Kragujevac Journal of Mathematics
Volume 45(2) (2021), Pages 267–274.

APPLICATION OF THE SUMUDU TRANSFORM TO SOLVE REGULAR FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

ZINEB KAISSERLI[1] AND DJILLALI BOUAGADA[1]

Abstract. In this work, Sumudu transform is used to establish the solution of a regular fractional continuous-time linear system based on Caputo fractional derivative-integral. First results of the proposed method are presented and compared to the existing ones.

1. Introduction

In recent years, fractional calculus and fractional systems appear and play a key role in several applications and domains [7, 9, 12, 13]. The use of the mathematical tools, theories and methods is required to solve such problems. Find the solution of a regular fractional continuous-time linear system with regular pencil is one of the most important problems in systems and control theory [3, 14].

In this paper, instead of Laplace transform [2, 7], the Sumudu transform [1, 16], which is a kind of the Laplace transform but does not require any conditions on the function to be transformable [15], is used to solve such a regular fractional continuous-time linear system based on the Caputo fractional derivative-integral [7]. The Sumudu transform is relatively new but it is as powerful as the Laplace transform and has some good features as for instance, unlike the Laplace transform, the Sumudu transform of the Heaviside step function is also Heaviside step function [6].

More than that, an interesting fact about this transformation is that the original function and its Sumudu transform have the same Taylor coefficients expect $n!$. Hence,

Key words and phrases. Regular fractional linear system, Caputo fractional derivative-integral, Sumudu Transform.

2010 *Mathematics Subject Classification.* Primary: 26A33. Secondary: 45F15, 44A10.

DOI 10.46793/KgJMat2102.267K

Received: March 03, 2018.

Accepted: December 04, 2018.

the Sumudu transform, can be viewed as a power series transformation as shown in [11,16].

Another very interesting property, which makes the Sumudu transform more advantageous then the Laplace transform is the scale and unit preserving properties which could provide convenience when solving differential equations. In other words, the Sumudu transform can be used to solve various mathematical and physical sciences problems without restoring to a new frequency domain [1,16].

Furthermore, the solution of a regular fractional continuous-time linear system, using the Sumudu transform, requires only some boundary conditions and compatibility requirements.

The rest of the present paper is organized as follows. Basic definitions and properties are recalled in Section 2. Then, in Section 3, the solution of the regular fractional continuous-time linear systems is proposed using Sumudu transform followed by some academic and real examples which are presented in Section 4. The obtained results are compared to the state-of-the-art methods [2,7]. Finally, the last Section summarizes and discusses the obtained results.

2. Preliminaries

In the present section, main definitions and properties are recalled.

Definition 2.1 ([4])**.** The function defined by:

$$\mathbf{D}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad x^{(n)}(\tau)=\frac{d^n x(\tau)}{d\tau^n}, \tag{2.1}$$

is called the Caputo fractional derivative-integral of the function $x(t)$, where $n-1<\alpha\leq n$, $n\in\mathbb{N}^*$, and Γ refers to the standard Gamma function.

Definition 2.2 ([1,16])**.** Let us consider the set of functions:

$$\mathcal{A}=\left\{x(t)\mid \text{exists } M,\, \tau_1,\, \tau_2>0,\, |x(t)|<Me^{-\frac{|t|}{\tau_j}},\ \text{ if } t\in(-1)^j\times[0,\infty)\right\}.$$

The Sumudu transform $X(v)$ of the function $x(t)$ is defined over the set of functions $\mathcal{A}$ by:

$$X(v)=\mathcal{S}[x(t)](v)=v^{-1}\int_0^{\infty}x(t)e^{-\frac{t}{v}}dv, \quad v\in(-\tau_1,\tau_2). \tag{2.2}$$

Theorem 2.1 ([10])**.** *The Sumudu transform of the fractional derivative-integral* (2.1) *for $n-1<\alpha\leq n$, $n\in\mathbb{N}^*$ has the form:*

$$\mathcal{S}[\mathbf{D}^{\alpha}x(t)](v)=v^{-\alpha}\left(X(v)-\sum_{k=1}^{n}v^{k-1}\left[x^{(k-1)}(t)\right]_{t=0}\right), \tag{2.3}$$

where $X(v)$ refers to the Sumudu transform of the function $x(t)$.

