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SOME REFINEMENTS OF THE NUMERICAL RADIUS INEQUALITIES VIA YOUNG INEQUALITY

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Abstract. In this paper, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ and refine generalized inequalities involving powers of the numerical radius for sums and products of Hilbert space operators.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall that for $A \in B(\mathcal{H})$, $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$, $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$ and $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, denote the numerical range, the numerical radius and the usual operator norm of A, respectively. Also an operator $A \in B(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for each $x \in \mathcal{H}$ and, in this case, is denoted by $A \geqslant 0$.

It is well-known that $\overline{W(A)}$ is a convex subset of the complex plane that contains the convex hull spectrum of A (see [4, p. 7]). It is known that $w(\cdot)$ defines a norm on $B(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$ [4, Theorem 1.3-1]. For $A \in B(\mathcal{H})$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

The inequalities in (1.1) have been improved by many mathematicians, (see [2, 7, 10, 13, 17–19]).

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Kittaneh in [7,8] showed that if $A \in B(\mathcal{H})$, then

$$w(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}), \tag{1.2}$$

where $|A|^2 = A^*A$, and

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

He also obtained the following generalizations of the first inequality in (1.2) and the second inequality in (1.3):

$$w^r(A) \leq \frac{1}{2}\left\||A|^{2\lambda r} + |A^*|^{2(1-\lambda)r}\right\| \tag{1.4}$$

and

$$w^{2r}(A) \leq \left\|\lambda|A|^{2r} + (1-\lambda)|A^*|^{2r}\right\|, \tag{1.5}$$

where $0 < \lambda < 1$, and $r \geq 1$ in [9, Theorem 1, Theorem 2], respectively.

In Section 2 of this paper, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ and refine inequality (1.4) for $r \geq 1$ and inequality (1.5) for $r \geq 2$, see ([3,12,16]). In addition, we establish some improvements of norm and numerical radius inequalities for sums and powers of operators acting on a Hilbert space in Section 3. For recent work on the numerical radius inequalities, we refer the reader to [13–15,18].

2. Refinements of the Hölder-McCarthy Operator Inequality

In this section, we obtain an improvement of Hölder-McCarthy's operator inequality in the case when $r \geq 1$ and get some improvements of numerical radius inequalities for Hilbert space operators. The following lemmas are essential for our investigation. The first lemma is a simple consequence of the Jensen inequality for convex function $f(t) = t^r$, where $r \geq 1$.

Lemma 2.1. ([13, Lemma 2.1]). *Let* $a, b \geq 0$ *and* $0 \leq \lambda \leq 1$. *Then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b \leq (\lambda a^r + (1-\lambda)b^r)^{\frac{1}{r}}, \quad \textit{for } r \geq 1.$$

The second lemma is known as a generalized mixed Schwarz inequality.

Lemma 2.2. ([8, Lemma 5]). *Let* $A \in B(\mathcal{H})$ *and* $x, y \in \mathcal{H}$ *be two vectors and* $0 \leq \lambda \leq 1$. *Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\lambda}x, x\rangle\langle |A^*|^{2(1-\lambda)}y, y\rangle.$$

The third lemma follows from the spectral theorem for positive operators and the Jensen inequality and is known as the Hölder McCarthy inequality.

Lemma 2.3. ([13, Lemma 2.2]). *Suppose that* A *is a positive operator in* $B(\mathcal{H})$ *and* $x \in \mathcal{H}$ *is any unit vector. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for* $r \geq 1$;

(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for* $0 < r \le 1$.

The last lemma is an improvement of Hölder-McCarthy's inequality.

Lemma 2.4. ([6, Corollary 3.1]). *Let* A *be a positive operator on* $\mathcal{H}$*. If* $x \in \mathcal{H}$ *is a unit vector, then*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle, \quad \text{for } r \ge 2.$$

The next theorem is a refinement of inequality (1.5) for $r \ge 2$.

