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MORE ABOUT PETROVIĆ'S INEQUALITY ON COORDINATES VIA m-CONVEX FUNCTIONS AND RELATED RESULTS

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Abstract. In this paper the authors extend Petrović's inequality for coordinated m-convex functions in the plane and also find Lagrange type and Cauchy type mean value theorems for Petrović's inequality for m-convex functions and coordinated m-convex functions. The authors consider functional due to Petrović's inequality in plane and discuss its properties for certain class of coordinated log-m-convex functions.

1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds, for all $x, y \in [a, b]$ and $t \in [0,1]$.

In [6], Dragomir gave the definition of convex functions on coordinates as follows.

Definition 1.1. Let $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a mapping. Define partial mappings

$$f_y : [a, b] \to \mathbb{R} \text{ by } f_y(u) = f(u, y) \tag{1.1}$$

and

$$f_x : [c, d] \to \mathbb{R} \text{ by } f_x(v) = f(x, v). \tag{1.2}$$

Then f is said to be convex on coordinates (or coordinated convex) in Δ if f_y and f_x are convex on $[a, b]$ and $[c, d]$ respectively for all $y \in [c, d]$ and $x \in [a, b]$. A mapping f is said to be strictly convex on coordinates (or strictly coordinated convex) in Δ

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if f_y and f_x are strictly convex on $[a,b]$ and $[c,d]$, respectively, for all $y \in [c,d]$ and $x \in [a,b]$.

In [22], G. Toader gave the definition of m-convexity as follows.

Definition 1.2. The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have

$$f\left(tx + m(1-t)y\right) \leqslant tf(x) + m(1-t)f(y),$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Remark 1.1. One can note that the notion of m-convexity reduces to convexity if we take $m = 1$. We get starshaped functions from m-convex functions if we take $m = 0$.

Definition 1.3. A function $f : [a,b] \to \mathbb{R}_+$ is called log-convex if

$$f\left(tx + (1-t)y\right) \leqslant f^t(x) + f^{(1-t)}(y)$$

holds, for all $x, y \in [0,b]$ and $t \in [0,1]$.

Log-convex functions have excellent closure properties. The sum and product of two log-convex functions is convex. If f is convex function and g is log-convex function then the functional composition $g \circ f$ is also log-convex.

In [1], Almori and Darus gave the definition of log-convex on coordinates as follows.

Definition 1.4. Let $\Delta = [a,b] \times [c,d]$ and let a function $f : \Delta \to \mathbb{R}_+$ is called log-convex on coordinates in Δ if partial mappings defined in (1.1) and (1.2) are log-convex on $[a,b]$ and $[c,d]$, respectively, for all $y \in [c,d]$ and $x \in [a,b]$.

In [8], Farid et al. gave the definition of coordinated m-convex functions as follows.

Definition 1.5. Let $\Delta = [0,b] \times [0,d] \subset [0,\infty)^2$, then a function $f : \Delta \to \mathbb{R}$ will be called m-convex on coordinates if the partial mappings

$$f_y : [0,b] \to \mathbb{R} \text{ defined by } f_y(u) = f(u,y)$$

and

$$f_x : [0,d] \to \mathbb{R} \text{ defined by } f_x(v) = f(x,v)$$

are m-convex on $[0,b]$ and $[0,d]$, respectively, for all $y \in [0,d]$ and $x \in [0,b]$.

In [17] (see also [15, p. 154]), M. Petrović proved the following result, which is known as Petrović's inequality in the literature.

Theorem 1.1. *Suppose that $(x_1, \dots, x_n)$ and $(p_1, \dots, p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^n p_k x_k \geq x_i$ for $i = 1, \dots, n$ and $\sum_{k=1}^n p_k x_k \in [0,a]$. If f is a convex function on $[0,a)$, then the inequality*

$$\sum_{k=1}^n p_k f(x_k) \leq f\left(\sum_{k=1}^n p_k x_k\right) + \left(\sum_{k=1}^n p_k - 1\right) f(0) \tag{1.3}$$

is valid.

Remark 1.2. Take $p_k = 1$, $k = 1, \dots, n$ the above inequality becomes

$$\sum_{k=1}^{n} f(x_k) \leq f\left(\sum_{k=1}^{n} x_k\right) + (n-1)f(0).$$

In [2], M. Bakula et al. gave the Petrović's inequality for m-convex function which is stated in the following theorem.

Theorem 1.2. *Let* $(x_1, \dots, x_n)$ *be non-negative* n*-tuples and* $(p_1, \dots, p_n)$ *be positive* n*-tuples such that*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_j \text{ for each } j = 1, \dots, n.$$

If $f : [0, \infty) \to \mathbb{R}$ *be an* m*-convex function on* $[0, \infty)$ *with* $m \in (0, 1]$*, then*

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant \min\left\{ m f\left(\tfrac{\tilde{x}_n}{m}\right) + (P_n - 1) f(0), f(\tilde{x}_n) + m(P_n - 1) f(0) \right\}. \tag{1.4}$$

Remark 1.3. If we take $m = 1$ in Theorem 1.2, we get famous Petrović's inequality stated in Theorem 1.1.

In [19], Rehman et al. gave the Petrović's inequality for coordinated convex functions, which is stated in the following theorem.

