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SOME MATRIX AND COMPACT OPERATORS OF THE ABSOLUTE FIBONACCI SERIES SPACES

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Abstract. In the present paper, we introduce the absolute Fibonacci space $|F_u|_k$, give some inclusion relations and investigate topological and algebraic structure such as BK-space, α-, β-, γ- duals and Schauder basis. Further, we characterize certain matrix and compact operators on these spaces, also determine their norms and Hausdroff meausures of noncompactness.

1. Introduction

Let ω be the set of all sequences of complex numbers. We write c, ℓ_∞, c_s, b_s and ℓ_k, $k \geq 1$, for the sequence space of all convergent, bounded sequences; for the spaces of all convergent, bounded, k-absolutely convergent series, respectively. Let X and Y be two subspaces of ω and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. If the series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

converges for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then, by $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$. Also, we say that A defines a matrix transformation from X into Y, and denote it by $A \in (X, Y)$ or $A : X \to Y$ if $Ax = (A_n(x)) \in Y$ for every $x \in X$. The α-, β-, γ- duals of X and the domain of the matrix A in X are defined by

$$X^\alpha = \{\epsilon \in \omega : (\epsilon_n x_n) \in \ell \text{ for all } x \in X\},$$
$$X^\beta = \{\epsilon \in \omega : (\epsilon_n x_n) \in c_s \text{ for all } x \in X\},$$

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$$X^{\gamma} = \{\epsilon \in \omega : (\epsilon_n x_n) \in b_s \text{ for all } x \in X\}$$

and

$$X_A = \{x = (x_n) \in \omega : A(x) \in X\}, \tag{1.1}$$

respectively. Further, X is said to be a BK-space if it is a complete normed space with continuous coordinates $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ for all $n \in \mathbb{N}$. If there exists unique sequence of coefficients (x_k) such that, for each $x \in X$,

$$\left\| x - \sum_{k=0}^{m} x_k b_k \right\| \to 0, \quad m \to \infty,$$

then, the sequence (b_k) is called the Schauder basis (or briefly basis) for a normed sequence space X, and in this case we write $x = \sum_{k=0}^{\infty} x_k b_k$. For instance, the sequence $(e^{(j)})$ is the Schauder basis of the space ℓ_k, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in jth place for each $j \in \mathbb{N}$.

Now take $\sum x_v$ as an infinite series with nth partial sum s_n and let (u_n) be a sequence of positive terms. Then, the series $\sum x_v$ is said to be summable $|A, u_n|_k$, $k \geq 1$, if (see [32])

$$\sum_{n=0}^{\infty} u_n^{k-1} |\Delta A_n(s)|^k < \infty,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, $A_{-1}(s) = 0$.

Note that this method includes some well known methods. For example, if A is the matrix of weighted mean $\left(\bar{N}, p_n\right)$ (resp. $u_n = P_n/p_n$), then it reduces to the summability $\left|\bar{N}, p_n, u_n\right|_k$ [36] (the summability $\left|\bar{N}, p_n\right|_k$ [10]). Also if we take A as the matrix of Cesàro mean of order $\alpha > -1$ and $u_n = n$, then we get summability $|C, \alpha|_k$ in Flett's notation [11].

A large literature has recently grown up, concerned with producing sequence spaces by means of matrix domain of a special limitation method and studying their algebraic, topological structure and matrix transformations (see [1–7, 15–18, 25]). Also, some series spaces have been derived and studied by absolute summability methods from a different point of view (see [9–14, 23–26, 28–34, 36]). The aim of this paper is to define the space $|F_u|_k$ combining absolute summability and Fibonacci matrix given by Kara [15], investigate some inclusion relation, construct their α-, β-, γ- duals, basis and characterize some matrix operators related to that space, and also determine their norms and Hausdroff measures of noncompactness.