Proposition 2.1 ([1]). *Let $x_1(t)$ and $x_2(t)$ be in $\mathcal{A}$, having the Sumudu transforms $X_1(v)$ and $X_2(v)$, respectively. Then, the Sumudu transform of the convolution product of x_1 and x_2*

$$(x_1 \star x_2)(t) = \int_0^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau,$$

is given by:

$$\mathcal{S}[(x_1 \star x_2)(t)](v) = v\, X_1(v)\, X_2(v).$$

Proposition 2.2 ([1]). *For any $a \in \mathbb{R}_+^*$, the Sumudu transform of $\frac{t^a}{\Gamma(a+1)}$ is:*

$$\mathcal{S}\left[\frac{t^a}{\Gamma(a+1)}\right](v) = v^a.$$

In the following, we denote by $\mathbb{R}^{m\times n}$, the set of real matrices with m rows and n columns and by $\mathbb{R}^m$, the set of real columns vectors.

Proposition 2.3 ([7]). *Let $F \in \mathbb{R}^{n\times n}$ be a real matrix. Then, for any $v \in \mathbb{C}$ and $n-1 < \alpha \leq n$, $n \in \mathbb{N}^*$, the Laurent series is given by:*

$$(I_n - v^{\alpha} F)^{-1} = \sum_{k=0}^{\infty} F^k\, v^{k\,\alpha}. \tag{2.4}$$

3. Main Results

Let us consider the following regular fractional continuous-time linear systems:

$$\mathbf{D}^{\alpha} x(t) = Ax(t) + Bu(t), \tag{3.1}$$

$$y(t) = Cx(t) + Du(t), \tag{3.2}$$

where $\mathbf{D}^{\alpha}$ is the Caputo fractional derivative-integral, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, the input and the output vectors of the model respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.

The boundary conditions of the system (3.1) are given by:

$$x(0) = x_0.$$

Furthermore, the solution $x(t)$ is impulse free which is equivalent to the following compatibility conditions:

- $v^{k+i\alpha}\, A^i\, x^{(k)}(0)$ exists for all $i \in \mathbb{N}$, and $0 \leq k \leq n-1$, $n \in \mathbb{N}^*$ and $v \in (-\tau_1, \tau_2)$;
- $u(t)$ is provided.

It is assumed that the pencil of the pair $(I_n,\, A)$ is regular, i.e.,

$$\det(I_n - v^{\alpha} A)^{-1} \neq 0, \tag{3.3}$$

for almost $v \in \mathbb{C}$.

By applying the Sumudu transform (formulas (2.2) and (2.3)) to the system (3.1), we obtain:

$$\mathcal{S}[\mathbf{D}^{\alpha} x(t)](v) = \mathcal{S}[Ax(t) + Bu(t)](v).$$

Let us denote $X(v)$ and $U(v)$ as the Sumudu transforms of $x(t)$ and $u(t)$ respectively. Then, the use of the formula (3.3), yields:

$$X(v) = (I_n - v^\alpha A)^{-1} \left(v^\alpha BU(t) + \sum_{k=1}^{n} v^{k-1} x^{(k-1)}(0) \right),$$

which is equivalent to:

$$X(v) = v^\alpha (I_n - v^\alpha A)^{-1} BU(t) + (I - v^\alpha A)^{-1} \sum_{k=0}^{n-1} v^k x^{(k)}(0).$$

By the Laurent series (2.4), we obtain:

$$X(v) = \sum_{i=0}^{\infty} A^i B\, v \left(v^{(i+1)\alpha-1} \right) \left(U(v) \right) + \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} A^i v^{i\alpha+k} x^{(k)}(0).$$

Finally, applying the convolution theorem (Proposition 2.1) and the inverse Sumudu transform (Proposition 2.2) give the following theorem.

Theorem 3.1. *The solution of the implicit fractional dynamical system* (3.1) *is given by:*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} \frac{A^i B}{\Gamma((i+1)\alpha)} \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau) d\tau \\ &+ \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} A^i \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} x^{(k)}(0), \end{aligned} \tag{3.4}$$

where α and Γ represent the fractional derivative-integral order and the standard Gamma function, respectively.

