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A NOTE ON ALMOST ANTI-PERIODIC FUNCTIONS IN BANACH SPACES

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ABSTRACT. The main aim of this note is to introduce the notion of an almost anti-periodic function in Banach space. We prove some characterizations for this class of functions, investigating also its relationship with the classes of anti-periodic functions and almost periodic functions in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

As mentioned in the abstract, the main aim of this note is to introduce the notion of an almost anti-periodic function in Banach space as well as to prove some characterizations for this class of functions. Any anti-periodic function is almost anti-periodic, and any almost anti-periodic function is almost periodic. Unfortunately, almost anti-periodic functions do not have a linear vector structure with the usually considered operations of pointwise addition of functions and multiplication with scalars. The main result of paper is Theorem 2.3, in which we completely profile the closure of linear span of almost anti-periodic functions in the space of almost periodic functions. We also prove some other statements regarding almost anti-periodic functions, and introduce the concepts of Stepanov almost anti-periodic functions, asymptotically almost anti-periodic functions and Stepanov asymptotically almost anti-periodic functions. We investigate the almost anti-periodic properties of convolution products, providing also a few elementary examples and applications.

Let $(X, \|\cdot\|)$ be a complex Banach space. By $C_b([0,\infty) : X)$ we denote the space consisting of all bounded continuous functions from $[0,\infty)$ into X, the symbol $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of functions

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vanishing at infinity. By $BUC([0,\infty) : X)$ we denote the space consisted of all bounded uniformly continuous functions from $[0,\infty)$ to X. This space becomes one of Banach's endowed with the sup-norm.

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [6–9] and [16] for more details on the subject). Let $I=\mathbb{R}$ or $I=[0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if

$$\|f(t+\tau)-f(t)\| \leq \epsilon, \quad t \in I.$$

The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, if and only if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$.

The space consisted of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$. Equipped with the sup-norm, $AP(I : X)$ becomes a Banach space.

For the sequel, we need some preliminary results appearing already in the pioneering paper [2] by H. Bart and S. Goldberg, who introduced the notion of an almost periodic strongly continuous semigroup there (see [1] for more details on the subject). The translation semigroup $(W(t))_{t\geq 0}$ on $AP([0,\infty) : X)$, given by $[W(t)f](s) := f(t+s)$, $t \geq 0$, $s \geq 0$, $f \in AP([0,\infty) : X)$ is consisted solely of surjective isometries $W(t)$ $(t \geq 0)$ and can be extended to a C_0-group $(W(t))_{t\in\mathbb{R}}$ of isometries on $AP([0,\infty) : X)$, where $W(-t) := W(t)^{-1}$ for $t > 0$. Furthermore, the mapping $E : AP([0,\infty) : X) \to AP(\mathbb{R} : X)$, defined by

$$[Ef](t) := [W(t)f](0), \quad t \in \mathbb{R},\ f \in AP([0,\infty) : X),$$

is a linear surjective isometry and Ef is the unique continuous almost periodic extension of a function $f(\cdot)$ from $AP([0,\infty) : X)$ to the whole real line. We have that $[E(Bf)] = B(Ef)$ for all $B \in L(X)$ and $f \in AP([0,\infty) : X)$.

The most intriguing properties of almost periodic vector-valued functions are collected in the following two theorems (in the case that $I=\mathbb{R}$, these assertions are well-known in the existing literature; in the case that $I=[0,\infty)$, then these assertions can be deduced by using their validity in the case $I=\mathbb{R}$ and the properties of extension mapping $E(\cdot)$; see [14] for more details).

Theorem 1.1. *Let $f \in AP(I : X)$. Then the following holds:*

(i) $f \in BUC(I : X)$;
(ii) *if* $g \in AP(I : X)$, $h \in AP(I : \mathbb{C})$, $\alpha,\ \beta \in \mathbb{C}$, *then* $\alpha f+\beta g$ *and* $hf \in AP(I : X)$;
(iii) *Bohr's transform of* $f(\cdot)$,

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-irs} f(s)\,ds,$$

exists for all $r \in \mathbb{R}$ *and*

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_{\alpha}^{t+\alpha} e^{-irs} f(s)\, ds,$$

for all $\alpha \in I,\ r \in \mathbb{R}$*;*

(iv) *if* $P_r(f) = 0$ *for all* $r \in \mathbb{R}$, *then* $f(t) = 0$ *for all* $t \in I$*;*

(v) $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ *is at most countable;*

