BASIC INEQUALITIES FOR (m, M)-Ψ-CONVEX FUNCTIONS WHEN $\Psi = -\ln$

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ABSTRACT. In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. INTRODUCTION

Assume that the function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the function $\Phi : I \to \mathbb{R}$ is m-Ψ-*lower convex* if $\Phi - m\Psi$ is a convex function on I. We may introduce the class of functions (see [1])

$$(1.1) \qquad \mathcal{L}(I, m, \Psi) := \{ \Phi : I \to \mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I \}.$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of M-Ψ-*upper convex* functions by (see [1])

$$(1.2) \qquad \mathcal{U}(I, M, \Psi) := \{ \Phi : I \to \mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I \}.$$

The intersection of these two classes will be called the class of (m, M)-Ψ-*convex functions* and will be denoted by (see [1])

$$(1.3) \qquad \mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi).$$

Remark 1.1. If $\Phi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [7], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on I. In [7], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of *g*-convex dominated functions can be obtained as a particular case from (m, M)-Ψ-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds (see [1]).

Lemma 1.1. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$, the interior of I and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$(1.4) \qquad m\left(\Psi\left(t\right) - \Psi\left(s\right) - \Psi'\left(s\right)\left(t - s\right)\right) \leq \Phi\left(t\right) - \Phi\left(s\right) - \Phi'\left(s\right)\left(t - s\right),$$

for all $t, s \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$(1.5) \qquad \Phi\left(t\right) - \Phi\left(s\right) - \Phi'\left(s\right)\left(t - s\right) \leq M\left(\Psi\left(t\right) - \Psi\left(s\right) - \Psi'\left(s\right)\left(t - s\right)\right),$$

for all $t, s \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both (1.4) and (1.5) hold.*

Another elementary fact for twice differentiable functions also holds (see [1]).

Lemma 1.2. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$(1.6) \qquad\qquad\qquad m\Psi''\left(t\right) \leq \Phi''\left(t\right), \quad \text{for all } t \in \mathring{I}.$$

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$(1.7) \qquad\qquad\qquad \Phi''\left(t\right) \leq M\Psi''\left(t\right), \quad \text{for all } t \in \mathring{I}.$$

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both (1.6) and (1.7) hold.*

For various inequalities concerning these classes of function, see the survey paper [3].

In what follows, we consider the class of functions $\mathcal{B}\left(I, m, M, -\ln\right)$ for $M, m \in \mathbb{R}$, with $M \geq m$ that is obtained for $\Psi : I \subseteq (0, \infty) \to \mathbb{R}$, $\Psi\left(t\right) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathring{I}$ then by Lemma 1.1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$(1.8) \qquad m\left(\ln s - \ln t - \frac{1}{s}(s - t)\right) \leq \Phi(t) - \Phi(s) - \Phi'(s)(t - s)$$

$$\leq M\left(\ln s - \ln t - \frac{1}{s}(s - t)\right),$$

for any $s, t \in \mathring{I}$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a twice differentiable function on $\mathring{I}$ then by Lemma 1.2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$(1.9) \qquad m \leq t^2 \Phi''(t) \leq M,$$

which is a convenient condition to verify in applications.

In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

For recent results concerning inequalities for weighted arithmetic mean and geometric mean (see [4,5] and [8–15]).

2. Some Inequalities From Definition of Convexity

We define the weighted arithmetic and geometric means

$$A_\nu(a, b) := (1 - \nu) a + \nu b \text{ and } G_\nu(a, b) := a^{1-\nu} b^\nu,$$

where $\nu \in [0, 1]$ and $a, b > 0$. If $\nu = \frac{1}{2}$, then we write for brevity $A(a, b)$ and $G(a, b)$, respectively.

The following double inequality holds, see also [6].

Theorem 2.1. *Let $M, m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$. Then for any $a, b > 0$ and $\nu \in [0, 1]$ we have*

$$(2.1) \qquad \ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^m \leq (1 - \nu)\Phi(a) + \nu\Phi(b) - \Phi((1 - \nu)a + \nu b)$$

$$\leq \ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^M.$$

Proof. Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m \ln$ is convex and by the definition of convexity, we have

$$\Phi((1 - \nu)a + \nu b) + m \ln A_\nu(a, b)$$
$$\leq (1 - \nu)(\Phi(a) + m \ln a) + \nu(\Phi(b) + m \ln b)$$
$$= (1 - \nu)\Phi(a) + \nu\Phi(b) + (1 - \nu)m \ln a + \nu m \ln b$$
$$= (1 - \nu)\Phi(a) + \nu\Phi(b) + m \ln G_\nu(a, b),$$

that is equivalent to

$$m \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b)$$

and the first inequality in (2.1) is proved.