Theorem 2.1. *If* $A \in B(\mathcal{H})$, $0 < \lambda < 1$ *and* $r \ge 2$, *then*

$$w^{2r}(A) \le \|\lambda |A|^{2r} + (1-\lambda)|A^*|^{2r}\| - \inf_{\|x\|=1} \zeta(x),$$

where

$$\zeta(x) = \left\langle \left(\lambda \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r + (1-\lambda) \left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r \right) x, x \right\rangle.$$

Proof. Let $x \in \mathcal{H}$ be a unit vector.

$$\begin{aligned}
|\langle Ax, x\rangle|^2 &\le \langle |A|^{2\lambda} x, x\rangle \langle |A^*|^{2(1-\lambda)} x, x\rangle \quad \text{(by Lemma 2.2)}\\
&\le \langle |A|^2 x, x\rangle^\lambda \langle |A^*|^2 x, x\rangle^{1-\lambda} \quad \text{(by Lemma 2.3 (ii))}\\
&\le (\lambda \langle |A|^2 x, x\rangle^r + (1-\lambda)\langle |A^*|^2 x, x\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma 2.1)}\\
&\le \Bigg(\lambda \left(\langle |A|^{2r} x, x\rangle - \left\langle \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r x, x \right\rangle\right)\\
&\quad + (1-\lambda)\left(\langle |A^*|^{2r} x, x\rangle - \left\langle \left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r x, x \right\rangle\right)\Bigg)^{\frac{1}{r}}\\
&\quad \text{(by Lemma 2.4).}
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} \le{}& \lambda \left(\langle |A|^{2r} x, x\rangle - \left\langle \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r x, x \right\rangle\right)\\
&+ (1-\lambda)\left(\langle |A^*|^{2r} x, x\rangle - \left\langle \left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r x, x \right\rangle\right).
\end{aligned}$$

By taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired relation. □

Recall that the Young inequality says that if $a, b \ge 0$ and $\lambda \in [0, 1]$, then

$$(1-\lambda)a + \lambda b \ge a^{1-\lambda} b^\lambda.$$

Many mathematicians improved the Young inequality and its reverse. Kober [11], proved that for $a, b > 0$

$$(1-\lambda)a + \lambda b \le a^{1-\lambda} b^\lambda + (1-\lambda)(\sqrt{a} - \sqrt{b})^2, \quad \lambda \ge 1. \tag{2.1}$$

By using (2.1), we obtain a refinement of the Hölder-McCarthy inequality.

Lemma 2.5. *Let $A \in B(\mathcal{H})$ be a positive operator. Then*

$$\langle Ax, x\rangle^{\lambda}\left(1+2(\lambda-1)\left(1-\frac{\langle A^{\frac{1}{2}}x, x\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}}\right)\right) \leq \langle A^{\lambda}x, x\rangle, \tag{2.2}$$

for any $\lambda \geq 1$ and $x \in \mathcal{H}$ with $\|x\| = 1$.

Proof. Applying functional calculus for the positive operator A in (2.1), we get

$$(1-\lambda)aI + \lambda A \leq a^{1-\lambda}A^{\lambda} + (1-\lambda)\left(aI + A - 2\sqrt{a}A^{\frac{1}{2}}\right).$$

The above inequality is equivalent to

$$(1-\lambda)a + \lambda\langle Ax, x\rangle \leq a^{1-\lambda}\langle A^{\lambda}x, x\rangle + (1-\lambda)\left(a + \langle Ax, x\rangle - 2\sqrt{a}\langle A^{\frac{1}{2}}x, x\rangle\right), \tag{2.3}$$

for any $x \in \mathcal{H}$ with $\|x\| = 1$. By substituting $a = \langle Ax, x\rangle$ in (2.3), we get

$$\langle Ax, x\rangle \leq \langle Ax, x\rangle^{1-\lambda}\langle A^{\lambda}x, x\rangle + 2(1-\lambda)\langle Ax, x\rangle\left(1-\frac{\langle A^{\frac{1}{2}}x, x\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}}\right).$$

By rearranging terms, we get the desired result (2.2). □

Note that by the Hölder-McCarthy inequality, $1 \geq 1 - \frac{\langle A^{\frac{1}{2}}x, x\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}} \geq 0$. Hence, the following chain of inequalities are true:

$$\langle Ax, x\rangle^{\lambda} \leq \langle Ax, x\rangle^{\lambda}\left(1+2(\lambda-1)\left(1-\frac{\langle A^{\frac{1}{2}}x, x\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}}\right)\right) \leq \langle A^{\lambda}x, x\rangle,$$

where A is positive and $\lambda \geq 1$. One can easily see that

$$1-\frac{\left\langle A^{\frac{1}{2}}x, x\right\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}} \geq \inf\left\{1-\frac{\left\langle A^{\frac{1}{2}}x, x\right\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

