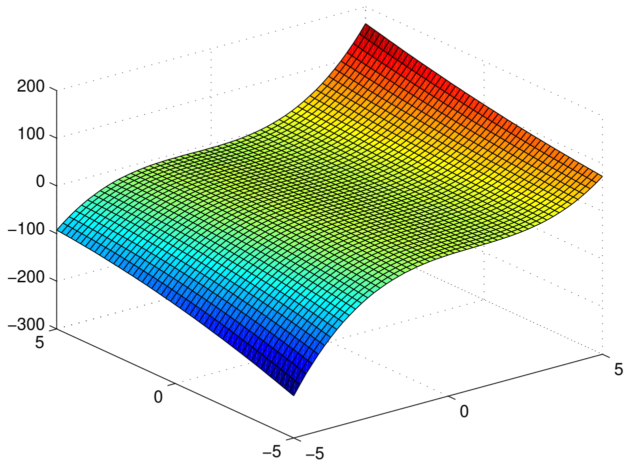

FIGURE 3. Surface plot of ${}_{\rho}\mathfrak{B}_{3,\frac{1}{2},\frac{1}{4}}(x,y)$

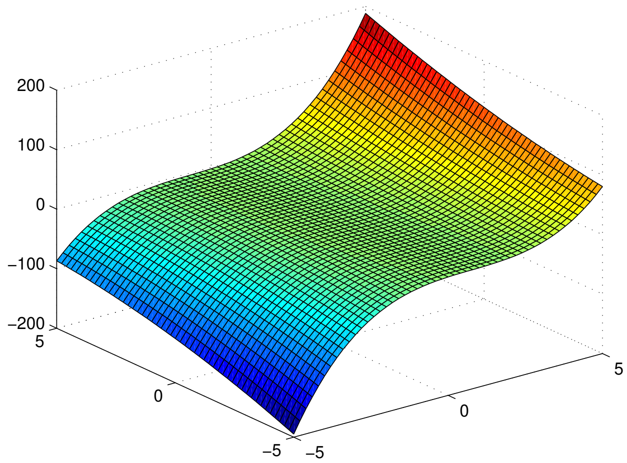

FIGURE 4. Surface plot of ${}_{\rho}\mathcal{E}_{3,\frac{1}{2},\frac{1}{4}}(x,y)$

5. Concluding Remarks

The Bernoulli, Euler and Genocchi numbers are among the most interesting and important number sequences in mathematics. These numbers are particularly important in number theory, they have deep connections with calculus of finite differences, combinatorics and other fields. Here, let us recall (p, q)-Bernoulli, (p, q)-Euler and (p, q)-Genocchi numbers.

We note that (see [6])

$$\begin{aligned} \mathfrak{B}_{m,p,q} &:= \mathfrak{B}_{m,p,q}(0), \quad (p,q)\text{-Bernoulli numbers,} \\ \mathcal{E}_{m,p,q} &:= \mathcal{E}_{m,p,q}(0), \quad (p,q)\text{-Euler numbers,} \\ \mathcal{G}_{m,p,q} &:= \mathcal{G}_{m,p,q}(0), \quad (p,q)\text{-Genocchi numbers.} \end{aligned}$$

Further, we note that

$$\rho_{m,p,q} := \rho_{m,p,q}(0), \quad (p,q)\text{-Bessel numbers.}$$

In this section, we introduce the numbers related to the polynomial families established in Sections 2 and 3.

Taking $x = 0$ in the generating functions of the ${}_\rho\mathfrak{B}_{m,p,q}(x)$, ${}_\rho\mathcal{E}_{m,p,q}(x)$ and ${}_\rho\mathcal{G}_{m,p,q}(x)$ given by equations (2.18), (2.22) and (2.25), the (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi numbers are obtained. These numbers are listed in Table 6.

TABLE 6. Certain members belonging to (p,q)-Bessel-Appell numbers

S. No.	**Notations**	**Generating Functions**	**Numbers**
I.	${}_\rho\mathfrak{B}_{m,p,q} := {}_\rho\mathfrak{B}_{m,p,q}(0)$	$\frac{t}{e_{p,q}(t)-1} = \sum\limits_{m=0}^{\infty} {}_\rho\mathfrak{B}_{m,p,q} \frac{t^m}{[m]_{p,q}!}$	The (p,q)-Bessel-Bernoulli numbers
II.	${}_\rho\mathcal{E}_{m,p,q} := {}_\rho\mathcal{E}_{m,p,q}(0)$	$\frac{[2]_{p,q}}{(e_{p,q}(t)+1)} = \sum\limits_{m=0}^{\infty} {}_\rho\mathcal{E}_{m,p,q} \frac{t^m}{[m]_{p,q}!}$	The (p,q)-Bessel-Euler numbers
III.	${}_\rho\mathcal{G}_{m,p,q} := {}_\rho\mathcal{G}_{m,p,q}(0)$	$\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)} = \sum\limits_{m=0}^{\infty} {}_\rho\mathcal{G}_{m,p,q} \frac{t^m}{[m]_{p,q}!}$,	The (p,q)-Bessel-Genocchi numbers

Similarly, on taking $x = y = 0$ in the generating functions of the ${}_\rho\mathfrak{B}_{m,p,q}(x,y)$, ${}_\rho\mathcal{E}_{m,p,q}(x,y)$ and ${}_\rho\mathcal{G}_{m,p,q}(x,y)$ given in Table 3 (I-III), we get the same numbers given in Table 6 (I-III).

We note that the class of numbers introduced in this section are actually the (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi numbers, respectively.

In this article, the (p,q)-analogue of Bessel polynomials and its hybrid form are introduced by means of series expansion and generating function. The determinant form related to these polynomials are derived, which can be helpful for computation purposes and can also be used in finding the solutions of general linear interpolation problems.

Some properties including addition theorem, difference equations and recurrence relations for the (p,q)-Appell family have been analyzed and established in [13] (see also [11]). This provides motivation to establish (p,q)-difference equations and other properties for (p,q)-Bessel-Appell polynomials and their generalized 2D form in future investigation.

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