Theorem 1.3. *Let* $(x_1, \dots, x_n) \in [0, a)^n$, $(y_1, \dots, y_n) \in [0, b)^n$ *and* $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ *be positive* n*-tuples such that* $\sum_{k=1}^{n} p_k x_k \in [0, a)$, $\sum_{j=1}^{n} q_j y_j \in [0, b)$, $\sum_{k=1}^{n} p_k \geq 1$,

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \text{ for each } i = 1, \dots, n,$$

and

$$Q_n := \sum_{j=1}^{n} q_j, \quad 0 \neq \tilde{y}_n = \sum_{j=1}^{n} q_j y_j \geq y_i \text{ for each } i = 1, \dots, n.$$

If $f : \Delta \to \mathbb{R}$ *be a coordinated convex, then*

$$\begin{aligned} \sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq & f(\tilde{x}_n, \tilde{y}_n) + (Q_n - 1) f(\tilde{x}_n, 0) \\ & + (P_n - 1)\left(f(0, \tilde{y}_n) + (Q_n - 1) f(0, 0)\right). \end{aligned} \tag{1.5}$$

By considering non-negative difference of (1.5), the authors in [19] defined the following functional

$$\begin{aligned} \Upsilon(f) = & f(\tilde{x}_n, \tilde{y}_n) + (Q_n - 1) f(\tilde{x}_n, 0) + (P_n - 1)\left[f(0, \tilde{y}_n) + (Q_n - 1) f(0, 0)\right] \\ & - \sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j). \end{aligned} \tag{1.6}$$

By considering non-negative difference of (1.3), the authors in [4] defined the following functional

$$\mathcal{P}(f) = f\left(\sum_{k=1}^{n} p_k x_k\right) - \left(\sum_{k=1}^{n} p_k f(x_k)\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0). \tag{1.7}$$

One of the generalizations of convex functions is m-convex functions and it is considered in literature by many researchers and mathematicians, for example, see [7, 10–12, 24] and references there in. In [17] (also see [15, p. 154]), M. Petrović gave the inequality for convex functions known as Petrović's inequality. Many authors worked on this inequality by giving results related to it, for example see [13, 15, 17] and it has been generalized for m-convex functions by M. Bakula et al. in [2]. In [19], Petrović's inequality was generalized on coordinate by using the definition of convex functions on coordinates given by Dragomir in [6].

In this paper the authors extend Petrović's inequality for coordinated m-convex functions in the plane and also find Lagrange type and Cauchy type mean value theorems for Petrović's inequality for m-convex functions and coordinated m-convex functions. The authors consider functional due to Petrović's inequality in plane and discuss its properties for certain class of coordinated log-m-convex functions.

2. Main Result

The following theorem consist the result for Petrović's inequality on coordinated m-convex functions.

Theorem 2.1. *Let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ be non-negative n-tuples and $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ be positive n-tuples such that $\sum\limits_{k=1}^{n} p_k \geq 1$,*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \text{ for each } i = 1, \dots, n$$

and

$$Q_n := \sum_{j=1}^{n} q_j, \quad 0 \neq \tilde{y}_n = \sum_{j=1}^{n} q_j y_j \geq y_i \text{ for each } i = 1, \dots, n.$$

If $f : [0, \infty)^2 \to \mathbb{R}$ be an m-convex function on coordinates with $m \in (0, 1]$, then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq \min\{&m \min\{G_{m,1}(\tilde{x}_n/m), G_{1,m}(\tilde{x}_n/m)\} + (P_n - 1) \\ &\times \min\{G_{m,1}(0), G_{1,m}(0)\}, \min\{G_{m,1}(\tilde{x}_n), G_{1,m}(\tilde{x}_n)\} \\ &+ m(P_n - 1)\min\{G_{m,1}(0), G_{1,m}(0)\}\},\end{aligned} \tag{2.1}$$

where

$$G_{m,\widetilde{m}}(t) = mf\left(t, \frac{\tilde{y}_n}{m}\right) + \widetilde{m}(Q_n - 1) f(t, 0). \tag{2.2}$$

Proof. Let $f_x : [0,\infty) \to \mathbb{R}$ and $f_y : [0,\infty) \to \mathbb{R}$ be mappings such that $f_x(v) = f(x,v)$ and $f_y(u) = f(u,y)$. Since f is coordinated m-convex on $[0,\infty)^2$, therefore f_y is m-convex on $[0,\infty)$, so by Theorem 1.2, one has

$$\sum_{k=1}^{n} p_k f_y(x_k) \leq \min\left\{ m f_y\left(\tilde{x}_n/m\right) + (P_n-1) f_y\left(0\right), f_y\left(\tilde{x}_n\right) + m(P_n-1) f_y\left(0\right)\right\}.$$

This is equivalent to

$$\sum_{k=1}^{n} p_k f(x_k,y) \leq \min\left\{ m f\left(\tilde{x}_n/m, y\right) + (P_n-1) f\left(0,y\right), \right.$$
$$\left. f\left(\tilde{x}_n, y\right) + m(P_n-1) f\left(0,y\right)\right\}.$$