(vi) *if* $c_0 \not\subseteq X$, *which means that* X *does not contain an isomorphic copy of* c_0, $I = \mathbb{R}$ *and* $g(t) = \int_0^t f(s)\, ds$ $(t \in \mathbb{R})$ *is bounded, then* $g \in AP(\mathbb{R} : X)$*;*

(vii) *if* $(g_n)_{n\in\mathbb{N}}$ *is a sequence in* $AP(I : X)$ *and* $(g_n)_{n\in\mathbb{N}}$ *converges uniformly to* g, *then* $g \in AP(I : X)$*;*

(viii) *if* $I = \mathbb{R}$ *and* $f' \in BUC(\mathbb{R} : X)$, *then* $f' \in AP(\mathbb{R} : X)$*;*

(ix) (*spectral synthesis*) $f \in \overline{\mathrm{Span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}}$*;*

(x) $R(f)$ *is relatively compact in* X*;*

(xi) *we have*

$$\|f\|_\infty = \sup_{t\geq t_0} \|f(t)\|, \quad t_0 \in I.$$

Theorem 1.2 (Bochner's criterion)**.** *Let* $f \in BUC(\mathbb{R} : X)$. *Then* $f(\cdot)$ *is almost periodic if and only if for any sequence* (b_n) *of numbers from* $\mathbb{R}$ *there exists a subsequence* (a_n) *of* (b_n) *such that* $(f(\cdot + a_n))$ *converges in* $BUC(\mathbb{R} : X)$.

Theorem 1.2 has served S. Bochner to introduce the notion of an almost automorphic function, which slightly generalize the notion of an almost periodic function [4]. For more details about almost periodic and almost automorphic solutions of abstract Volterra integro-differential equations, we refer the reader to the monographs by T. Diagana [6], G. M. N'Guérékata [9], M. Kostić [14] and M. Levitan, V. V. Zhikov [16].

By either $AP(\Lambda : X)$ or $AP_\Lambda(I : X)$, where Λ is a non-empty subset of I, we denote the vector subspace of $AP(I : X)$ consisting of all functions $f \in AP(I : X)$ for which the inclusion $\sigma(f) \subseteq \Lambda$ holds good. It can be easily seen that $AP(\Lambda : X)$ is a closed subspace of $AP(I : X)$ and therefore Banach space itself.

2. Almost Anti-Periodic Functions

Assume that $I = \mathbb{R}$ or $I = [0, \infty)$, as well as that $f : I \to X$ is continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-antiperiod for $f(\cdot)$ if and only if

$$\|f(t + \tau) + f(t)\| \leq \epsilon, \quad t \in I. \tag{2.1}$$

In what follows, by $\vartheta_{ap}(f, \epsilon)$ we denote the set of all ϵ-antiperiods for $f(\cdot)$.

We introduce the notion of an almost anti-periodic function as follows.

Definition 2.1. It is said that $f(\cdot)$ is almost anti-periodic if and only if for each $\epsilon > 0$ the set $\vartheta_{ap}(f, \epsilon)$ is relatively dense in I.

Suppose that $\tau > 0$ is an ϵ-antiperiod for $f(\cdot)$. Applying (2.1) twice, we get that

$$\begin{aligned}\|f(t+2\tau)-f(t)\| &= \|[f(t+2\tau)+f(t+\tau)]-[f(t+\tau)+f(t)]\| \\ &\leq \|f(t+2\tau)+f(t+\tau)\|+\|f(t+\tau)+f(t)\| \leq 2\epsilon, \quad t \in I.\end{aligned}$$

Taking this inequality in account, we obtain almost immediately from elementary definitions that $f(\cdot)$ needs to be almost periodic. Further on, assume that $f : I \to X$ is anti-periodic, i.e, there exists $\omega > 0$ such that $f(t + \omega) = -f(t)$, $t \in I$. Then we obtain inductively that $f(t + (2k + 1)\omega) = -f(t)$, $k \in \mathbb{Z}$, $t \in I$. Since the set $\{(2k + 1)\omega : k \in \mathbb{Z}\}$ is relatively dense in I, the above implies that $f(\cdot)$ is almost anti-periodic. Therefore, we have proved the following theorem.