If $\alpha = 1$, its remain to the following.

Corollary 3.1. *For $\alpha = 1$, we get:*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} A^i B \frac{1}{\Gamma(i+1)} \int_0^t (t-\tau)^i\, u(\tau)\, d\tau \\ &+ \sum_{i=0}^{\infty} A^i \frac{t^i}{\Gamma(i+1)} x_0. \end{aligned}$$

4. Experimental Results

This section present academic and real examples. In both cases the obtained results are compared to the existing ones.

Example 4.1. Find the solution of the system (3.1) for $0 < \alpha \leq 1$ and:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u(t) = \mathbb{I}(t).$$

Using (3.4), it follows that:

$$x(t) = \begin{pmatrix} 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix},$$

which is the same result obtained in [7].

Example 4.2. Let us consider the following system:

$$\mathbf{D}^{\alpha} Ex(t) = Ax(t) + Bu(t), \tag{4.1}$$

with $0 < \alpha \leq 1$ and the matrices:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}, \quad D = 0 \tag{4.2}$$

and the initial conditions:

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

It is clear that $\det E \neq 0$. Therefore, the system (4.1) becomes:

$$\mathbf{D}^{\alpha} x(t) = \tilde{A}x(t) + \tilde{B}u(t),$$

where

$$\tilde{A} = E^{-1}A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = E^{-1}B = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

It follows then

$$x(t) = \begin{pmatrix} 1 \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau - \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} - \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{pmatrix}.$$

Finally, using the systems (3.2) and (4.2), and the state $x(t)$ the output result is:

$$y(t) = 3 + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

the same result is obtained using Laplace transform [2].

Example 4.3. Let us consider the following regular fractional continuous-time system:

$$\mathbf{D}^{1.5}x(t) = Ax(t) + Bu(t), \tag{4.3}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -w_h^{1.5} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad w_h = 20.$$

The fractional continuous-time linear system (4.3) is derived from the one degree of freedom model of a passive car suspension as shown in Figure 1.

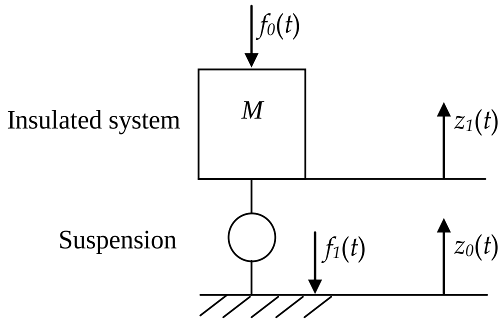

FIGURE 1. One degree of freedom general model of a car suspension [5].

M represents the car quarter mass, z_0 is the profile of the road, f_0 is the efforts applied on the suspension, and z_1, f_1 are the force generated by the suspension and the vertical movement of the mass respectively.

In this example $\alpha = 1.5$. Then, using (3.4), the solution is:

$$\begin{aligned} x(t) = {} & \frac{2\sqrt{\pi}}{\pi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\int_0^t (t-\tau)^{0.5}u(\tau)d\tau + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -w_h^{1.5} \end{pmatrix}\int_0^t (t-\tau)^2 u(\tau)d\tau \\ & + \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix} + \frac{4\sqrt{\pi}}{3\pi}\begin{pmatrix} x_{0,2} \\ x_{0,3} \\ -x_{0,3}\, w_h^{1.5} \end{pmatrix} t^{1.5} + \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ x'_{0,3} \end{pmatrix} t \\ & + \frac{8\sqrt{\pi}}{15\pi}\begin{pmatrix} x'_{0,2} \\ x'_{0,3} \\ -x'_{0,3}\, w_h^{1.5} \end{pmatrix} t^{2.5} \\ & + \sum_{i=2}^{\infty}\begin{pmatrix} (-w_h^{1.5})^{i-2} \\ (-w_h^{1.5})^{i-1} \\ (-w_h^{1.5})^{i} \end{pmatrix}\Bigg[\frac{1}{(1.5i+0.5)\,\Gamma(1.5i+0.5)}\int_0^t (t-\tau)^{1.5i+0.5}u(\tau)d\tau \\ & + \frac{x_{0,3}}{1.5i\Gamma(1.5i)}t^{1.5i} + \frac{x'_{3,0}}{(2.25i^2+1.5i)\Gamma(1.5i)}t^{1.5i+1}\Bigg], \end{aligned}$$

where

$$x(0) = x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix} \quad \text{and} \quad x'(0) = \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ x'_{0,3} \end{pmatrix}.$$