By setting $y = y_j$, we have

$$\sum_{k=1}^{n} p_k f(x_k,y_j) \leq \min\left\{ m f\left(\tilde{x}_n/m, y_j\right) + (P_n-1) f\left(0,y_j\right), \right.$$
$$\left. f\left(\tilde{x}_n, y_j\right) + m(P_n-1) f\left(0,y_j\right)\right\},$$

this gives

$$\begin{aligned} \sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k,y_j) \leq \min\Bigg\{ & m\sum_{j=1}^{n} q_j f\left(\tilde{x}_n/m, y_j\right) + (P_n-1)\sum_{j=1}^{n} q_j f\left(0,y_j\right), \\ & \sum_{j=1}^{n} q_j f\left(\tilde{x}_n, y_j\right) + m(P_n-1)\sum_{j=1}^{n} q_j f\left(0,y_j\right)\Bigg\}. \end{aligned} \tag{2.3}$$

Now again by Theorem 1.2, one has

$$\sum_{j=1}^{n} q_j f\left(\tilde{x}_n/m, y_j\right) \leq \min\left\{ m f\left(\tilde{x}_n/m, \tilde{y}_n/m\right) + (Q_n-1) f\left(\tilde{x}_n/m, 0\right), \right.$$
$$\left. f\left(\tilde{x}_n/m, \tilde{y}_n\right) + m(Q_n-1) f\left(\tilde{x}_n/m, 0\right)\right\},$$

$$\sum_{j=1}^{n} q_j f\left(0, y_j\right) \leq \min\left\{ m f\left(0, \tilde{y}_n/m\right) + (Q_n-1) f\left(0,0\right), \right.$$
$$\left. f\left(0, \tilde{y}_n\right) + m(Q_n-1) f\left(0,0\right)\right\}$$

and

$$\sum_{j=1}^{n} q_j f\left(\tilde{x}_n, y_j\right) \leq \min\left\{ m f\left(\tilde{x}_n, \tilde{y}_n/m\right) + (Q_n-1) f\left(\tilde{x}_n, 0\right), \right.$$
$$\left. f\left(\tilde{x}_n, \tilde{y}_n\right) + m(Q_n-1) f\left(\tilde{x}_n, 0\right)\right\}.$$

Putting these values in inequality (2.3), and using the notation in (2.2), one has the required result. □

Remark 2.1. If we take $m = 1$ in Theorem 2.1, we get Theorem 1.3.

In the following corollary, we gave new Petrović's type inequality for m-convex functions.

Corollary 2.1. *Let* $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ *be non-negative* n*-tuples and* $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ *be positive* n*-tuples such that* $\sum\limits_{k=1}^{n} p_k \geq 1$ *and*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \text{ for each } i = 1, \dots, n.$$

If $f : [0, \infty)^2 \to \mathbb{R}$ *be an* m*-convex function on coordinates with* $m \in (0, 1]$*, then one has*

$$\begin{aligned}(2.4) \quad \sum_{k=1}^{n} np_k f(x_k) \leq \min \{ & m \min \{(m+n-1)f(\tilde{x}_n/m), (mn-m+1)f(\tilde{x}_n/m)\} \\ & + (P_n - 1) \min \{(m+n-1)f(0), (mn-m+1)f(0)\}, \\ & \min \{(m+n-1)f(\tilde{x}_n), (mn-m+1)f(\tilde{x}_n)\} \\ & + m(P_n - 1) \min \{(m+n-1), (mn-m+1)f(0)\}\}.\end{aligned}$$

Proof. If we put $y_j = 0$ and $q_j = 1$, $j = 1, \dots, n$ with $f(x, 0) \mapsto f(x)$ in inequality (2.1), we get the required result. □

Remark 2.2. If we take $m = 1$ in inequality (2.4), we get the inequality (1.3).

Let $f : [0, b] \to \mathbb{R}$ be a function. Then we define

$$(2.5) \qquad P_{a,m,f}(x) := \frac{f(x) - mf(a)}{x - ma},$$

for all $x \in [0, b] \backslash \{ma\}$, for fixed $a \in [0, b]$. Also define

$$(2.6) \qquad r_m(x_1, x_2, x_3; f) := \frac{P_{x_1,m}(x_3) - P_{x_1,m}(x_2)}{x_3 - x_2},$$

where $x_1, x_2, x_3 \in [0, b]$, $(x_2 - mx_1)(x_3 - mx_1) > 0$, $x_2 \neq x_3$.

In [11] (see also [7, p. 294]), V. G. Mihesan considered the functions defined in (2.5), (2.6) and proved the following result.

Remark 2.3. If we take $m = 1$ in (2.5) and (2.6), we get divided differences of first and second order respectively.

By considering non-negative difference of (1.4), we defined following functional

$$(2.7) \quad \mathcal{P}_m(f) = \min \left\{ mf\left(\frac{\tilde{x}_n}{m}\right) + (P_n - 1)f(0), f(\tilde{x}_n) + m(P_n - 1)f(0) \right\} - \sum_{k=1}^{n} p_k f(x_k).$$

Also by considering non-negative difference of (2.1), we defined following functional

$$\begin{aligned}(2.8) \quad \Upsilon_m(f) = \min \{ & m \min \{G_{m,1}(\tilde{x_n}/m), G_{1,m}(\tilde{x_n}/m)\} \\ & + (P_n - 1) \min \{G_{m,1}(0), G_{1,m}(0)\}, \min \{G_{m,1}(\tilde{x_n}), G_{1,m}(\tilde{x_n})\} \\ & + m(P_n - 1) \min \{G_{m,1}(0), G_{1,m}(0)\}\} - \sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j).\end{aligned}$$

If we take $m = 1$ in the above (2.8), we get $\Upsilon_1(f) = \Upsilon(f)$.