Theorem 2.1. (i) *Assume $f : I \to X$ is almost anti-periodic. Then $f : I \to X$ is almost periodic.*
(ii) *Assume $f : I \to X$ is anti-periodic. Then $f : I \to X$ is almost anti-periodic.*

It is well known that any anti-periodic function $f : I \to X$ is periodic since, with the notation used above, we have that $f(t + 2k\omega) = f(t)$, $k \in \mathbb{Z} \setminus \{0\}$, $t \in I$. But, the constant non-zero function is a simple example of a periodic function (therefore, almost periodic function) that is neither anti-periodic nor almost anti-periodic.

Example 2.1. (i) Consider the function $f(t) := \sin(\pi t) + \sin(\pi t\sqrt{2})$, $t \in \mathbb{R}$. This is an example of an almost anti-periodic function that is not a periodic function. This can be verified as it has been done by A. S. Besicovitch [3, Introduction, p. ix].
(ii) The function $g(t) := f(t) + 5$, $t \in \mathbb{R}$, where $f(\cdot)$ is defined as above, is almost periodic, not almost anti-periodic and not periodic.

We continue by noting the following simple facts. Let $f : I \to X$ be continuous, and let $\epsilon' > \epsilon > 0$. Then the following holds true.

(i) $\vartheta_{ap}(f, \epsilon) \subseteq \vartheta_{ap}(f, \epsilon')$.
(ii) If $I = \mathbb{R}$ and (2.1) holds with some $\tau > 0$, then (2.1) holds with $-\tau$.
(iii) If $I = \mathbb{R}$ and $\tau_1, \tau_2 \in \vartheta_{ap}(f, \epsilon)$, then $\tau_1 \pm \tau_2 \in \vartheta(f, \epsilon)$.

Furthermore, the argumentation contained in the proofs of structural results of [3, pp. 3-4] shows that the following holds.

Theorem 2.2. *Let $f : I \to X$ be almost anti-periodic. Then we have the following.*
(i) *$cf(\cdot)$ is almost anti-periodic for any $c \in \mathbb{C}$.*
(ii) *If $X = \mathbb{C}$ and $\inf_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is almost anti-periodic.*
(iii) *If $(g_n : I \to X)_{n\in\mathbb{N}}$ is a sequence of almost anti-periodic functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is almost anti-periodic.*

Concerning products and sums of almost anti-periodic functions, the situation is much more complicated than for the usually examined class of almost periodic functions.

Example 2.2. (i) The product of two scalar almost anti-periodic functions need not be almost anti-periodic. To see this, consider the functions $f_1(t) = f_2(t) = \cos t$, $t \in \mathbb{R}$, which are clearly (almost) anti-periodic. Then $f_1(t) \cdot f_2(t) = \cos^2 t$, $t \in \mathbb{R}$, $\cos^2(t+\tau) + \cos^2 t \geq \cos^2 t$, τ, $t \in \mathbb{R}$, and therefore $\vartheta_{ap}(f_1 \cdot f_2, \epsilon) = \emptyset$ for any $\epsilon \in (0,1)$.

(ii) The sum of two scalar almost anti-periodic functions need not be almost anti-periodic, so that the almost anti-periodic functions do not form a vector space. To see this, consider the functions $f_1(t) = 2^{-1}\cos 4t$ and $f_2(t) = 2\cos 2t$, $t \in \mathbb{R}$, which are clearly (almost) anti-periodic. Then

$$f_1(t) + f_2(t) = 4\cos^4 t - \frac{3}{2}, \quad t \in \mathbb{R}.$$

Asssume that $f_1 + f_2$ is almost anti-periodic. Then the above identity implies that the function $t \mapsto 8\cos^4 t - 3$, $t \in \mathbb{R}$, is almost anti-periodic, as well. This, in particular, yields that for any $\epsilon \in (0,1)$ we can find $\tau \in \mathbb{R}$ such that

$$\Big| 8\cos^4(t+\tau) + 8\cos^4 t - 6 \Big| \leq \epsilon, \quad t \in \mathbb{R}.$$

Plugging $t = \pi$, we get that $8\cos^4 \tau + 2 \leq \epsilon$, which is a contradiction. Finally, we would like to point out that there exists a large number of much simpler examples which can be used for verification of the statement clarified in this part; for example, the interested reader can easily check that the function $t \mapsto \cos t + \cos 2t$, $t \in \mathbb{R}$, is not almost anti-periodic.

Assume that $f : I \to X$ is almost anti-periodic. Then it can be easily seen that $f(\cdot + a)$ and $f(b\, \cdot)$ are likewise almost anti-periodic, where $a \in I$ and $b \in I \setminus \{0\}$.