Firstly, we mention some properties of Fibonacci numbers as follows: the sequence (f_n) of Fibonacci numbers is given by the relations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2$$

that is, each term is equal to the sum of the previous two terms. The sequences of Fibonacci numbers have been important for artist, architects, physicists and mathematicians since the old. The ratio of Fibonacci numbers converges to the golden ratio which is one of the most interesting irrationals having an important role in number theory, algorithms, network theory, etc. Also, Fibonacci numbers have the following properties [19]:

$$\sum_n \frac{1}{f_n} \text{ converges,}$$
$$f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}, \quad n \geq 1,$$
$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = 1.61803398875....$$

Fibonacci matrix $F = (\hat{f}_{nv})$ has recently been defined by Kara [15] as follows:

$$\hat{f}_{nv} = \begin{cases} \frac{-f_{n+1}}{f_n}, & v = n-1, \\ \frac{f_n}{f_{n+1}}, & v = n, \\ 0, & v > n \text{ or } 0 \leq v < n-1, \end{cases}$$

where f_n be the nth Fibonacci number for every $n \in \mathbb{N}$. Note that if we take the Fibonacci matrix instead of A, then $|A, u_n|_k$ summability reduces to the absolute Fibonacci summability. On the other hand, since (s_n) is a sequence of partial sum of the series $\sum x_v$, we get

$$A_n(s) = \sum_{v=0}^{n} \hat{f}_{nv} s_v = \sum_{j=0}^{n} x_j \sum_{v=j}^{n} \hat{f}_{nv} = x_n \hat{f}_{nn} + \sum_{j=0}^{n-1} (\hat{f}_{nn} + \hat{f}_{n,n-1}) x_j$$

and so,

$$\begin{aligned} \Delta A_n(s) =& x_n \frac{f_n}{f_{n+1}} + x_{n-1} \left(\frac{(-1)^n}{f_n f_{n+1}} - \frac{f_{n+1}}{f_n} \right) + \sum_{j=0}^{n-2} (-1)^n \frac{f_{n-1} + f_{n+1}}{f_{n-1} f_n f_{n+1}} x_j \\ =& \sum_{j=0}^{n} \sigma_{nj} x_j, \end{aligned}$$

where

$$\sigma_{nj} = \begin{cases} \frac{f_n}{f_{n+1}}, & j = n, \\ \frac{(-1)^n}{f_n f_{n+1}} - \frac{f_{n+1}}{f_n}, & j = n-1, \\ (-1)^n \frac{f_{n-1} + f_{n+1}}{f_{n-1} f_n f_{n+1}}, & 0 \leq j \leq n-2, \\ 0, & j > n. \end{cases}$$

Now, we introduce the absolute Fibonacci space as follows:

$$|F_u|_k = \left\{ x \in \omega : \sum_{n=0}^{\infty} u_n^{k-1} \left| \sum_{j=0}^{n} \sigma_{nj} x_j \right|^k < \infty \right\}.$$

Also, it may be written that

$$(E^{(k)} \circ T)_n(x) = u_n^{1/k^*} (T_n(x) - T_{n-1}(x)), \tag{1.2}$$

where

$$t_{nv}=\begin{cases}\dfrac{f_n}{f_{n+1}}, & v=n,\\ \dfrac{f_n^2-f_{n+1}^2}{f_nf_{n+1}}, & 0\le v\le n-1,\\ 0, & v>n,\end{cases}$$

$$e_{nv}^{(k)}=\begin{cases}u_n^{1/k^*}, & v=n,\\ -u_n^{1/k^*}, & v=n-1,\\ 0, & v\ne n,n-1,\end{cases}$$

and k^* is the conjugate of k, i.e., $1/k+1/k^*=1$ for $k>1$, and $1/k^*=0$ for $k=1$. With these matrices $T=(t_{nv})$ and $E^{(k)}=(e_{nv}^{(k)})$, according to the notation (1.1), it is obvious that $|F_u|_k=(\ell_k)_{E^{(k)}\circ T}$. Further, since every triangle matrix has a unique inverse which also is a triangle [37], T and $E^{(k)}$ have a unique inverse $\tilde{T}=(\tilde{t}_{nv})$ and $\tilde{E}^{(k)}=(\tilde{e}_{nv})$ given by