So,

$$1+2(\lambda-1)\left(1-\frac{\left\langle A^{\frac{1}{2}}x, x\right\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}}\right) \geq 1+2(\lambda-1)\inf\left\{1-\frac{\left\langle A^{\frac{1}{2}}x, x\right\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}} : x \in \mathcal{H}, \|x\| = 1\right\}. \tag{2.4}$$

If we denote the right-hand side of inequality (2.4) by $\zeta(x)$, then from inequality (2.2), we get

$$\langle Ax, x\rangle^{\lambda} \leq \frac{1}{\zeta}\langle A^{\lambda}x, x\rangle, \qquad \lambda \geq 1. \tag{2.5}$$

The following theorem is an improvement of inequality (1.4).

Theorem 2.2. *Let $A \in B(\mathcal{H})$ be an invertible operator, $0 < \lambda < 1$ and $r > 1$. If for each unit vector $x \in \mathcal{H}$*

$$\zeta(x) = \left(1+2(r-1)\left(1-\frac{\langle |A|^{\lambda}x, x\rangle}{\langle |A|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right)\right)$$

and

$$\gamma(x)=\left(1+2(r-1)\left(1-\frac{\langle |A^*|^{(1-\lambda)}x,x\rangle}{\langle |A^*|^{2(1-\lambda)}x,x\rangle^{\frac{1}{2}}}\right)\right),$$

then

$$w^r(A)\leq \frac{1}{2\mu}\left\||A|^{2\lambda r}+|A^*|^{2(1-\lambda)r}\right\|,$$

where $\zeta=\inf_{\|x\|=1}\zeta(x)$, $\gamma=\inf_{\|x\|=1}\gamma(x)$ *and* $\mu=\min\{\zeta,\gamma\}$.

Proof. Let $x\in\mathcal{H}$ be a unit vector. Then

$$\begin{aligned}|\langle Ax,x\rangle| &\leq \left\langle |A|^{2\lambda}x,x\right\rangle^{\frac{1}{2}}\left\langle |A^*|^{2(1-\lambda)}x,x\right\rangle^{\frac{1}{2}}\\ &\leq \left(\frac{\left\langle |A|^{2\lambda}x,x\right\rangle^{r}+\left\langle |A^*|^{2(1-\lambda)}x,x\right\rangle^{r}}{2}\right)^{\frac{1}{r}}\\ &\leq \left(\frac{1}{2}\left(\frac{1}{\zeta}\left\langle |A|^{2r\lambda}x,x\right\rangle+\frac{1}{\gamma}\left\langle |A^*|^{2r(1-\lambda)}x,x\right\rangle^{r}\right)\right)^{\frac{1}{r}}.\end{aligned}$$

Hence,

$$|\langle Ax,x\rangle|^r\leq \frac{1}{2\mu}\left\langle (|A|^{2\lambda r}+|A^*|^{2(1-\lambda)r})x,x\right\rangle.$$

By taking supremum over $x\in\mathcal{H}$ with $\|x\|=1$, we get the desired relation. ☐

3. Inequalities for Sums and Products of Operators

In this section, we recall that some general result for the product of operators from [5].

If $A,B\in B(\mathcal{H})$, then

$$w(AB)\leq 4w(A)w(B).$$

If A is an isometry and $AB=BA$, or a unitary operator that commutes with another operator B, then

$$w(AB)\leq w(B),$$

(see [4, Corollary 2.5-3]). Dragomir in [1, Theorem 2] showed that for $A,B\in B(\mathcal{H})$, any $\lambda\in(0,1)$ and $r\geq 1$

$$|\langle Ax,By\rangle|^{2r}\leq \lambda\langle (A^*A)^{\frac{r}{\lambda}}x,x\rangle+(1-\lambda)\langle (B^*B)^{\frac{r}{1-\lambda}}y,y\rangle, \tag{3.1}$$

where $x,y\in\mathcal{H}$, with $\|x\|=\|y\|=1$.

Let $A,B\in B(\mathcal{H})$. The Schwarz inequality states that

$$|\langle Ax,By\rangle|^2\leq \langle Ax,Ax\rangle\langle By,By\rangle,\quad \text{for all } x,y\in\mathcal{H}.$$

We get the following refinements of inequality (3.1) for $r\geq 2$.