Denote now by $ANP_0(I : X)$ the linear span of almost anti-periodic functions $I \mapsto X$. By Theorem 2.1(i), $ANP_0(I : X)$ is a linear subspace of $AP(I : X)$. Let $ANP(I : X)$ be the linear closure of $ANP_0(I : X)$ in $AP(I : X)$. Then, clearly, $ANP(I : X)$ is a Banach space. Furthermore, we have the following result.

Theorem 2.3. $ANP(I : X) = AP_{\mathbb{R} \setminus \{0\}}(I : X)$.

Proof. Since the mapping $E : AP([0,\infty) : X) \to AP(\mathbb{R} : X)$ is a linear surjective isometry, it suffices to consider the case in which $I = \mathbb{R}$. Assume first that $f \in AP_{\mathbb{R} \setminus \{0\}}(I : X)$. By spectral synthesis (see Theorem 1.1(ix)), we have that

$$f \in \overline{\mathrm{Span}\{e^{i\mu \cdot} x : \mu \in \sigma(f),\ x \in R(f)\}},$$

where the closure is taken in the space $AP(\mathbb{R} : X)$. Since $\sigma(f) \subseteq \mathbb{R} \setminus \{0\}$ and the function $t \mapsto e^{i\mu t}$, $t \in \mathbb{R}$, $\mu \in \mathbb{R} \setminus \{0\}$ is anti-periodic, we have that $\mathrm{Span}\{e^{i\mu \cdot} x : \mu \in \sigma(f),\ x \in R(f)\} \subseteq ANP_0(\mathbb{R} : X)$. Hence, $f \in ANP(\mathbb{R} : X)$. The converse statement immediately follows if we prove that, for any fixed function $f \in ANP(\mathbb{R} : X)$, we have that $P_0(f) = 0$, i.e.,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(s)\, ds = 0. \tag{2.2}$$

By almost periodicity of $f(\cdot)$, the limit in (2.2) exists. Hence, it is enough to show that for any given number $\epsilon > 0$ we can find a sequence $(\omega_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty} \omega_n = \infty$ and

$$\left\| \frac{1}{2\omega_n} \int_0^{2\omega_n} f(s)\, ds \right\| \leq \frac{\epsilon}{2}, \quad n \in \mathbb{N}. \tag{2.3}$$

By definition of almost anti-periodicity, we have the existence of a number $l > 0$ such that any interval $I_n = [nl, (n+1)l]$ $(n \in \mathbb{N})$ contains a number ω_n that is anti-period for $f(\cdot)$. The validity of (2.3) is a consequence of the following computation:

$$\begin{aligned} \left\| \int_0^{2\omega_n} f(s)\, ds \right\| &= \left\| \int_0^{\omega_n} f(s)\, ds + \int_{\omega_n}^{2\omega_n} f(s)\, ds \right\| \\ &= \left\| \int_0^{\omega_n} [f(s) + f(s+\omega_n)]\, ds \right\| \\ &\leq \int_0^{\omega_n} \|f(s) + f(s+\omega_n)\|\, ds \leq \epsilon\omega_n, \quad n \in \mathbb{N}, \end{aligned}$$

finishing the proof of theorem. □

Let $f \in AP(I : X)$ and $\emptyset \neq \Lambda \subseteq \mathbb{R}$. Since

$\sigma(f) \subseteq \Lambda$ if and only if $P_r(f) = 0$, $r \in \mathbb{R}\backslash\Lambda$ if and only if $P_0\Big(e^{-ir\cdot} f(\cdot)\Big) = 0$, $r \in \mathbb{R}\backslash\Lambda$,

we have the following corollary of Theorem 2.3 (see also [1, Corollary 4.5.9]).

Corollary 2.1. *Let $f \in AP(I : X)$ and $\emptyset \neq \Lambda \subseteq \mathbb{R}$. Then $f \in AP_\Lambda(I : X)$ if and only if $e^{-ir\cdot} f(\cdot) \in ANP(I : X)$ for all $r \in \mathbb{R} \setminus \Lambda$.*

Further on, Theorem 2.3 combined with the obvious equality $\sigma(Ef) = \sigma(f)$ immediately implies that the unique ANP extension of a function $f \in ANP([0,\infty) : X)$ to the whole real axis is $Ef(\cdot)$. As the next proposition shows, this also holds for almost anti-periodic functions.