$$\tilde{t}_{nv}=\begin{cases}\dfrac{f_{n+1}}{f_n}, & v=n,\\ \dfrac{f_{n+1}^2-f_n^2}{f_vf_{v+1}}, & 0\le v\le n-1,\\ 0, & v>n,\end{cases}\tag{1.3}$$

$$\tilde{e}_{nv}^{(k)}=\begin{cases}u_v^{-1/k^*}, & 0\le v\le n,\\ 0, & v>n,\end{cases}\tag{1.4}$$

respectively.

Before the main theorems, we point out some well known lemmas which are needed in the proofs of theorems.

Lemma 1.1 ([35])**.** *Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$\|A\|_{(\ell_k,\ell)}=\sup_{N\in\mathfrak{F}}\left\{\sum_{v=0}^{\infty}\left|\sum_{n=0}^{\infty}a_{nv}\right|^{k^*}\right\}^{1/k^*},$$

where $\mathfrak{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.1 exposes a rather difficult condition to apply in applications. So the following lemma is more useful in many cases, which gives equivalent norm.

Lemma 1.2 ([29])**.** *Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$\|A\|'_{(\ell_k,\ell)}=\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}\right\}^{1/k^*}<\infty.$$

Moreover, since

$$\|A\|_{(\ell_k,\ell)}\le\|A\|'_{(\ell_k,\ell)}\le 4\,\|A\|_{(\ell_k,\ell)},$$

there exists $1\le\xi\le 4$ such that $\|A\|'_{(\ell_k,\ell)}=\xi\,\|A\|_{(\ell_k,\ell)}$.

Lemma 1.3 ([20]). *Let* $1 \leq k < \infty$. *Then,* $A \in (\ell, \ell_k)$ *if and only if*

$$\|A\|_{(\ell,\ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{\frac{1}{k}}.$$

Lemma 1.4 ([35]).

(a) $A \in (\ell, c) \Leftrightarrow (i)$ $\lim_n a_{nv}$ *exists for* $v \geq 0$, (ii) $\sup_{n,v} |a_{nv}| < \infty$;

(b) $A \in (\ell, \ell_\infty) \Leftrightarrow (ii)$ *holds;*

(c) *If* $1 < k < \infty$, *then,* $A \in (\ell_k, c) \Leftrightarrow (i)$ *holds,* (iii) $\sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$;

(d) *If* $1 < k < \infty$, *then,* $A \in (\ell_k, \ell_\infty) \Leftrightarrow (iii)$ *holds.*

2. The Hausdorff Measure of Noncompactness

If S and H are subsets of a metric space (X, d) and, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$ then, S is called an ε-net of H; if S is finite, then the ε-net S of H is called a finite ε-net of H. Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $\mathcal{C}(X, Y)$. If Q is a bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } X\},$$

and χ is called the Hausdorff measure of noncompactness.

The following lemma is very important to calculate the Hausdorff measure of noncompactness of a bounded subset of the space ℓ_k.

Lemma 2.1 ([27]). *Let* Q *be a bounded subset of the normed space* X *where* $X = \ell_k$ *for* $1 \leq k < \infty$ *or* $X = c_0$. *If* $P_n : X \to X$ *is the operator defined by* $P_n(x) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ *for all* $x \in X$, *then*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - P_r)(x)\| \right).$$