Remark 2.4. Under the suppositions of Theorem 2.1, if f is coordinated m-convex in Δ^2, then $\Upsilon_m(f) \geq 0$.

Here we state an important lemma that is very helpful in proving mean value theorems related to the non-negative functional of Petrović's inequality for m-convex functions.

Lemma 2.1. *Let $f : [0, b] \to \mathbb{R}$ be a function such that*

$$m_1 \leqslant \frac{(x - ma)f'(x) - f(x) + mf(a)}{x^2 - 2max + ma^2} \leqslant M_1,$$

for all $x \in [0, b]\backslash\{ma\}$, $a \in (0, b)$ and $m \in (0, 1)$.

Consider the functions $\psi_1, \psi_2 : [0, b] \to \mathbb{R}$ defined as

$$\psi_1(x) = M_1x^2 - f(x)$$

and

$$\psi_2(x) = f(x) - m_1x^2,$$

then ψ_1 and ψ_2 are m-convex in $[0, b]$.

Proof. Suppose

$$\begin{aligned} P_{a,m,\psi_1}(x) =& \frac{\psi_1(x) - m\psi_1(a)}{x - ma} \\ =& \frac{M_1x^2 - f(x) - mf(a) + mM_1a^2}{x - ma} \\ =& \frac{M_1(x^2 - ma^2)}{x - ma} - \frac{f(x) - mf(a)}{x - ma}. \end{aligned}$$

So we have

$$P'_{a,m,\psi_1}(x) = M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} - \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2}.$$

Since

$$x^2 - 2max + ma^2 = (x - ma)^2 + m(1 - m)a^2 > 0,$$

by given condition, we have

$$M_1(x^2 - 2max + ma^2) \geq (x - ma)f'(x) - f(x) + mf(a).$$

This leads to

$$M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} \geq \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2},$$

$$M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} - \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2} \geq 0.$$

This implies

$$P'_{a,m,\psi_1}(x) \geq 0, \quad \text{for all } x \in [0, ma) \cup (ma, b].$$

Similarly, one can show that

$$P'_{a,m,\psi_2}(x) \geq 0, \qquad \text{for all } x \in [0, ma) \cup (ma, b].$$

This gives P_{a,m,ψ_1} and P_{a,m,ψ_2} are increasing on $x \in [0, ma) \cup (ma, b]$ for all $a \in [0, b]$. Hence by Lemma 2.1, $\psi_1(x)$ and $\psi_2(x)$ are m-convex in $[0, b]$. □

Here we give mean value theorems related to functional defined for Petrović's inequality for m-convex functions.

Theorem 2.2. *Let $(x_1, \ldots, x_n) \in [0, b]$, $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \ldots, n$. Also, let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}_m(f) = \frac{(\xi - ma) f'(\xi) - f(\xi) + m f(a)}{\xi^2 - 2ma\xi + ma^2} \mathcal{P}_m(\phi), \tag{2.9}$$

provided that $\mathcal{P}_m(\phi)$ is non zero and $a \in (0, b)$.

Proof. As $f \in C^1([0, b])$, so there exists real numbers m_1 and M_1 such that

$$m_1 \leqslant \frac{(x - ma) f'(x) - f(x) + m f(a)}{x^2 - 2max + ma^2} \leqslant M_1,$$

for each $x \in [0, b]$, $a \in (0, b)$ and $m \in (0, 1)$.

Now let us consider the functions ψ_1 and ψ_2 defined in Lemma 2.1. As ψ_1 is m-convex in $[0, b]$,

$$\mathcal{P}_m(\psi_1) \geq 0,$$

that is

$$\mathcal{P}_m(M_1 x^2 - f(x)) \geq 0,$$

which gives

$$M_1 \mathcal{P}_m(\phi) \geq \mathcal{P}_m(f). \tag{2.10}$$

Similarly ψ_2 is m-convex in $[0, b]$, therefore one has

$$m_1 \mathcal{P}_m(\phi) \leqslant \mathcal{P}_m(f). \tag{2.11}$$

By assumption $\mathcal{P}_m(\phi)$ is non zero, combining inequalities (2.10) and (2.11), one has

$$m_1 \leqslant \frac{\mathcal{P}_m(f)}{\mathcal{P}_m(\phi)} \leqslant M_1.$$

Hence, there exists $\xi \in (0, b)$ such that

$$\frac{\mathcal{P}_m(f)}{\mathcal{P}_m(\phi)} = \frac{(\xi - ma) f'(\xi) - f(\xi) + m f(a)}{\xi^2 - 2ma\xi + ma^2}.$$

Hence, we get the required result. □

Corollary 2.2. *Let $(x_1, \dots, x_n) \in [0, b]$, $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \dots, n$. Also let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}(f) = \frac{(\xi - a) f'(\xi) - f(\xi) + f(a)}{(\xi - a)^2} \mathcal{P}(\phi),$$

provided that $\mathcal{P}(\phi)$ is non zero and $a \in (0, b)$.