Proposition 2.1. *Suppose that $f : [0,\infty) \to X$ is almost anti-periodic. Then $Ef : \mathbb{R} \to X$ is a unique almost anti-periodic extension of $f(\cdot)$ to the whole real axis.*

Proof. The uniqueness of an almost anti-periodic extension of $f(\cdot)$ follows from the uniqueness of an almost periodic extension of $f(\cdot)$. It remains to be proved that $Ef : \mathbb{R} \to X$ is almost anti-periodic. To see this, let $\epsilon > 0$ be given. Then there exists $l > 0$ such that any interval $I \subseteq [0,\infty)$ of length l contains a number $\tau \in I$ such that $\|f(s+\tau) + f(s)\| \leq \epsilon$, $s \geq 0$. We only need to prove that any interval $I \subseteq \mathbb{R}$ of length $2l$ contains a number $\tau \in I$ such that

$$\|[Ef](t+\tau) + [Ef](t)\| = \|[W(t+\tau)f + W(t)f](0)\| \leq \epsilon, \quad t \in \mathbb{R}.$$

If $I \subseteq [0,\infty)$, then the situation is completely clear. Suppose now that $I \subseteq (-\infty, 0]$. Then $-I \subseteq [0,\infty)$ and there exists a number $-\tau \in -I$ such that

$$\sup_{s\geq 0} \|f(s-\tau) + f(s)\| \leq \epsilon.$$

Then the conclusion follows from the computation

$$\begin{aligned}\|[W(t+\tau)f+W(t)f](0)\| &\leq \|W(t+\tau)f+W(t)f\|_{L^\infty([0,\infty))}\\ &\leq \|W(t+\tau)\|_{L^\infty([0,\infty))}\|W(-\tau)f+f\|_{L^\infty([0,\infty))}\\ &= \sup_{s\geq 0}\|f(s-\tau)+f(s)\| \leq \epsilon, \quad t\in\mathbb{R}.\end{aligned}$$

Finally, if $I = I_1 \cup I_2$, where $I_1 = [a, 0]$ $(a < 0)$ and $I_2 = [0, b]$ $(b > 0)$, then $|a| \geq l$ or $b \geq l$. In the case that $|a| \geq l$, then the conclusion follows similarly as in the previously considered case. If $b \geq l$, then the conclusion follows from the computation

$$\begin{aligned}\|[W(t+\tau)f+W(t)f](0)\| &\leq \|W(t+\tau)f+W(t)f\|_{L^\infty([0,\infty))}\\ &\leq \|W(t)\|_{L^\infty([0,\infty))}\|W(\tau)f+f\|_{L^\infty([0,\infty))}\\ &= \sup_{s\geq 0}\|f(s+\tau)+f(s)\| \leq \epsilon, \quad t\in\mathbb{R},\end{aligned}$$

where $\tau \in I_2$ is an ϵ-antiperiod of $f(\cdot)$. □

For various generalizations of almost periodic functions, we refer the reader to [14]. In the following definition, we introduce the notion of a Stepanov almost anti-periodic function.

Definition 2.2. Let $1 \leq p < \infty$, and let $f \in L^p_{loc}(I : X)$. Then we say that $f(\cdot)$ is Stepanov p-almost anti-periodic function, S^p-almost anti-periodic shortly, if and only if the function $\hat{f} : I \to L^p([0, 1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I,\ s \in [0, 1],$$

is almost anti-periodic.

It can be easily seen that any almost anti-periodic function needs to be S^p-almost anti-periodic, as well as that any S^p-almost anti-periodic function has to be S^p-almost periodic, $1 \leq p < \infty$.

3. Almost Anti-Periodic Properties of Convolution Products

Since almost anti-periodic functions do not form a vector space, we will focus our attention here to the almost anti-periodic properties of finite and infinite convolution product, which is undoubtedly a safe and sound way for providing certain applications to abstract PDEs.

Proposition 3.1. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. If $g : \mathbb{R} \to X$ is S^p-almost anti-periodic, then the function $G(\cdot)$, given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \in \mathbb{R}, \tag{3.1}$$

is well-defined and almost anti-periodic.