Lemma 3.1. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$*

$$\begin{aligned} |\langle Ax, By\rangle|^{2r} \leq & \lambda\langle (A^*A)^{\frac{r}{\lambda}}x, x\rangle - \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle + (1-\lambda) \\ & \times \langle (B^*B)^{\frac{r}{1-\lambda}}y, y\rangle - (1-\lambda)\left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle, \end{aligned} \tag{3.2}$$

for any $x, y \in \mathcal{H}$, with $\|x\| = \|y\| = 1$.

Proof. For any unit vectors $x, y \in \mathcal{H}$, we have

$$\begin{aligned} |\langle (B^*A)x, y\rangle|^2 \leq & \langle (A^*A)x, x\rangle\langle (B^*B)y, y\rangle \quad \text{(by Schwarz inequality)} \\ = & \langle ((A^*A)^{\frac{1}{\lambda}})^{\lambda}x, x\rangle\langle ((B^*B)^{\frac{1}{1-\lambda}})^{1-\lambda}y, y\rangle \\ \leq & \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle^{\lambda}\langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle^{1-\lambda} \quad \text{(by Lemma 2.3)} \\ \leq & (\lambda\langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle^r + (1-\lambda)\langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma 2.1)} \\ \leq & \Bigg(\lambda\langle (A^*A)^{\frac{r}{\lambda}}x, x\rangle - \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle \\ & + (1-\lambda)\langle (B^*B)^{\frac{r}{1-\lambda}}y, y\rangle \\ & - (1-\lambda)\left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle\Bigg)^{\frac{1}{r}} \quad \text{(by Lemma 2.4).} \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle Ax, By\rangle|^{2r} \leq & \lambda\langle (A^*A)^{\frac{r}{\lambda}}x, x\rangle - \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle \\ & + (1-\lambda)\langle (B^*B)^{\frac{r}{1-\lambda}}y, y\rangle \\ & - (1-\lambda)\left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle, \end{aligned}$$

for any $x, y \in \mathcal{H}$, with $\|x\| = \|y\| = 1$. □

Theorem 3.1. *Let $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$. Then*

$$\|B^*A\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda)\|(B^*B)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\zeta(x) - \inf_{\|y\|=1}\zeta(y), \tag{3.3}$$

where

$$\begin{aligned} \zeta(x) = & \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle, \\ \zeta(y) = & (1-\lambda)\left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle. \end{aligned}$$

In addition,

$$w^{2r}(B^*A) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)(B^*B)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\gamma(x), \tag{3.4}$$

where

$$\gamma(x) = \left\langle \left(\lambda\left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r + (1-\lambda)\left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}}x, x\rangle\right|^r\right) x, x\right\rangle.$$

Proof. By taking supremum over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ in inequality (3.2), we get the required inequality (3.3).

Putting $x = y$ in inequality (3.2), we obtain the numerical radius inequality (3.4). $\square$

Corollary 3.1. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$, the following inequalities hold:*

$$|\langle Ax, y\rangle|^{2r} \leq \lambda\langle (A^*A)^{\frac{r}{\lambda}}x, x\rangle - \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle + (1-\lambda),$$

$$\begin{aligned}|\langle A^2x, y\rangle|^{2r} \leq{}& \lambda\langle (A^*A)^{\frac{r}{\lambda}}x, x\rangle - \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle \\ &+ (1-\lambda)\langle (AA^*)^{\frac{r}{1-\lambda}}y, y\rangle - (1-\lambda)\left\langle \left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle,\end{aligned}$$

where $x, y \in \mathcal{H}$, $\|x\| = \|y\| = 1$.

Corollary 3.2. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$, the following norm inequalities and numerical radius inequalities hold:*

(i) $\|A\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda) - \inf_{\|x\|=1}\zeta(x)$;

(ii) $\|A^2\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda)\|(AA^*)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\zeta(x) - \inf_{\|y\|=1}\zeta(y)$;

(iii) $w^{2r}(A) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)I\| - \inf_{\|x\|=1}\zeta(x)$, *where*

$$\begin{aligned}\zeta(x) ={}& \lambda\left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r x, x\right\rangle, \\ \zeta(y) ={}& (1-\lambda)\left\langle \left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}}y, y\rangle\right|^r y, y\right\rangle;\end{aligned}$$

(iv) $w^{2r}(A^2) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)(AA^*)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\zeta(x)$, *where*

$$\zeta(x) = \left\langle \left(\lambda\left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}}x, x\rangle\right|^r + (1-\lambda)\left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}}x, x\rangle\right|^r\right) x, x\right\rangle.$$

We are going to establish a refinement of a numerical inequality for Hilbert space operators. We need the following lemmas. The first lemma is a generalization of the mixed Schwarz inequality.

