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BEURLING'S THEOREM FOR THE Q-FOURIER-DUNKL TRANSFORM

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Abstract. The Q-Fourier-Dunkl transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. By using the heat kernel associated to the Q-Fourier-Dunkl operator, we establish an analogue of Beurling's theorem for the Q-Fourier-Dunkl transform $\mathcal{F}_Q$ on $\mathbb{R}$.

1. Introduction and Preliminaries

There are many known theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. This principle has several version which were proved by A. Beurling [3]. The Beurling theorem for the classical Fourier transform on $\mathbb{R}$ which was proved by L. Hörmander [5], says that for any non trivial function f in $L^2(\mathbb{R})$, the function $f(x)\mathcal{F}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|xy|}dxdy$. A far reaching generalization of this result has been recently proved in [4]. In this paper the author proved that a square integrable function f on $\mathbb{R}$ satisfying for an integer N

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}(y)|}{(1+|x|+|y|)^N} e^{|xy|}dxdy < \infty,$$

has the form $f(x) = P(x)e^{-rx^2}$, where P is a polynomial of degree strictly lower than $\frac{N-1}{2}$ and $r > 0$. Many authors have established the analogous of Beurling's theorem in other various setting of harmonic analysis (see for instance [1, 6]). In this paper we study an analogue of Beurling's theorem, in the next we deduce an analogue of Gelfand-Shilov, for the Q-Fourier-Dunkl transform.

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The outline of the content of this paper is as follows. Section 2 is dedicated to some properties and results concerning the Q-Fourier-Dunkl transform. In Section 3 we give an analogue of Beurling's theorem and Gelfand-Shilov theorems for the Q-Fourier-Dunkl transform. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper $\alpha > -\frac{1}{2}$,

- $Q(x) = \exp\left(-\int_0^x q(t)dt\right)$, $x \in \mathbb{R}$, where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$;
- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

  and $\|f\|_{\infty,\alpha} = \|f\|_\infty = \text{esssup}_{x\in\mathbb{R}} |f(x)|$.
- $L^p_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where Q is given by $Q(x) = \exp\left(-\int_0^x q(t)dt\right)$, $x \in \mathbb{R}$.

We consider the first singular differential-difference operator Λ defined on $\mathbb{R}$

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x} + q(x) f(x),$$

where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}.$$

1.1. **Q-Fourier-Dunkl Transform.** The following statements are proved in [2].

Lemma 1.1. (a) *For each $\lambda \in \mathbb{C}$, the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1,$$

admits a unique $\mathcal{C}^\infty$ solution on $\mathbb{R}$, denoted by Ψ_λ, given by

$$\Psi_\lambda(x) = Q(x) e_\alpha(i\lambda x),$$

where e_α denotes the one-dimensional Dunkl kernel defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z), \quad z \in \mathbb{C},$$

and j_α being the normalized spherical Bessel function of index α given by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

(b) *For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, \ldots$, we have*

$$\left|\frac{\partial^n}{\partial \lambda^n} \Psi_\lambda(x)\right| \le Q(x) |x|^n e^{|\text{Im}(\lambda)|\cdot|x|}.$$

In particular,

$$|\Psi_\lambda(x)| \le Q(x) e^{|\text{Im}(\lambda)|\cdot|x|}.$$

(c) *For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, we have the Laplace type integral representation*

$$\Psi_\lambda(x) = a_\alpha Q(x) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} (1+t) e^{i\lambda x t} dt,$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

Definition 1.1. The Q-Fourier-Dunkl transform associated with Λ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$\mathcal{F}_Q(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-\lambda}(x) |x|^{2\alpha+1} dx.$$

Theorem 1.1. (a) *Let $f \in L^1_Q(\mathbb{R})$ such that $\mathcal{F}_Q(f) \in L^1_\alpha(\mathbb{R})$. Then for almost $x \in \mathbb{R}$ we have the inversion formula*

$$f(x) \left(Q(x)\right)^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_Q(f)(\lambda) \Psi_\lambda(x) |\lambda|^{2\alpha+1} d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)} (\Gamma(\alpha+1))^2}.$$

(b) *For every $f \in L^2_Q(\mathbb{R})$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 \left(Q(x)\right)^2 |x|^{2\alpha+1} dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_Q(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda.$$

(c) *The Q-Fourier-Dunkl transform $\mathcal{F}_Q$ extends uniquely to an isomorphism from $L^2_Q(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$.*

The heat kernel $N(x,s)$, $x \in \mathbb{R}$, $s > 0$, associated with the Q-Fourier-Dunkl transform is given by

$$N(x,s) = m_\alpha \frac{e^{-\frac{x^2}{4s}}}{(2s)^{\alpha+\frac{1}{2}} Q(x)}.$$

Some basic properties of $N(x,s)$ are the following:

- $N(x,s)Q^2(x) = m_\alpha \int_{\mathbb{R}} e^{-sy^2} \Psi_y(x) |y|^{2\alpha+1} dy$;
- $\mathcal{F}_Q(N(.,s))(x) = e^{-sx^2}$.