Proof. If we put $m = 1$ in (2.9), we get the required result. □

Corollary 2.3. *Let $(x_1, \dots, x_n) \in [0, b]$, $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \dots, n$ and $a \in (0, b)$. Also let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}(f) = f''(a) \mathcal{P}(\phi).$$

Proof. If we put $m = 1$ in (2.9), we get

$$\begin{aligned} \frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} &= \frac{(\xi - a) f'(\xi) - f(\xi) + f(a)}{(\xi - a)^2} \\ &= \frac{1}{\xi - a} \left(f'(\xi) - \frac{f(a) - f(\xi)}{a - \xi} \right). \end{aligned}$$

Take limit as $\xi \to a$, we get

$$\begin{aligned} \frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} &= \lim_{\xi \to a} \frac{1}{\xi - a} \left(f'(\xi) - \frac{f(a) - f(\xi)}{a - \xi} \right) \\ &= \lim_{\xi \to a} \frac{1}{\xi - a} \left(f'(\xi) - f'(a) \right). \end{aligned}$$

Again taking limit as $\xi \to a$, we get

$$\frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} = f''(a).$$

Hence, we get the required result. □

Theorem 2.3. *Let $(x_1, \dots, x_n) \in [0, b]$, $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \dots, n$. Also, let $\phi(x) = x^2$.*

If $f_1, f_2 \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\frac{\mathcal{P}_m(f_1)}{\mathcal{P}_m(f_2)} = \frac{(\xi - ma) f_1'(\xi) - f_1(\xi) + m f_1(a)}{(\xi - ma) f_2'(\xi) - f_2(\xi) + m f_2(a)},$$

provided that the denominators are non-zero and $a \in (0, b)$.

Proof. Suppose a function $k \in C^1([0, b])$ be defined as

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined as

$$c_1 = \mathcal{P}_m(f_2),$$
$$c_2 = \mathcal{P}_m(f_1).$$

Then using Theorem 2.2 with $f = k$, one has

$$(\xi - ma)((c_1 f_1 - c_2 f_2)(\xi))' - (c_1 f_1 - c_2 f_2)(\xi) + m(c_1 f_1 - c_2 f_2)(a) = 0,$$

that is

$$(\xi - ma)(c_1 f_1'(\xi) - c_2 f_2'(\xi)) - c_1 f_1(\xi) + c_2 f_2(\xi) + mc_1 f_1(a) - mc_2 f_2(a) = 0,$$

which gives

$$(\xi - ma)c_1 f_1'(\xi) - (\xi - ma)c_2 f_2'(\xi) - c_1 f_1(\xi) + c_2 f_2(\xi) + mc_1 f_1(a) - mc_2 f_2(a) = 0,$$

which implies

$$c_1 \{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)\} - c_2 \{(\xi - ma)f_2'(\xi) + f_2(\xi) - mf_2(a)\} = 0,$$
$$c_1 \{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)\} = c_2 \{(\xi - ma)f_2'(\xi) - f_2(\xi) + mf_2(a)\}$$

and

$$\frac{c_2}{c_1} = \frac{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)}{(\xi - ma)f_2'(\xi) - f_2(\xi) + mf_2(a)}.$$

After putting the values of c_1 and c_2, we get the required result. □

Here we state an important lemma that is very helpful in proving mean value theorems related to the non-negative functional of Petrović's inequality for coordinated m-convex functions.

Lemma 2.2. *Let $\Delta = [0, b] \times [0, d]$, $m \in (0, 1)$. Also let $f : \Delta \to \mathbb{R}$ be a function such that*

$$m_1 \leqslant \frac{(x - ma)\frac{\partial}{\partial x} f(x, y) - f(x, y) + mf(a, y)}{(x^2 - 2max + ma^2)y^2} \leqslant M_1$$

and

$$m_2 \leqslant \frac{(y - mc)\frac{\partial}{\partial y} f(x, y) - f(x, y) + mf(x, c)}{(y^2 - 2mcy + mc^2)x^2} \leqslant M_2,$$

for all $x \in [0, b] \backslash \{ma\}$, $a \in (0, b)$ and $y \in [0, d] \backslash \{mc\}$, $c \in (0, d)$.

Consider the functions $\alpha_y : [0, b] \to \mathbb{R}$, and $\alpha_x : [0, d] \to \mathbb{R}$, defined as

$$\alpha(x, y) = \max\{M_1, M_2\}x^2 y^2 - f(x, y)$$

and

$$\beta(x, y) = f(x, y) - \min\{m_1, m_2\}x^2 y^2.$$

Then α and β are coordinated m-convex in Δ.