Lemma 3.2. ([17, Lemma 2.1]). *Let $A \in B(\mathcal{H})$ and f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|,$$

for all $x, y \in H$.

The next lemma is a consequence of the convexity of the function $f(t) = t^r$, $r \geq 1$.

Lemma 3.3. ([17, Lemma 2.3]). *Let a_i, $i = 1, 2, \ldots, n$, be positive real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r, \quad \textit{for } r \geq 1.$$

The following theorem is a generalization of the inequalities (1.3) and (1.4).

Theorem 3.2. ([17, Lemma 2.5]). *Let $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2}\left\| \sum_{i=1}^{n}((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r)\right\|, \quad r \geq 1.$$

We refine the above inequality for $r \geq 1$ by applying a refinement of the Hölder-McCarthy inequality. To achieve our next result, we utilize the strategy of [17, Lemma 2.5].

Theorem 3.3. *Let $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, be invertible operators and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy in $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then, for all $r > 1$,*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\| \sum_{i=1}^{n}(B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$, $\zeta = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle (B_i^* f^2(|X_i|)B_i)^{\frac{1}{2}}x, x\right\rangle}{\left\langle (B_i^* f^2(|X_i|)B_i)x, x\right\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}$

and $\gamma = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle (A_i^* g^2(|X_i^*|)A_i)^{\frac{1}{2}}x, x\right\rangle}{\left\langle (A_i^* g^2(|X_i^*|)A_i)x, x\right\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}$.

Proof. For every unit vector $x \in H$, we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)x, x\right\rangle\right|^r = \left|\sum_{i=1}^{n}\langle(A_i^* X_i B_i)x, x\rangle\right|^r \\
\leq &\left(\sum_{i=1}^{n}|\langle A_i^* X_i B_i x, x\rangle|\right)^r = \left(\sum_{i=1}^{n}|\langle X_i B_i x, A_i x\rangle|\right)^r \\
\leq &\left(\sum_{i=1}^{n}\langle f^2(|X_i|)B_i x, B_i x\rangle^{\frac{1}{2}}\langle g^2(|X_i^*|)A_i x, A_i x\rangle^{\frac{1}{2}}\right)^r \\
&\text{(by Lemma 3.2)} \\
\leq &n^{r-1}\sum_{i=1}^{n}\left\langle B_i^* f^2(|X_i|)B_i x, x\right\rangle^{\frac{r}{2}}\left\langle A_i^* g^2(|X_i^*|)A_i x, x\right\rangle^{\frac{r}{2}} \\
&\text{(by Lemma 3.3)} \\
= &n^{r-1}\sum_{i=1}^{n}\left(\langle B_i^* f^2(|X_i|)B_i x, x\rangle^r\right)^{\frac{1}{2}}\left(\langle A_i^* g^2(|X_i^*|)A_i x, x\rangle^r\right)^{\frac{1}{2}} \\
\leq &\frac{n^{r-1}}{2}\left(\sum_{i=1}^{n}\left(\langle B_i^* f^2(|X_i|)B_i x, x\rangle^r + \langle A_i^* g^2(|X_i^*|)A_i x, x\rangle^r\right)\right) \\
&\text{(by AM} - \text{GM)}
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{n^{r-1}}{2}\left(\sum_{i=1}^{n}\left(\frac{1}{\zeta(x)}\langle (B_i^* f^2(|X_i|)B_i)^r x, x\rangle + \frac{1}{\gamma(x)}\langle (A_i^* g^2(|X_i^*|)A_i)^r x, x\rangle\right)\right)\\
& \text{(by (2.5))}\\
\leq & \frac{n^{r-1}}{2\mu}\sum_{i=1}^{n}\left\langle\left((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right)x, x\right\rangle\\
= & \frac{n^{r-1}}{2\mu}\left\langle\sum_{i=1}^{n}\left((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right)x, x\right\rangle.
\end{aligned}
$$

Therefore, by taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we have the desired relation. □

If we assume that $f(t) = t^{\lambda}$ and $g(t) = t^{1-\lambda}$, $0 < \lambda < 1$, in Theorem 3.3, then we get the following corollary.