We define the heat functions W_l, $l \in \mathbb{N}$, as

$$Q^2(x) W_l(x,s) = \int_{\mathbb{R}} y^l e^{-\frac{y^2}{4s}} \Psi_y(x) |y|^{2\alpha+1} dy, \tag{1.1}$$

$$\mathcal{F}_Q(W_l(.,s)) = i^l y^l e^{-sy^2}. \tag{1.2}$$

The intertwining operators associated with a Q-Fourier-Dunkl transform on the real line is given by

$$X_Q(f)(x) = a_\alpha Q(x) \int_{-1}^{1} f(tx)(1-t^2)^{\alpha-\frac{1}{2}} dt,$$

its dual is given by

$${}^tX_Q(f)(y) = a_\alpha \int_{|x| \geq |y|} f(x) Q(x) \operatorname{sgn}(x) (x^2-y^2)^{\alpha-\frac{1}{2}} (x+y) dx. \tag{1.3}$$

Proposition 1.1. *If $f \in L^1_Q(\mathbb{R})$, then ${}^tX_Q(f) \in L^1(\mathbb{R})$ and $\|{}^tX_Q(f)\|_1 \leq \|f\|_{1,Q}$.*

For every $f \in L^1_Q(\mathbb{R})$ we have

$$\mathcal{F}_Q = \mathcal{F} \circ^t X_Q(f), \tag{1.4}$$

where $\mathcal{F}$ is the usual Fourier transform defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx.$$

2. Beurling's Theorem for the Q-Fourier-Dunkl Transform

Theorem 2.1. *Let $N \in \mathbb{N}$ and $f \in L^2_Q(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}_Q(f)(y)|Q(x)}{(1+|x|+|y|)^N} e^{|x||y|}|x|^{2\alpha+1}dxdy < \infty. \tag{2.1}$$

If $N > 1$, then $f(y) = \Sigma_{|s|<\frac{N-1}{2}} b_sW_s(r,y)$ a.e. where $r > 0$, $b_s \in \mathbb{C}$ and $W_s(r,\cdot)$ is given by (1.1)*. Otherwise, $f(y) = 0$ a.e.*

Proof. We start with the following lemma.

Lemma 2.1. *We suppose that $f \in L^2_Q(\mathbb{R})$ satisfies* (2.1)*. Then $f \in L^1_Q(\mathbb{R})$.*

Proof. We may suppose that $f \neq 0$ in $L^2_Q(\mathbb{R})$. (2.1) and Fubini theorem imply that for almost every $y \in \mathbb{R}$,

$$\frac{|\mathcal{F}_Q(f)(y)|}{(1+|y|)^N}\int_{\mathbb{R}} \frac{Q(x)|f(x)|}{(1+|x|)^N} e^{|x||y|}|x|^{2\alpha+1}dx < \infty.$$

Since $\mathcal{F}_Q(f) \neq 0$, there exist $y_0 \in \mathbb{R}$, $y_0 \neq 0$, such that $\mathcal{F}_Q(f)(y_0) \neq 0$.

Therefore,

$$\int_{\mathbb{R}} \frac{Q(x)|f(x)|}{(1+|x|)^N} e^{|x||y_0|}|x|^{2\alpha+1}dx < \infty. \tag{2.2}$$

Since $\frac{e^{|x||y_0|}}{(1+|x|)^N} \geq 1$ for large $|x|$, it follows that $\int_{\mathbb{R}} Q(x)|f(x)||x|^{2\alpha+1}dx < \infty$. □

This Lemma and Proposition 1.1 imply that ${}^tX_Q(f)$ is well-defined almost everywhere on $\mathbb{R}$. We shall prove that we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|{}^tX_Q(f)(x)||\mathcal{F}({}^tX_Q)(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|}dxdy < \infty. \tag{2.3}$$

Take y_0 as in Lemma 2.1, we write the above integral as a sum of the following integrals

$$I = \int_{\mathbb{R}}\int_{|y|\leq|y_0|} \frac{e^{|x||y|}}{(1+|x|+|y|)^N}|{}^tX_Qf(x)||\mathcal{F}({}^tX_Q(f))(y)|dydx$$

and

$$J = \int_{\mathbb{R}}\int_{|y|\geq|y_0|} \frac{e^{|x||y|}}{(1+|x|+|y|)^N}|{}^tX_Qf(x)||\mathcal{F}({}^tX_Q(f))(y)|dydx.$$

We will prove that I and J are finite, which implies (2.3).