Proof. Consider the partial mappings $\alpha_y : [0,b] \to \mathbb{R}$ and $\alpha_x : [0,d] \to \mathbb{R}$ defined by $\alpha_y(x) := \alpha(x,y)$ for all $x \in (0,b]$ and $\alpha_x(y) := \alpha(x,y)$ for all $y \in (0,d]$.

$$\begin{aligned} P_{a,m,\alpha_y}(x) &= \frac{\alpha_y(x) - m\alpha_y(a)}{x - ma} \\ &= \frac{\alpha(x,y) - m\alpha(a,y)}{x - ma} \\ &= \frac{M_1x^2y^2 - f(x,y) - mM_1a^2y^2 + mf(a,y)}{x - ma} \\ &= M_1\frac{(x^2 - ma^2)y^2}{x - ma} - \frac{f(x,y) - mf(a,y)}{x - ma}. \end{aligned}$$

So we have

$$\begin{aligned} P'_{a,m,\alpha_y}(x) &= M_1\frac{\partial}{\partial x}\left(\frac{(x^2 - ma^2)y^2}{x - ma}\right) - \frac{\partial}{\partial x}\left(\frac{f(x,y) - mf(a,y)}{x - ma}\right) \\ &= M_1y^2\frac{(x^2 - 2max + ma^2)}{(x - ma)^2} - \frac{(x - ma)\frac{\partial}{\partial x}f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2}. \end{aligned}$$

Since

$$M_1 \geq \frac{(x - ma)\frac{\partial}{\partial x}f(x,y) - f(x,y) + mf(a,y)}{(x^2 - 2max + ma^2)y^2},$$

by given conditions, we have

$$(x^2 - 2max + ma^2)y^2 > 0.$$

This implies

$$\begin{aligned} M_1y^2\frac{(x^2 - 2max + ma^2)}{(x - ma)^2} &\geq \frac{(x - ma)\frac{\partial}{\partial x}f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2} \\ M_1y^2\frac{(x^2 - 2max + ma^2)}{(x - ma)^2} &- \frac{(x - ma)\frac{\partial}{\partial x}f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2} \geq 0. \end{aligned}$$

This implies

$$P'_{a,m,\alpha_y}(x) \geq 0 \text{ for all } x \in [0,ma) \cup (ma,b].$$

Similarly, one can show that

$$P'_{a,m,\alpha_x}(y) \geq 0 \text{ for all } x \in [0,mc) \cup (mc,d].$$

This ensures that P_{a,m,α_y} is increasing on $[0,ma) \cup (ma,b]$ for all $a \in [0,b]$ and P_{a,m,α_x} is increasing on $[0,mc) \cup (mc,d]$ for all $c \in [0,d]$. Hence, by Lemma 2.1, α is m-convex in Δ.

Similarly, one can show that β is m-convex in Δ. □

Here we give mean value theorems related to the functional defined by Petrović's inequality for coordinated m-convex functions.

Theorem 2.4. *Let* $\Delta = [0,b] \times [0,d]$, $(x_1,\dots,x_n) \in [0,b]$, $(y_1,\dots,y_n) \in [0,d]$ *be non-negative n-tuples and* $(q_1,\dots,q_n)$, $(p_1,\dots,p_n)$ *be positive n-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_j$ *for each* $j = 1,2,\dots,n$. *Also, let* $\varphi(x,y) = x^2y^2$.

If $f \in C^1(\Delta)$, *then there exists* (ξ_1,η_1) *and* (ξ_2,η_2) *in the interior of* Δ, *such that*

$$\Upsilon_m(f) = \frac{(\xi_1 - ma)\frac{\partial}{\partial x} f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + mf(a,\eta_1)}{(\xi_1^2 - 2ma\xi_1 + ma^2)\eta_1^2}\Upsilon_m(\varphi) \tag{2.12}$$

and

$$\Upsilon_m(f) = \frac{(\xi_2 - ma)\frac{\partial}{\partial y} f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + mf(a,\eta_2)}{(\xi_2^2 - 2ma\xi_2 + ma^2)\eta_2^2}\Upsilon_m(\varphi), \tag{2.13}$$

and provided that $\Upsilon_m(\varphi)$ *is non-zero and* $a \in (0,b)$.

Proof. As f has continuous first order partial derivative in Δ, so there exists real numbers m_1, m_2, M_1 and M_2 such that

$$m_1 \leqslant \frac{(x - ma)\frac{\partial}{\partial x} f(x,y) - f(x,y) + mf(a,y)}{(x^2 - 2max + ma^2)y^2} \leqslant M_1$$

and

$$m_2 \leq \frac{(y - ma)\frac{\partial}{\partial y} f(x,y) - f(x,y) + mf(x,a)}{(y^2 - 2may + ma^2)x^2} \leq M_2,$$

for all $x \in (0,b]$, $y \in (0,d]$, $a \in (0,b)$ and $m \in (0,1)$.

Now let us consider the functions α and β defined in Lemma 2.2.

As α is m-convex in Δ, then

$$\Upsilon_m(\alpha) \geq 0,$$

that is

$$\Upsilon_m(M_1x^2y^2 - f(x,y)) \geq 0,$$

which gives

$$M_1\Upsilon_m(\varphi) \geq \Upsilon_m(f). \tag{2.14}$$

Similarly β is m-convex in Δ, therefore one has

$$m_1\Upsilon_m(\varphi) \leqslant \Upsilon_m(f). \tag{2.15}$$

By the assumption $\Upsilon_m(\varphi)$ is non-zero. Combining inequalities (2.14) and (2.15), one has

$$m_1 \leqslant \frac{\Upsilon_m(f)}{\Upsilon_m(\varphi)} \leqslant M_1.$$

Hence there exists (ξ_1,η_1) in the interior of Δ, such that

$$\Upsilon_m(f) = \frac{(\xi_1 - ma)\frac{\partial}{\partial x} f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + mf(a,\eta_1)}{(\xi_1^2 - 2ma\xi_1 + ma^2)\eta_1^2}\Upsilon_m(\varphi).$$