Proof. It can be easily seen that, for every $t \in \mathbb{R}$, we have $G(t) = \int_0^\infty R(s)g(t-s)\,ds$. Since $g(\cdot)$ is S^p-almost periodic, we can apply [15, Proposition 2.11] in order to see that $G(\cdot)$ is well-defined and almost periodic. It remains to be proved that $G(\cdot)$ is almost anti-periodic. Let a number $\epsilon > 0$ be given in advance. Then we can find a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\int_t^{t+1} \|g(s+\tau) + g(s)\|^p\,ds \leq \epsilon^p$, $t \in \mathbb{R}$. Applying Hölder inequality and this estimate, similarly as in the proof of above-mentioned proposition, we get that

$$\begin{aligned}
\|G(t+\tau) + G(t)\| &\leq \int_0^\infty \|R(r)\| \cdot \|g(t+\tau-r) + g(t-r)\|\,dr \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|g(t+\tau-r) + g(t-r)\|\,dr \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left(\int_k^{k+1} \|g(t+\tau-r) + g(t-r)\|^p\,dr \right)^{1/p} \\
&= \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left(\int_{t-k-1}^{t-k} \|g(s+\tau) + g(s)\|^p\,ds \right)^{1/p} \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \epsilon = M\epsilon, \quad t \in \mathbb{R},
\end{aligned}$$

which clearly implies that the set of all ϵ-antiperiods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. □

In order to relax our exposition, we shall introduce the notion of an asymptotically (S^p-)almost anti-periodic function in the following way (cf. also [12, Lemma 1.1]):

Definition 3.1. (i) Let $f \in C_b([0,\infty) : X)$. Then we say that $f(\cdot)$ is asymptotically almost anti-periodic if and only if there are two locally functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:
- (a) g is almost anti-periodic;
- (b) q belongs to the class $C_0([0,\infty) : X)$;
- (c) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

(ii) Let $1 \leq p < \infty$, and let $f \in L^p_{loc}([0,\infty) : X)$. Then we say that $f(\cdot)$ is asymptotically Stepanov p-almost anti-periodic, asymptotically S^p-almost anti-periodic shortly, if and only if there are two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:
- (a) g is S^p-almost anti-periodic;
- (b) $\hat{q}$ belongs to the class $C_0([0,\infty) : L^p([0,1] : X))$;
- (c) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

Keeping in mind Proposition 3.1 and the proof of [15, Propostion 2.13], we can simply clarify the following result.

Proposition 3.2. *Suppose that* $1 \leq p < \infty$, $1/p + 1/q = 1$ *and* $(R(t))_{t>0} \subseteq L(X)$ *is a strongly continuous operator family satisfying that, for every* $s \geq 0$, *we have that*

$$m_s := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[s+k,s+k+1]} < \infty.$$

Suppose, further, that $f : [0,\infty) \to X$ *is asymptotically* S^p*-almost anti-periodic as well as that the locally* p*-integrable functions* $g : \mathbb{R} \to X$, $q : [0,\infty) \to X$ *satisfy the conditions from* Definition 3.1(ii). *Let there exist a finite number* $M > 0$ *such that the following holds:*

(i) $\lim_{t\to+\infty} \int_t^{t+1} \Big[\int_M^s \|R(r)\| \|q(s-r)\| \, dr\Big]^p \, ds = 0$;

(ii) $\lim_{t\to+\infty} \int_t^{t+1} m_s^p \, ds = 0$.

Then the function $H(\cdot)$, *given by*

$$H(t) := \int_0^t R(t-s)f(s)\, ds, \quad t \geq 0,$$

is well-defined, bounded and asymptotically S^p*-almost anti-periodic.*

Before providing some applications, we want to note that our conclusions from [15, Remark 2.14] and [14, Proposition 2.7.5] can be reformulated for asymptotical almost anti-periodicity.

It is clear that we can apply results from this section in the study of existence and uniqueness of almost anti-periodic solutions of fractional Cauchy inclusion

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$ and $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator (see [8] for the notion). Furthermore, we can analyze the existence and uniqueness of asymptotically (S^p-) almost anti-periodic solutions of fractional Cauchy inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), & t \geq 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$ and $f : [0,\infty) \to X$, satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator (cf. [14] for more details). Arguing so, we can analyze the existence and uniqueness of (asymptotically S^p-) almost anti-periodic solutions of the fractional Poisson heat equations

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \in \mathbb{R}, \ x \in \Omega, \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \geq 0, \ x \in \Omega, \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $\gamma \in (0, 1)$ and $1 < p < \infty$, see [8] and [14] for further information in this direction.

For some other references regarding the existence and uniqueness of anti-periodic and Bloch periodic solutions of certain classes of abstract Volterra integro-differential equations, we refer the reader to [5, 7, 10–13, 17, 18].

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