- As the functions $|\mathcal{F}_Q(f)(y)|$ is continuous in the compact $\{y \in \mathbb{R} \mid |y| \leq |y_0|\}$, so we get
$$I \leq C \int_{\mathbb{R}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx.$$
Writing the integral of the second member as $I_1 + I_2$ with
$$I_1 = \int_{|x| \leq \frac{N}{|y_0|}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx$$
and
$$I_2 = \int_{|x| \geq \frac{N}{|y_0|}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx.$$
Therefore, we have the following results.
  - As the function $x \rightarrow \frac{e^{|x||y_0|}}{(1+|x|)^N}$ is continuous in the compact $\left\{x \in \mathbb{R} \mid |x| \leq \frac{N}{|y_0|}\right\}$, and $f \in L^1_Q(\mathbb{R})$, we deduce by using proposition (1.1) that $|{}^tX_Q(f)|$ belongs to $L^1(\mathbb{R})$. Hence, I_1 is finite.
  - On the other hand, for $t > \frac{N}{|y_0|}$, the function $t \mapsto \frac{e^{t|y_0|}}{(1+t)^N}$ is increasing, so we obtain by using Proposition 1.1 that
  $$I_2 \leq \int_{\mathbb{R}} \frac{Q(\xi)e^{|\xi||y_0|}}{(1+|\xi|)^N} |f(\xi)||\xi|^{2\alpha+1} d\xi.$$
  The inequality (2.2) assert that I_2 is finite. This proves that I is finite.
- We suppose $|y_0| \leq N$. Then $J = J_1 + J_2 + J_3$, with
$$J_1 = \int_{|x| \leq \frac{N}{|y_0|}} \int_{|y_0| \leq |y| \leq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx,$$
$$J_2 = \int_{|x| \geq \frac{N}{|y_0|}} \int_{|y_0| \leq |y| \leq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx,$$
$$J_3 = \int_{\mathbb{R}} \int_{|y| \geq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx.$$
  - As the function $(x, y) \mapsto \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |\mathcal{F}_Q(f)(y)|$ is bounded in the compact $\left\{x \in \mathbb{R} \mid |x| \leq \frac{N}{|y_0|}\right\} \times \{\xi \in \mathbb{R} \mid |y_0| \leq |\xi| \leq N\}$ and ${}^tX_Q(|f|)(x)$ is Lebesgue-integrable on $\mathbb{R}$, then J_1 is finite.
  - Let $\lambda > 0$. As the function $t \mapsto \frac{e^{\lambda t}}{(1+t+\lambda)^N}$ is increasing for $t > \frac{N}{\lambda}$. Thus, for all $(x, y) \in C(\xi, y_0, N)$ we have the inequality
  $$\frac{e^{|x||y|}}{(1+|x|+|y|)^N} \leq \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N},$$
  with
  $$C(\xi, y_0, N) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \;\middle|\; \frac{N}{|y_0|} \leq |x| \leq |\xi| \text{ et } |y_0| \leq |y| \leq N\right\}.$$

Therefore, from Fubini-Tonelli's theorem and Proposition 1.1 we get

$$J_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |Q(\xi) f(\xi)||\mathcal{F}_Q(f)(y)| \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N} |\xi|^{2\alpha+1} d\xi dy.$$

Taking account of the condition (2.1), we deduce that J_2 is finite.

– For $|y| > N$, the function $t \mapsto \frac{e^{t|y|}}{(1+t+|y|)^N}$ is increasing. We deduce, by using Fubini-Tonelli's theorem and Proposition 1.1, that

$$J_3 \leq \int_{\mathbb{R}} \int_{|y|\geq N} |(f)(\xi)||F_Q(f)(y)| \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N} dy |\xi|^{2\alpha+1} d\xi < +\infty.$$

This implies that J_3 is finite. Finally for $|y_0| > N$, we have $J \leq J_3 < \infty$. This completes the proof of the relation (2.3).

According to Corollary 3.1, ii) of [4], we conclude that

$${}^t X_Q(f)(x) = R(x) e^{-\delta x^2}, \quad \text{for all } x \in \mathbb{R},$$

with $\delta > 0$ and R a polynomial of degree strictly lower than $\frac{N-1}{2}$.