Similarly, one can show that

$$\Upsilon_m(f) = \frac{(\xi_2 - ma)\frac{\partial}{\partial y} f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + mf(a,\eta_2)}{(\xi_2^2 - 2ma\xi_2 + ma^2)\eta_2^2}\Upsilon_m(\varphi),$$

which is the required result. □

Corollary 2.4. *Let* $\Delta = [0,b] \times [0,d]$, $(x_1,\dots,x_n) \in [0,b]$, $(y_1,\dots,y_n) \in [0,d]$ *be non-negative* n*-tuples and* $(q_1,\dots,q_n)$, $(p_1,\dots,p_n)$ *be positive* n*-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_j$ *for each* $j = 1,2,\dots,n$. *Also, let* $\varphi(x,y) = x^2y^2$.

If $f \in C^1(\Delta)$, *then there exists* (ξ_1,η_1) *and* (ξ_2,η_2) *in the interior of* Δ, *such that*

$$\Upsilon(f) = \frac{(\xi_1 - a)\frac{\partial}{\partial x} f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + f(a,\eta_1)}{(\xi_1 - a)^2\eta_1^2}\Upsilon(\varphi)$$

and

$$\Upsilon(f) = \frac{(\xi_2 - a)\frac{\partial}{\partial y} f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + f(a,\eta_2)}{(\xi_2 - a)^2\eta_2^2}\Upsilon(\varphi),$$

provided that $\Upsilon(\varphi)$ *is non-zero and* $a \in (0,b)$.

Proof. If we put m=1 in (2.12) and (2.13), we get the required result. □

Theorem 2.5. *Let* $\Delta = [0,b] \times [0,d]$, $(x_1,\dots,x_n) \in [0,b]$, $(y_1,\dots,y_n) \in [0,d]$ *be non-negative* n*-tuples and* $(q_1,\dots,q_n)$, $(p_1,\dots,p_n)$ *be positive* n*-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_j$ *for each* $j = 1,2,\dots,n$. *Also, let* $\varphi(x,y) = x^2y^2$.

If $f_1, f_2 \in C^1(\Delta)$, *then there exists* (ξ_1,η_1) *and* (ξ_2,η_2) *in the interior of* Δ, *such that*

$$\frac{\Upsilon_m(f_1)}{\Upsilon_m(f_2)} = \frac{(\xi_1 - ma)\frac{\partial}{\partial x} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial x} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)}$$

and

$$\frac{\Upsilon_m(f_1)}{\Upsilon_m(f_2)} = \frac{(\xi_1 - ma)\frac{\partial}{\partial y} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial y} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)},$$

provided that the denominators are non-zero and $a \in (0,b)$.

Proof. Suppose

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined by

$$\begin{aligned} c_1 =& \Upsilon_m(f_2), \\ c_2 =& \Upsilon_m(f_1). \end{aligned}$$

Then using Theorem 2.4 with $f = k$, we get

$$
\begin{aligned}
&(\xi - ma)\frac{\partial}{\partial x}(c_1 f_1 - c_2 f_2)(\xi,\eta) - (c_1 f_1 - c_2 f_2)(\xi,\eta) + m(c_1 f_1 - c_2 f_2)(a,\eta) = 0,\\
&(\xi - ma)c_1\frac{\partial}{\partial x}f_1(\xi,\eta) - (\xi - ma)c_2\frac{\partial}{\partial x}f_2(\xi,\eta) - c_1 f_1(\xi,\eta) + c_2 f_2(\xi,\eta)\\
&+ mc_1 f_1(a,\eta) - mc_2 f_2(a,\eta) = 0,\\
&c_1\left\{(\xi - ma)\frac{\partial}{\partial x}f_1(\xi,\eta) - f_1(\xi,\eta) + mf_1(a,\eta)\right\} - c_2\left\{(\xi - ma)\frac{\partial}{\partial x}f_2(\xi,\eta)\right.\\
&\left.+ f_2(\xi,\eta) - mf_2(a,\eta)\right\} = 0,\\
&c_1\left\{(\xi - ma)\frac{\partial}{\partial x}f_1(\xi,\eta) - f_1(\xi,\eta) + mf_1(a,\eta)\right\} = c_2\left\{(\xi - ma)\frac{\partial}{\partial x}f_2(\xi,\eta)\right.\\
&\left.- f_2(\xi,\eta) + mf_2(a,\eta)\right\},
\end{aligned}
$$

and

$$
\frac{c_2}{c_1} = \frac{(\xi_1 - ma)\frac{\partial}{\partial x}f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial x}f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)}.
$$

Similarly, one can show that

$$
\frac{c_2}{c_1} = \frac{(\xi_1 - ma)\frac{\partial}{\partial y}f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial y}f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)}.
$$