Similarly, by the convexity of $\Phi_M := -M \ln -\Phi$ we get the second part of (2.1). $\square$

For m, M with $M > m > 0$ we define

(2.2) $M_p := \begin{cases} M^p, & \text{if } p > 1, \\ m^p, & \text{if } p < 0, \end{cases}$ and $m_p := \begin{cases} m^p, & \text{if } p > 1, \\ M^p, & \text{if } p < 0. \end{cases}$

Consider the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$. This is a convex function and $\Phi''(t) = p(p-1)t^{p-2}$, $t > 0$. Consider $\kappa(t) := t^2\Phi''(t) = p(p-1)t^p$. We observe that

$$\max_{t\in[m,M]} \kappa(t) = p(p-1)M_p \text{ and } \min_{t\in[m,M]} \kappa(t) = p(p-1)m_p.$$

Corollary 2.1. *Let m, M with $M > m > 0$ and $p \in (-\infty, 0) \cup (1, \infty)$. Then for any a, $b \in [m, M]$ and $\nu \in [0, 1]$ we have*

(2.3) $\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)m_p} \leq (1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p$

$$\leq \ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p},$$

where M_p and m_p are defined by (2.2).

By taking the exponential in (2.3) we get the equivalent inequality

(2.4) $\exp\left(\frac{(1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p}{p(p-1)M_p}\right)$

$$\leq \frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$\leq \exp\left(\frac{(1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p}{p(p-1)m_p}\right),$$

for any $p \in (-\infty, 0) \cup (1, \infty)$, $\nu \in [0, 1]$ and any a, $b \in [m, M]$.

If we take $p = 2$ in (2.4) and perform the calculations, then we get

(2.5) $\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{M^2}\right) \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{m^2}\right),$

for any $a, b \in [m, M]$.

If a, $b > 0$ then by taking $M = \max\{a, b\}$ and $m = \min\{a, b\}$ in (2.5) we have

(2.6) $\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max^2\{a,b\}}\right) \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min^2\{a,b\}}\right).$

Since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{b-a}{\max\{a,b\}}\right)^2 = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{b-a}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2$$

for any $a,\, b > 0$, then (2.6) can be written as

$$(2.7) \qquad \exp\left(\frac{1}{2}(1-\nu)\nu\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right)$$

$$\leq \frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$\leq \exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right).$$

This inequality was obtained in a different way in [5].

If we take $p = -1$ in (2.4) and perform the calculations, then we get

$$(2.8) \quad \exp\left(\frac{1}{2}(1-\nu)\nu\frac{m(b-a)^2}{abA_\nu(a,b)}\right) \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{M(b-a)^2}{abA_\nu(a,b)}\right),$$

for any $a,\, b \in [m, M]$ and $\nu \in [0,1]$.

If $a,\, b > 0$ then by taking $M = \max\{a,b\}$ and $m = \min\{a,b\}$ in (2.8) and since $ab = \max\{a,b\}\min\{a,b\}$ we have

$$(2.9) \qquad \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)$$

$$\leq \frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$\leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right),$$

for any $\nu \in [0,1]$.

Since

$$\frac{1}{\max\{a,b\}} \leq \frac{1}{A_\nu(a,b)} \leq \frac{1}{\min\{a,b\}},$$

hence,

$$\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2\right) \leq \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)$$

and

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right) \leq \exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right),$$

showing that the double inequality (2.9) is better than (2.7).

3. Some Perturbed Inequalities

Recall the following result obtained by Dragomir in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$(3.1) \qquad n \min_{j \in \{1,2,\ldots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right)$$

$$\leq \frac{1}{P_n} \sum_{j=1}^{n} p_j f(x_j) - f\left(\frac{1}{P_n} \sum_{j=1}^{n} p_j x_j \right)$$

$$\leq n \max_{j \in \{1,2,\ldots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right),$$

where $f : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j \in \{1,2,\ldots,n\}}$ are vectors in C and $\{p_j\}_{j \in \{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (3.1) that

$$(3.2) \qquad 2r \left(\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2} \right) \right)$$

$$\leq \nu f(x) + (1 - \nu) f(y) - f(\nu x + (1 - \nu) y)$$

$$\leq 2R \left(\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2} \right) \right),$$

for any $x, y \in C$ and $\nu \in [0,1]$, where $r := \min \{\nu, 1 - \nu\}$ and $R := \max \{\nu, 1 - \nu\}$.