Corollary 3.3. *Let* $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, *be invertible operators,* $r > 1$ *and* $0 < \lambda < 1$. *Then*

$$
w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}(B_i^*|X_i|^{2\lambda}B_i)^r + (A_i^*|X_i^*|^{2(1-\lambda)}A_i)^r\right\|,
$$

where $\mu = \min\{\zeta, \gamma\}$,

$$
\zeta = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle\left(B_i^*|X_i|^{2\lambda}B_i\right)^{\frac{1}{2}}x, x\right\rangle}{\left\langle\left(B_i^*|X_i|^{2\lambda}B_i\right)x : x\right\rangle^{\frac{1}{2}}}\right) : \ \|x\| = 1\right\},
$$

$$
\gamma = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle\left(A_i^*|X_i|^{2(1-\lambda)}A_i\right)^{\frac{1}{2}}x, x\right\rangle}{\left\langle\left(A_i^*|X_i|^{2(1-\lambda)}A_i\right)x, x\right\rangle^{\frac{1}{2}}}\right) : \ \|x\| = 1\right\}.
$$

In particular,

$$
w\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}(B_i^*|X_i|B_i + A_i^*|X_i^*|A_i)\right\|.
$$

Setting $A_i = B_i = I$, $i = 1, 2, \cdots, n$, in Theorem 3.3, the following inequalities for sums of operators are obtained.

Corollary 3.4. *Let* $X_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, *be invertible operators and* f *and* g *be continuous nonnegative functions on* $[0, \infty)$, *such that* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then, for* $r > 1$,

$$
w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}(f^{2r}(|X_i|) + g^{2r}(|X_i^*|))\right\|,
$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\langle f(|X_i|)x, x\rangle}{\langle f^2(|X_i|)x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$
$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\langle g(|X_i^*|)x, x\rangle}{\langle g^2(|X_i^*|)x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

In particular,

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |X_i|^{2\lambda r} + |X_i^*|^{2(1-\lambda)r}\right\|, \quad \lambda \in (0,1),$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X_i|^{\lambda}x, x\right\rangle}{\langle |X_i|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$
$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X_i^*|^{(1-\lambda)}x, x\right\rangle}{\langle |X_i^*|^{2(1-\lambda)}x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

If $\lambda = \frac{1}{2}$ in above inequality, we get

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |X_i|^{r} + |X_i^*|^{r}\right\|, \quad r \geq 1,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X_i|^{\frac{1}{2}}x, x\right\rangle}{\langle |X_i|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$
$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X_i^*|^{\frac{1}{2}}x, x\right\rangle}{\langle |X_i^*|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

Letting $n = 1$ in inequality (3.3), we obtain

$$w^r(X) \leq \frac{1}{2\mu}\left\||X|^r + |X^*|^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X|^{\frac{1}{2}}x, x\right\rangle}{\langle |X|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$
$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle |X^*|^{\frac{1}{2}}x, x\right\rangle}{\langle |X^*|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

Next, we present some numerical radius inequalities for products of operators. Put $X_i = I$, $i = 1, 2, \ldots, n$, in Theorem 3.3, to get the following.

Corollary 3.5. *Let $A_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \dots, n$, be invertible operators and $r \geq 1$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |B_i|^{2r} + |A_i|^{2r}\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\langle |B_i| x, x\rangle}{\langle |B_i| x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\langle |A_i| x, x\rangle}{\langle |A_i| x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

In particular,

$$w\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n} (B_i^* B_i + A_i^* A_i)\right\|.$$

Remark 3.1. If we set $n = 1$ in Corollary 3.5, then

$$w^r(A^*B) \leq \frac{1}{2\mu}\left\|(B^*B)^r + (A^*A)^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle (B^*B)^{\frac{1}{2}} x, x\right\rangle}{\langle (B^*B) x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle (A^*A)^{\frac{1}{2}} x, x\right\rangle}{\langle (A^*A) x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

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