Let X and Y be Banach space and χ_1 and χ_2 be Hausdorff measures on X and Y, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Y and there exists a positive constant M such that $\chi_2(L(Q)) \leq M\chi_1(L(Q))$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf \{M > 0 : \chi_2(L(Q)) \leq M\chi_1(L(Q)) \text{ for all bounded set } Q \subset X\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$ then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 2.2 ([22]). *Let X and Y be Banach spaces and $L \in \mathcal{B}(X,Y)$. Also $S_x = \{x \in X : \|x\| \leq 1\}$ be the unit sphere in X. Then,*

$$\|L\|_{\chi} = \chi\left(L\left(S_x\right)\right)$$

and

$$L \in \mathcal{C}(X,Y) \Leftrightarrow \|L\|_{\chi} = 0.$$

Lemma 2.3 ([21]). *Let X be a normed sequence space, $T = (t_{nv})$ be an infinite triangle matrix, χ_T and χ denote the Hausdroff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.*

3. Absolute Fibonacci Space $|F_u|_k$

In this section, we investigate some inclusion relations, topological and algebraic structures of the space $|F_u|_k$. Also we characterize some classes of compact matrix operators on that space and compute their norms and Hausdroff measure of noncompactness.

Firstly, since $|F_u|_k$ is generated from ℓ_k, to explain a relation between the spaces ℓ_k and $|F_u|_k$, we begin with the following theorem.

Theorem 3.1. *Let $u = (u_n) \in \ell_\infty$ and $1 \leq k < \infty$. Then, $\ell_k \subset |F_u|_k$.*

Proof. To prove the inclusion $\ell_k \subset |F_u|_k$, it is sufficient to show that

$$\|x\|_{|F_u|_k} \leq O(1) \|x\|_{\ell_k},$$

for all $x \in \ell_k$. The proof is clear for the case $k = 1$, and so it is omitted. Let $k > 1$. Then, since the series $\sum_n \frac{1}{f_n}$ is convergent and $(\frac{1}{f_n})$ is decreasing sequence, it follows from Abel's Theorem, $\frac{n}{f_n} \to 0$ as $n \to \infty$, $\sum_{v=0}^{n} |\sigma_{nv}| = O(1)$ and $\sum_{n=v}^{\infty} |\sigma_{nv}| = O(1)$. Now applying Hölder's inequality, we get

$$\begin{aligned}
\|x\|_{|F_u|_k} &= \left\{\sum_{n=0}^{\infty} u_n^{k-1} \left|\sum_{v=0}^{n} \sigma_{nv} x_v\right|^k\right\}^{1/k} \\
&\leq \left\{\sum_{n=0}^{\infty} u_n^{k-1} \sum_{v=0}^{n} |\sigma_{nv}| \, |x_v|^k \left(\sum_{v=0}^{n} |\sigma_{nv}|\right)^{k/k^*}\right\}^{1/k} \\
&= O(1) \left\{\sum_{v=0}^{\infty} |x_v|^k \sum_{n=v}^{\infty} |\sigma_{nv}|\right\}^{1/k} \\
&= O(1) \left\{\sum_{v=0}^{\infty} |x_v|^k\right\}^{1/k} = O(1) \|x\|_{\ell_k},
\end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $1 \leq k \leq q < \infty$. If there is a constant $M > 0$ such that $u_n \leq M$ for all $n \in \mathbb{N}$, then $|F_u|_k \subset |F_u|_q$.*

Proof. Take $x \in |F_u|_k$. Since $\ell_k \subset \ell_q$, then $\left(u_n^{\frac{1}{k^*}} \sum_{j=0}^{n} \sigma_{nj} x_j\right) \in \ell_q$ and also, since $u_n \leq M$ for all $n \in \mathbb{N}$,

$$M^{\frac{q}{k^*}-\frac{q}{q^*}} \left| u_n^{\frac{1}{q^*}} \sum_{j=0}^{n} \sigma_{nj} x_j \right|^q \leq \left| u_n^{1/k^*} \sum_{j=0}^{n} \sigma_{nj} x_j \right|^q,$$

where k^* and q^* are the conjugate of exponent of k and q, respectively. So this gives that $x \in |F_u|_q$, which completes the proof. ☐