Theorem 3.1. *Let M, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}\left((0, \infty), m, M, -\ln \right)$. Then for any a, $b > 0$ and $\nu \in [0,1]$ we have*

$$(3.3) \quad \ln \left(\frac{A_\nu(a,b)}{G_\nu(a,b)} \left(\frac{G(a,b)}{A(a,b)} \right)^{2r} \right)^m$$

$$\leq (1 - \nu) \Phi(a) + \nu \Phi(b) - \Phi((1 - \nu) a + \nu b) - 2r \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2} \right) \right)$$

$$\leq \ln \left(\left(\frac{G(a,b)}{A(a,b)} \right)^{2r} \frac{A_\nu(a,b)}{G_\nu(a,b)} \right)^M$$

and

$$(3.4)$$

$$\ln \left(\left(\frac{A(a,b)}{G(a,b)} \right)^{2R} \frac{G_\nu(a,b)}{A_\nu(a,b)} \right)^m$$

$$\leq 2R \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi \left(\frac{a+b}{2} \right) \right) - \left(\nu \Phi(a) + (1 - \nu) \Phi(b) - \Phi(\nu a + (1 - \nu) b) \right)$$

$$\leq \ln \left(\frac{G_\nu(a, b)}{A_\nu(a, b)} \left(\frac{A(a, b)}{G(a, b)} \right)^{2R} \right)^M,$$

where $r := \min \{\nu, 1 - \nu\}$ *and* $R := \max \{\nu, 1 - \nu\}$.

Proof. Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $f_m := \Phi + m \ln$ is convex and by (3.2) we have

$$(3.5) \qquad 2r \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi \left(\frac{a+b}{2} \right) \right) - 2rm \ln \frac{A(a, b)}{G(a, b)}$$

$$\leq \nu \Phi(a) + (1 - \nu) \Phi(b) - \Phi(\nu a + (1 - \nu) b) - m \ln \frac{A_\nu(a, b)}{G_\nu(a, b)}$$

$$\leq 2R \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi \left(\frac{a+b}{2} \right) \right) - 2Rm \ln \frac{A(a, b)}{G(a, b)},$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by (3.2) we have

$$(3.6) \qquad 2r \left(\Phi \left(\frac{a+b}{2} \right) - \frac{\Phi(a) + \Phi(b)}{2} \right) + 2rM \ln \frac{A(a, b)}{G(a, b)}$$

$$\leq \Phi(\nu a + (1 - \nu) b) - \nu \Phi(a) - (1 - \nu) \Phi(b) + M \ln \frac{A_\nu(a, b)}{G_\nu(a, b)}$$

$$\leq 2R \left(\Phi \left(\frac{a+b}{2} \right) - \frac{\Phi(a) + \Phi(b)}{2} \right) + 2RM \ln \frac{A(a, b)}{G(a, b)},$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

From the first inequality in (3.5) we have

$$\ln \left(\frac{A_\nu(a, b)}{G_\nu(a, b)} \left(\frac{G(a, b)}{A(a, b)} \right)^{2r} \right)^m$$

$$\leq \nu \Phi(a) + (1 - \nu) \Phi(b) - \Phi(\nu a + (1 - \nu) b) - 2r \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi \left(\frac{a+b}{2} \right) \right),$$

while from the first inequality in (3.6) we also have

$$\nu \Phi(a) + (1 - \nu) \Phi(b) - \Phi(\nu a + (1 - \nu) b) - 2r \left(\frac{\Phi(a) + \Phi(b)}{2} - \Phi \left(\frac{a+b}{2} \right) \right)$$

$$\leq \ln \left(\left(\frac{G(a, b)}{A(a, b)} \right)^{2r} \frac{A_\nu(a, b)}{G_\nu(a, b)} \right)^M,$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

These prove the desired result (3.3).