After putting the values of c_1 and c_2, we get the required result. □

3. Log Convexity

Here we have defined some families of parametric functions which we use in sequal. Let $I = [0,a), J = [0,b) \subseteq \mathbb{R}$ be intervals and $f_t : I \times J \to \mathbb{R}$ represents some parametric mapping for $t \in (c,d) \subseteq \mathbb{R}$. We define functions

$$
f_{t,y} : I \to \mathbb{R} \text{ by } f_{t,y}(u) = f_t(u,y)
$$

and

$$
f_{t,x} : J \to \mathbb{R} \text{ by } f_{t,x}(v) = f_t(x,v),
$$

where $x \in I$ and $y \in J$. Suppose $\mathcal{H}_1$ denotes the class of functions $f_t : I \times J \to \mathbb{R}$ for $t \in (c,d)$ such that the functions

$$
t \mapsto r_m(u_0,u_1,u_2,f_{t,y}), \quad \text{for all } u_0,u_1,u_2 \in I
$$

and

$$
t \mapsto r_m(v_0,v_1,v_2,f_{t,x}), \quad \text{for all } v_0,v_1,v_2 \in J
$$

are log-convex functions in Jensen sense on (c,d).

The following lemma is given in [16].

Lemma 3.1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0, \infty)$ is log-convex in J-sense on I, that is, for each $r, t \in I$*

$$f(r)f(t) \geq f^2\left(\frac{t+r}{2}\right)$$

if and only if the relation

$$m^2 f(t) + 2mnf\left(\frac{t+r}{2}\right) + n^2 f(r) \geq 0$$

holds, for each $m, n \in \mathbb{R}$ and $r, t \in I$.

Our next result comprises properties of functional defined in Theorem 2.1.

Theorem 3.1. *Let $f_t \in \mathcal{H}_1$ and Υ_m be the functional defined in (2.8). Then the function $t \mapsto \Upsilon_m(f_t)$ is log-convex in Jensen sense for each $t \in (c, d)$.*

Proof. Let

$$h(u, v) = m^2 f_t(u, v) + 2mnf_{\frac{t+r}{2}}(u, v) + n^2 f_r(u, v),$$

where $m, n \in \mathbb{R}$ and $t, r \in (c, d)$. Also we can consider that

$$h_y(u) = m^2 f_{t,y}(u) + 2mnf_{\frac{t+r}{2},y}(u) + n^2 f_{r,y}(u)$$

and

$$h_x(v) = m^2 f_{t,x}(v) + 2mnf_{\frac{t+r}{2},x}(v) + n^2 f_{r,x}(v),$$

which gives

$$\begin{aligned} r_m(u_0, u_1, u_2, h_y) = & m^2 r_m(u_0, u_1, u_2, f_{t,y}) + 2mnr_m(u_0, u_1, u_2, f_{\frac{t+r}{2},y}) \\ & + n^2 r_m(u_0, u_1, u_2, f_{r,y}). \end{aligned}$$

As $r_m[u_0, u_1, u_2, f_{t,y}]$ is log-convex in Jensen sense so by using Lemma 3.1, the right hand side of the above expression is non negative so h_y is m-convex, similarly h_x is also m-convex, so h is m-convex on coordinates, which implies $r_m(h) \geq 0$ and

$$m^2 r_m(f_t) + 2mnr_m(f_{\frac{t+r}{2}}) + n^2 r_m(f_r) \geq 0.$$

Hence, $t \mapsto \Upsilon_m(f_t)$ is log-convex in Jensen sense. □

Theorem 3.2. *Assume that f_t is of class $\mathcal{H}_1$ and Υ_m be the functional defined in (2.8). If the function $\Upsilon_m(f_t)$ is continuous for each $t \in (c, d)$, then $\Upsilon_m(f_t)$ is log-convex for each $t \in (c, d)$.*

Proof. If a function is continuous and log-convex in Jensen sense, then it is log-convex (see [3, p. 48]). It is given that $\Upsilon_m(f_t)$ is continuous for each $t \in (c, d)$, hence $\Upsilon_m(f_t)$ is log-convex for each $t \in (c, d)$. □

Lemma 3.2. *If f is a convex function for all x_1, x_2, x_3 of an open interval I for which $x_1 < x_2 < x_3$, then*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0.$$

Theorem 3.3. *Let $f_t \in \mathcal{H}_1$ and Υ_m be the functional defined in* (2.8). *If $\Upsilon_m(f_t)$ is positive, then for some $r < s < t$, where $r, s, t \in (c, d)$, one has*

$$[\Upsilon_m(f_s)]^{t-r} \leq [\Upsilon_m(f_r)]^{t-s} [\Upsilon_m(f_t)]^{s-r}.$$

Proof. Consider the functional $\Upsilon_m(f_t)$. Also let $r < s < t$, where $r, s, t \in (c, d)$, since $\Upsilon_m(f_t)$ is log-convex, that is, $\log \Upsilon_m(f_t)$ is convex. By taking $f = \log \Upsilon_m$ in Lemma 3.2, we have

$$(t-s)\log \Upsilon_m(f_r) + (r-t)\log \Upsilon_m(f_s) + (s-r)\log \Upsilon_m(f_t) \geq 0,$$

which can be written as

$$[\Upsilon_m(f_s)]^{t-r} \leq [\Upsilon_m(f_r)]^{t-s} [\Upsilon_m(f_t)]^{s-r}.$$

□

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