Theorem 3.3. *Let $1 \leq k < \infty$. Then, $|F_u|_k$ is BK-space with respect to the norm*

$$\|x\|_{|F_u|_k} = \left\| E^{(k)} \circ T(x) \right\|_{\ell_k}.$$

Also, the sequence $b^{(j)} = (b_n^{(j)})$ is a Schauder basis for the space $|F_u|_k$, where

$$b_n^{(j)} = \begin{cases} u_j^{-1/k^*} \dfrac{f_{n+1}}{f_n} + u_j^{-1/k^*} \sum\limits_{r=j}^{n-1} \dfrac{f_{n+1}^2 - f_n^2}{f_r f_{r+1}}, & 0 \leq j \leq n-1 \\ u_n^{-1/k^*} \dfrac{f_{n+1}}{f_n}, & j = n \\ 0, & j > n. \end{cases}$$

Proof. We note that ℓ_k is a BK-space for $1 \leq k < \infty$. Further, since $E^{(k)} \circ T$ is a triangle matrix, it follows from Theorem 4.3.2 of [37], $|F_u|_k = (\ell_k)_{E^{(k)} \circ T}$ is a BK-space. Since the sequence $(e^{(j)})$ is the Schauder basis of the space ℓ_k, it can be written from Theorem 2.3 in [14] that $b^{(j)} = (\widetilde{T}_n(\widetilde{E}^{(k)}(e^{(j)})))$ is a Schauder basis of the space $|F_u|_k$. ☐

Theorem 3.4. *Let $1 \leq k < \infty$. Then, the space $|F_u|_k$ is isomorphic to the space ℓ_k that is, $|F_u|_k \cong \ell_k$.*

Proof. To prove the theorem, we should show that there exists a linear bijection between the spaces $|F_u|_k$ and ℓ_k where $1 \leq k < \infty$. Let consider the transformations $T : |F_u|_k \to (\ell_k)_{E^{(k)}}, E^{(k)} : (\ell_k)_{E^{(k)}} \to \ell_k$ given in (1.3) and (1.4). Since the matrices corresponding these transformations are triangles, it can be easily seen that T and $E^{(k)}$ are linear bijections. So, the composite function $E^{(k)} \circ T$ is a linear bijective operator. Furthermore,

$$\|x\|_{|F_u|_k} = \left\| E^{(k)} \circ T(x) \right\|_{\ell_k},$$

i.e., it preserves the norm. So the proof is completed. ☐

In the following theorems, for the simplicity of presentation we take

$$\xi_{vr} = \left(\frac{f_{v+1}}{f_v} + \left(f_{v+1}^2 - f_v^2\right) \sum_{j=r}^{v-1} \frac{1}{f_j f_{j+1}} \right)$$

and define

$$
\begin{aligned}
D_1 &= \left\{ \epsilon \in \omega : \sum_{v=r+1}^{\infty} \xi_{vr}\epsilon_v \ \text{ exists for all } \ r \right\}, \\
D_2 &= \left\{ \epsilon \in \omega : \sup_m \left(\frac{1}{u_m} \left| \epsilon_m \frac{f_{m+1}}{f_m} \right|^{k^*} + \sum_{r=0}^{m-1} \frac{1}{u_r} \left| \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr}\epsilon_v \right|^{k^*} \right) < \infty \right\}, \\
D_3 &= \left\{ \epsilon \in \omega : \sup_{m,r} \left(\left| \epsilon_m \frac{f_{m+1}}{f_m} \right| + \left| \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr}\epsilon_v \right| \right) < \infty \right\}, \\
D_4 &= \left\{ \epsilon \in \omega : \sum_{r=0}^{\infty} \frac{1}{u_r} \left(\sum_{v=r+1}^{\infty} |\xi_{vr}\epsilon_v| + \left| \epsilon_r \frac{f_{r+1}}{f_r} \right| \right)^{k^*} < \infty \right\}, \\
D_5 &= \left\{ \epsilon \in \omega : \sup_r \left(\sum_{v=r+1}^{\infty} |\xi_{vr}\epsilon_v| + \left| \epsilon_r \frac{f_{r+1}}{f_r} \right| \right) < \infty \right\}.
\end{aligned}
$$