From the second inequality in (3.5) we have

$$
\ln\left(\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^m\right)
$$

$$
\leq 2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)),
$$

while from the second inequality in (3.6) we also have

$$
2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b))
$$

$$
\leq\ln\left(\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^M\right),
$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.4). □

Corollary 3.1. *Let m,M with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any $a,b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$
(3.7)\qquad \ln\left(\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^{p(p-1)m_p}\right)
$$

$$
\leq(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p-2r\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)
$$

$$
\leq\ln\left(\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p}\right)
$$

and

$$
(3.8)\qquad \ln\left(\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{p(p-1)m_p}\right)
$$

$$
\leq 2R\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)-((1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p)
$$

$$
\leq\ln\left(\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{p(p-1)M_p}\right),
$$

where $r:=\min\{\nu,1-\nu\}$ and $R:=\max\{\nu,1-\nu\}$ and M_p and m_p are defined by (2.2).

Observe, by simple calculation, we have that

$$(3.9) \qquad (1 - \nu) a^2 + \nu b^2 - ((1 - \nu) a + \nu b)^2 - 2r \left(\frac{a^2 + b^2}{2} - \left(\frac{a + b}{2} \right)^2 \right)$$

$$= (1 - \nu) \nu (b - a)^2 - \frac{r}{2} (b - a)^2 = r \left(R - \frac{1}{2} \right) (b - a)^2$$

and

$$(3.10) \qquad 2R \left(\frac{a^2 + b^2}{2} - \left(\frac{a + b}{2} \right)^2 \right) - \left((1 - \nu) a^2 + \nu b^2 - ((1 - \nu) a + \nu b)^2 \right)$$

$$= \frac{R}{2} (b - a)^2 - (1 - \nu) \nu (b - a)^2 = R \left(\frac{1}{2} - r \right) (b - a)^2 ,$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

If we write the inequalities (3.7) and (3.8) for $p = 2$, then we get

$$(3.11) \qquad \ln \left(\frac{A_\nu (a, b)}{G_\nu (a, b)} \left(\frac{G (a, b)}{A (a, b)} \right)^{2r} \right)^{2m^2} \leq r \left(R - \frac{1}{2} \right) (b - a)^2$$

$$\leq \ln \left(\left(\frac{G (a, b)}{A (a, b)} \right)^{2r} \frac{A_\nu (a, b)}{G_\nu (a, b)} \right)^{2M^2}$$

and

$$(3.12) \qquad \ln \left(\left(\frac{A (a, b)}{G (a, b)} \right)^{2R} \frac{G_\nu (a, b)}{A_\nu (a, b)} \right)^{2m^2} \leq R \left(\frac{1}{2} - r \right) (b - a)^2$$

$$\leq \ln \left(\frac{G_\nu (a, b)}{A_\nu (a, b)} \left(\frac{A (a, b)}{G (a, b)} \right)^{2R} \right)^{2M^2} ,$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

From the first inequality in (3.11) we have

$$(3.13) \qquad \frac{A_\nu (a, b)}{G_\nu (a, b)} \leq \left(\frac{A (a, b)}{G (a, b)} \right)^{2r} \exp \left(\frac{1}{2m^2} r \left(R - \frac{1}{2} \right) (b - a)^2 \right) ,$$

while from the second inequality in (3.11) we have

$$(3.14) \qquad \left(\frac{A (a, b)}{G (a, b)} \right)^{2r} \exp \left(\frac{1}{2M^2} r \left(R - \frac{1}{2} \right) (b - a)^2 \right) \leq \frac{A_\nu (a, b)}{G_\nu (a, b)} .$$

From the first inequality in (3.12) we have

$$(3.15) \qquad \left(\frac{A (a, b)}{G (a, b)} \right)^{2R} \exp \left(- \frac{1}{2m^2} R \left(\frac{1}{2} - r \right) (b - a)^2 \right) \leq \frac{A_\nu (a, b)}{G_\nu (a, b)} ,$$

while from the second inequality in (3.12) we have

$$(3.16) \qquad \frac{A_\nu(a,b)}{G_\nu(a,b)} \leq \left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right).$$

In conclusion, from (3.13)-(3.16) we have the following result:

$$(3.17) \qquad \max\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right.$$

$$\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\}$$

$$\leq \frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$\leq \min\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right.$$

$$\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\},$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

We need the following lemma (see [4]).

Lemma 3.1. *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a, b \in \mathring{I}$ and $\nu \in [0, 1]$ we have*

$$(3.18) \qquad 0 \leq (1-\nu)f(a) + \nu f(b) - f((1-\nu)a + \nu b)$$

$$\leq \nu(1-\nu)(b-a)(f'(b) - f'(a)).$$

We have the following theorem.