5. Discussion and Conclusion

In this paper, a new method for solving regular fractional continuous-time linear systems is presented which has been already introduced in [8]. The main idea consist on using the Sumudu transform to solve such a system. Thanks to the interesting properties of the Sumudu transform, the result can be derived easily and the method can be used for several practical applications.

The first results obtained are promising and encourage us to extend the method to singular fractional continuous-time linear systems, and to other type of systems and circuits, and also to other applications as for example to crone suspension which are one of our future research topics and will be discussed in a separate paper.

References

[1] F. M. Belgacem, A. A. Karaballi and S. Kalla, *Analytical investigations of the sumudu transform and applications to integral production equations*, Problems in Engineering, Hindawi Publishing Corporation **3** (2003), 103–118.

[2] D. Bouagada and P. Van Dooren, *State space solution of implicit fractional continuous-time systems*, Fract. Calc. Appl. Anal. **15**(3) (2012), 356–361.

[3] R. Caponetto, G. Dongola, L. Fortuna and I. Petras, *Fractional Order Systems: Modeling and Control Applications*, World Scientific Series on Nonlinear Science, Series A **72**, World Scientific, Singapore, 2010.

[4] M. Caputo, *Elasticitá e Dissipazione*, Zanichelli, Bologna, 1969.

[5] L. Fadiga, C. Farges, J. Sabatier and M. Moze, *On computation of H_∞ norm for commensurate fractional order systems*, 50^{th} IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA, 2011, 8231–8236.

[6] F. Jarad, K. Bayram, T. Abdeljawad and D. Baleanu, *On the discrete Sumudu transform*, Romanian Reports in Physics **64**(2) (2012), 347–356.

[7] T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*, Studies in Systems, Decision and Control **13**, Springer International Publishing, Switzerland, 2015.

[8] Z. Kaisserli and D. Bouagada, *State space solution of fractional continuous time systems using sumudu transform*, Bulletin d'Information de l'Académie de Hassan II des Sciences et Techniques, MADEV Conference, Rabat, Morocco, 2017, 53–57.

[9] S. Karunathilaka, N. Hampson, R. Leek and T. Sinclair, *The impedance of the alkaline zinc-manganese dioxide cell. II an interpretation of the data*, Journal of Applied Electrochemistry **11**(6) (1981), 715–721.

[10] Q. D. Katatbeh and F. M. Belgacem, *Application of the Sumudu transform to fractional differential equation*, Nonlinear Stud. **18**(1) (2011), 99–112.

[11] A. Kiliçman, N. Eltayeb and R. A. M. Atan, *A note on the comparison between Laplace and Sumudu transforms*, Bull. Iranian Math. Soc. **37** (2011), 131–141.

[12] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1998.

[13] V. H. Schmidt and J. E. Drumheller, *Dielectric properties of lithium hydrazinium sulfate*, Phys. Rev. **4** (1971), 4582–4597.

[14] D. Valèrio and J. Costa, *A method for identifying digital models and its application to non-integer order control*, Controlo-6th Portuguese Conference on Automatic Control, APCA, Faro, Junho, 2004.

[15] J. Vashi and M.G. Timol, *Laplace and Sumudu transforms and their application*, International Journal of Innovation Science, Engineering & Technology, **3**(8) (2016), 538–542.

[16] G. K. Watugala, *Sumudu transform - a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Tech. **24** (1993), 35–43.

[1]ACSY Team LMPA,
Mathematical and Computer Science Division,
Abdelhamid Ibn Badis University - Mostaganem (UMAB),
P. O. Box 227 University of Mostaganem, 27000 Mostaganem, Algeria
Email address: zineb.kaisserli@univ-mosta.dz
Email address: djillali.bouagada@univ-mosta.dz