Theorem 3.5. *Let $1 < k < \infty$ and $u = (u_n)$ be a sequence of positive numbers. Then,*

(i) $\{|F_u|\}^\alpha = D_5$, $\{|F_u|_k\}^\alpha = D_4$;

(ii) $\{|F_u|\}^\beta = D_1 \cap D_3$, $\{|F_u|_k\}^\beta = D_1 \cap D_2$;

(iii) $\{|F_u|\}^\gamma = D_3$, $\{|F_u|_k\}^\gamma = D_2$.

Proof. (ii) Let's recall that $\epsilon \in \{|F_u|_k\}^\beta$ if and only if $\epsilon x = (\epsilon_n x_n) \in c_s$ for all $x \in |F_u|_k$. By (1.3) and (1.4), it can be seen immediately that

$$
\begin{aligned}
\sum_{v=0}^{m} \epsilon_v x_v =& \epsilon_0 x_0 + \sum_{v=1}^{m} \epsilon_v \left(\frac{f_{v+1}}{f_v} y_v + \left(f_{v+1}^2 - f_v^2 \right) \sum_{j=0}^{v-1} \frac{y_j}{f_j f_{j+1}} \right) \\
=& \sum_{r=0}^{m} u_r^{-1/k^*} \sum_{v=r}^{m} \epsilon_v \frac{f_{v+1}}{f_v} z_r + \sum_{r=0}^{m-1} \left(\sum_{v=r+1}^{m} \epsilon_v (f_{v+1}^2 - f_v^2) \sum_{j=r}^{v-1} \frac{1}{f_j f_{j+1}} \right) u_r^{-1/k^*} z_r \\
=& u_m^{-1/k^*} \epsilon_m \frac{f_{m+1}}{f_m} z_m + \sum_{r=0}^{m-1} u_r^{-1/k^*} \left(\epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \epsilon_v \xi_{vr} \right) z_r \\
=& \sum_{r=0}^{m} h_{mr} z_r \quad (y = T(x), z = E^{(k)}(y))
\end{aligned}
$$

where $H = (h_{mr})$ is defined by

$$
h_{mr} = \begin{cases} u_r^{-1/k^*} \left(\epsilon_r \dfrac{f_{r+1}}{f_r} + \sum\limits_{v=r+1}^{m} \epsilon_v \xi_{vr} \right), & 0 \le r \le m-1, \\ u_m^{-1/k^*} \dfrac{f_{m+1}}{f_m} \epsilon_m, & r = m \\ 0, & r > m. \end{cases}
$$

Therefore, $\epsilon \in \{|F_u|_k\}^\beta$ if and only if $H \in (\ell_k, c)$. Applying Lemma 1.4 to the matrix H, we get $\{|F_u|_k\}^\beta = D_1 \cap D_2$, which completes the proof.

The proofs of other parts can similarly be proved, so we omit. □

Theorem 3.6. *Let $1 \leq k < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for each $n, v \in \mathbb{N}$ and define the matrix $B^{(n)} = \left(b^{(n)}_{mr}\right)$ by*

$$b^{(n)}_{mr} = \begin{cases} a_{nr}\dfrac{f_{r+1}}{f_r} + \sum\limits_{v=r+1}^{m} a_{nv}\xi_{vr}, & 0 \leq r \leq m-1, \\ \dfrac{f_{m+1}}{f_m} a_{nm}, & r = m, \\ 0, & r > m. \end{cases}$$