Theorem 3.2. *Let $M, m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any $a, b > 0$ and $\nu \in [0, 1]$ we have*

$$(3.19) \qquad m\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)$$

$$\leq \nu(1-\nu)(b-a)(\Phi'(b) - \Phi'(a))$$

$$- ((1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b))$$

$$\leq M\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right).$$

Proof. Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then $f_m := \Phi + m\ln$ is convex and by (3.18) we have

$$0 \leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) - m\ln\frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$\leq \nu (1 - \nu) (b - a) \left(\Phi' (b) - \Phi' (a) + \frac{m}{b} - \frac{m}{a} \right)$$

$$= \nu (1 - \nu) (b - a) (\Phi' (b) - \Phi' (a)) - \frac{m}{ab} \nu (1 - \nu) (b - a)^2 ,$$

that is equivalent to

$$m \left(\nu (1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu (a, b)}{G_\nu (a, b)} \right)$$

$$\leq \nu (1 - \nu) (b - a) (\Phi' (b) - \Phi' (a)) - ((1 - \nu) \Phi (a) + \nu \Phi (b) - \Phi ((1 - \nu) a + \nu b)),$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ and the first inequality in (3.19) is proved.

Since $\Phi \in \mathcal{B} ((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by (3.18) we have

$$0 \leq - (1 - \nu) \Phi (a) - \nu \Phi (b) + f ((1 - \nu) a + \nu b) + M \ln \frac{A_\nu (a, b)}{G_\nu (a, b)}$$

$$\leq - \nu (1 - \nu) (b - a) (\Phi' (b) - \Phi' (a)) + M \nu (1 - \nu) \frac{(b - a)^2}{ab},$$

that is equivalent to

$$\nu (1 - \nu) (b - a) (\Phi' (b) - \Phi' (a)) - (1 - \nu) \Phi (a) - \nu \Phi (b) + f ((1 - \nu) a + \nu b)$$

$$\leq M \left(\nu (1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu (a, b)}{G_\nu (a, b)} \right),$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ and the second inequality in (3.19) is proved. $\square$

Corollary 3.2. *Let m, M with $M > m > 0$ and $p \in (-\infty, 0) \cup (1, \infty)$. Then for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ we have*

$$(3.20) \qquad p (p - 1) m_p \left(\nu (1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu (a, b)}{G_\nu (a, b)} \right)$$

$$\leq p \nu (1 - \nu) (b - a) \left(b^{p-1} - a^{p-1} \right) - ((1 - \nu) a^p + \nu b^p - ((1 - \nu) a + \nu b)^p)$$

$$\leq p (p - 1) M_p \left(\nu (1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu (a, b)}{G_\nu (a, b)} \right),$$

where M_p and m_p are defined by (2.2).

The case $p = 2$ is of interest. Observe that

$$2 \nu (1 - \nu) (b - a)^2 - \left((1 - \nu) a^2 + \nu b^2 - ((1 - \nu) a + \nu b)^2 \right)$$

$$= 2 \nu (1 - \nu) (b - a)^2 - \nu (1 - \nu) (b - a)^2 = \nu (1 - \nu) (b - a)^2$$

and by (3.20) we have

$$2 m^2 \left(\nu (1 - \nu) \frac{(b - a)^2}{ab} - \ln \frac{A_\nu (a, b)}{G_\nu (a, b)} \right) \leq \nu (1 - \nu) (b - a)^2$$

$$\leq 2M^2 \left(\nu \left(1 - \nu \right) \frac{\left(b - a \right)^2}{ab} - \ln \frac{A_\nu \left(a, b \right)}{G_\nu \left(a, b \right)} \right),$$

which is equivalent to

(3.21)
$$\exp \left(\nu \left(1 - \nu \right) \left(b - a \right)^2 \left(\frac{1}{ab} - \frac{1}{2m^2} \right) \right) \leq \frac{A_\nu \left(a, b \right)}{G_\nu \left(a, b \right)}$$
$$\leq \exp \left(\nu \left(1 - \nu \right) \left(b - a \right)^2 \left(\frac{1}{ab} - \frac{1}{2M^2} \right) \right),$$

for any a, $b \in [m, M]$ and $\nu \in [0, 1]$.

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