Using this relation and (1.4), we deduce that

$$\mathcal{F}_Q(f)(y) = \mathcal{F} \circ^t X_Q(f)(y) = \mathcal{F}(R(x) e^{-\delta x^2})(y), \quad \text{for all } x \in \mathbb{R},$$

but

$$\mathcal{F}(P(x) e^{-\delta x^2})(y) = S(y) e^{\frac{-y^2}{4\delta}}, \quad \text{for all } x \in \mathbb{R},$$

with S a polynomial of degree strictly lower than $\frac{N-1}{2}$.
Thus from (1.2) we obtain

$$\mathcal{F}_Q(f)(y) = \mathcal{F}_Q\left(\sum_{|s|<\frac{N-1}{2}} b_s W_s\left(\frac{1}{4\delta}, \cdot\right)\right)(y), \quad \text{for all } x \in \mathbb{R}.$$

The injectivity of the transform $\mathcal{F}_Q$ implies

$$f(x) = \sum_{|s|<\frac{N-1}{2}} b_s W_s\left(\frac{1}{4\delta}, \cdot\right)(x) \text{ a.e}, \quad \text{for all } x \in \mathbb{R},$$

and the theorem is proved. □

As an application of Beurling's Theorem, we can deduce a Gelfand-Shilov type theorem for the Q-Fourier-Dunkl transform.

Theorem 2.2. *Let $N \in \mathbb{N}$, $a, b > 0$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^2_Q(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \frac{Q(x)|f(x)| e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} |x|^{2\alpha+1} dx < \infty \tag{2.4}$$

and

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_Q(f)(y)|e^{\frac{(2b)^q}{q}|y|^q}}{(1+|y|)^N} dy < \infty. \tag{2.5}$$

If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(x) = 0$ a.e. If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then $f(x) = \Sigma_{|s|<\frac{N-1}{2}} b_s W_s(r,\cdot))(x)$, whenever $N > 1$ and $r = 2b^2$. Otherwise, $f(x) = 0$ a.e.

Proof. Since

$$4ab|x||y| \leq \frac{(2a)^p}{p}|x|^p + \frac{(2b)^q}{q}|y|^q,$$

it follows from (2.4) and (2.5) that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{Q(x)|f(x)||\mathcal{F}_Q(f)(y)|}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|}|x|^{2\alpha+1} dxdy < \infty.$$

Then (2.1) is satisfied, because $4ab \leq 1$. Especially, according to the proof of Theorem 2.1, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)|}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|} dxdy < \infty,$$

and ${}^tX_Q(f)$ and f are of the forms ${}^tX_Q(f) = R(x)e^{-\frac{x^2}{4r}}$ and $\mathcal{F}_Q(f)(y) = S(y)e^{-ry^2}$, where $r > 0$ and S, R are polynomials of the same degree strictly lower than $\frac{2N-1}{2}$. Therefore, substituting these, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{r}|y|-\frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1)|x||y|} R(x)S(y)}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|} dxdy < \infty. \tag{2.6}$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Indeed, as $ab > \frac{1}{4}$, there exists $\varepsilon > 0$ such that $4ab - 1 - \varepsilon > 0$. If R is non null, S is also non null and we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|R(x)||S(y)|}{(1+|x|+|y|)^{2N}} e^{-(\sqrt{r}|y|-\frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1)|x||y|} dxdy \\ \geq& C\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(\sqrt{r}|y|-\frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1-\varepsilon)|x||y|} dxdy,\end{aligned}$$

where C is a positive constant. But the function

$$e^{-(\sqrt{r}|y|-\frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1-\varepsilon)|x||y|}$$

is not integrable, (2.6) does not hold. Hence, $f(x) = 0$ a.e.

Moreover, it follows from (2.4) and (2.5) that

$$\int_{\mathbb{R}} \frac{|f(x)|Q(x)e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N}|x|^{2\alpha+1} dx = \int_{\mathbb{R}} \frac{e^{-\frac{1}{4}x^2} e^{\frac{(2a)^p}{p}|x|^p} R(x)Q(x)}{(1+|x|)^N}|x|^{2\alpha+1} dx < \infty \tag{2.7}$$

and

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_Q(f)(y)|e^{\frac{(2b)^q}{q}|y|^q}}{(1+|y|)^N} dy = \int_{\mathbb{R}} \frac{e^{-ry^2} e^{\frac{(2b)^q}{q}|y|^q} S(y)}{(1+|y|)^N} dy < \infty. \tag{2.8}$$

Hence, one of these integrals is not finite unless $(p, q) = (2, 2)$. When $4ab = 1$ and $(p, q) = (2, 2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 2.1. □

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