Further, let $\bar{B} = (\bar{b}_{nv})$ be a matrix given by $\bar{b}_{nv} = \lim\limits_m b^{(n)}_{mv}$ and $\widetilde{B} = E^{(k)} \circ T \circ \bar{B}$. Then, $A \in (|F_u|, |F_u|_k)$ if and only if

$$\sum_{v=r+1}^{\infty} \xi_{vr} a_{nv} \ \text{ exists for all } \ r \tag{3.1}$$

$$\sup_{m,r} \left\{ \left| a_{nm} \frac{f_{m+1}}{f_m} \right| + \left| a_{nr}\frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr} a_{nv} \right| \right\} < \infty, \tag{3.2}$$

$$\sup_r \sum_{n=0}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right|^k < \infty, \tag{3.3}$$

If $A \in (|F_u|, |F_u|_k)$, then A is a bounded linear operator,

$$\|A\|_{\left(|F_u|, |F_u|_k\right)} = \left\| \widetilde{B} \right\|_{(l, l_k)}$$

and

$$\|A\|_{\chi} = \lim_{v \to \infty} \left\{ \sup_r \sum_{n=v+1}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right|^k \right\}^{\frac{1}{k}}.$$

Proof. $A \in (|F_u|, |F_u|_k)$ if and only if $(a_{nv})_{v=0}^{\infty} \in \{|F_u|\}^{\beta}$ and $A(x) \in |F_u|_k$ for all $x \in |F_u|$. Now, it can be easily seen from Theorem 3.5, $(a_{nv})_{v=0}^{\infty} \in \{|F_u|\}^{\beta}$ if and only if (3.1) and (3.2) hold. On the other hand, if a matrix $R = (r_{nv}) \in (\ell, c)$, then the series $R_n(x) = \sum_{v=0}^{\infty} r_{nv} x_v$ converges uniformly in n, because, the remaining term of the series tends to zero uniformly in n, since

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \sup_v |r_{nv}| \sum_{v=m}^{\infty} |x_v| \to 0, \quad m \to \infty.$$

So we obtain

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.4}$$

Using (1.3), (1.4) and (3.4) it can be written that

$$A_n(x) = \lim_m \sum_{k=0}^{m} a_{nk} x_k = \lim_m \sum_{r=0}^{m} b^{(n)}_{mr} z_r = \sum_{r=0}^{\infty} \bar{b}_{nr} z_r.$$

Besides, according to Theorem 3.4, since $|F_u|_k \cong \ell_k$ for $1 \leq k < \infty$, it follows that $A(x) \in |F_u|_k$ for all $x \in |F_u|$ if and only if $\bar{B} \in (\ell, |F_u|_k)$, or equivalently, since $|F_u|_k = (\ell_k)_{E^{(k)} \circ T}$, $\widetilde{B} \in (\ell, \ell_k)$. Also, it is clear that the terms of matrix $\widetilde{B}$ can be expressed as

$$\begin{aligned}
\hat{b}_{nr} &= \sum_{v=0}^{n} t_{nv} \bar{b}_{vr} = \frac{f_n}{f_{n+1}} \bar{b}_{nr} + \sum_{v=0}^{n-1} \frac{f_n^2 - f_{n+1}^2}{f_n f_{n+1}} \bar{b}_{vr}, \\
\widetilde{b}_{nr} &= u_r^{1/k^*} \left(\hat{b}_{nr} - \hat{b}_{n-1,r} \right), \quad n \geq 1 \text{ and } \widetilde{b}_{0r} = \bar{b}_{0r}.
\end{aligned}$$

Hence, applying Lemma 1.3 to the matrix $\widetilde{B}$, we have (3.3), which completes the first part of the proof.

Also, if $A \in (|F_u|, |F_u|_k)$, then, since the spaces $|F_u|_k$ and $|F_u|$ are BK-spaces, it is a bounded operator. In order to determine the operator norm of A, consider the isomorphisms $T : |F_u|_k \to (\ell_k)_{E^{(k)}}, E^{(k)} : (\ell_k)_{E^{(k)}} \to \ell_k$ defined as in Theorem 3.4. Then, it is easy to see that $A = \widetilde{T} \circ \widetilde{E}^{(k)} \circ \widetilde{B} \circ E^{(1)} \circ T$ and so,

$$\begin{aligned}
\|A\|_{(|F_u|, |F_u|_k)} &= \sup_{x \neq 0} \frac{\|A(x)\|_{|F_u|_k}}{\|x\|_{|F_u|}} = \sup_{x \neq 0} \frac{\left\| \widetilde{T} \circ \widetilde{E}^{(k)} \circ \widetilde{B} \circ E^{(1)} \circ T(x) \right\|_{|F_u|_k}}{\|x\|_{|F_u|}} \\
&= \sup_{z \neq 0} \frac{\left\| \widetilde{B}(z) \right\|_{\ell_k}}{\|z\|_{\ell}} = \left\| \widetilde{B} \right\|_{(\ell, \ell_k)} (z = E^{(1)} \circ T(x)).
\end{aligned}$$

Finally, assume that Q is a unique ball in $|F_u|$. Since $E^{(k)} \circ T \circ AQ = \widetilde{B} \circ E^{(1)} \circ TQ$, we get that

$$\begin{aligned}
\|A\|_{\chi} = \chi(AQ) &= \chi \left(E^{(k)} \circ T \circ AQ \right) = \chi \left(\widetilde{B} \circ E^{(1)} \circ TQ \right) \\
&= \lim_{v \to \infty} \left(\sup_{z \in E^{(1)}(T(Q))} \left\| (I - P_v) \left(\widetilde{B}(z) \right) \right\| \right) \\
&= \lim_{v \to \infty} \left\{ \sup_r \left(\sum_{n=v+1}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right| \right)^k \right\}^{\frac{1}{k}}.
\end{aligned}$$

This completes the proof. □

By Theorem 3.6 and Lemma 2.2, the compact operators in this class are characterized as follows.

Corollary 3.1. *Under the hypothesis of Theorem* 3.6

$$A \in (|F_u|, |F_u|_k) \text{ is compact if and only if } \lim_{v \to \infty} \left\{ \sup_r \left(\sum_{n=v+1}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right| \right)^k \right\}^{\frac{1}{k}} = 0.$$

Theorem 3.7. *Let $1 < k < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$ and $B^{(n)} = \left(b_{mv}^{(n)} \right)$ be as in Theorem* 3.6*. Besides, define $\bar{H} = (h_{nv})$*

by $\bar{h}_{nv} = \lim_m u_v^{-1/k^*} b_{mv}^{(n)}$ *and* $\widetilde{H} = E^{(1)} \circ T \circ \bar{H}$*. Then,* $A \in (|F_u|_k, |F_u|)$ *if and only if*

$$\sum_{v=r+1}^{\infty} \xi_{vr} a_{nv} \text{ exist for all } r, \tag{3.5}$$

$$\sup_m \left\{ \frac{1}{u_m} \left| a_{nm} \frac{f_{m+1}}{f_m} \right|^{k^*} + \sum_{r=0}^{m-1} \frac{1}{u_r} \left| a_{nr} \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr} a_{nv} \right|^{k^*} \right\} < \infty, \tag{3.6}$$

$$\sum_{r=0}^{\infty} \left(\sum_{n=0}^{\infty} \left| \widetilde{h}_{nr} \right| \right)^{k^*} < \infty. \tag{3.7}$$

Moreover, if $A \in (|F_u|_k, |F_u|)$*, then* A *is a bounded linear operator,*

$$\|A\|_{\left(|F_u|_k, |F_u|\right)} = \left\| \widetilde{H} \right\|_{(\ell_k, \ell)}$$

and

$$\|A\|_{\chi} = \frac{1}{\xi} \lim_{v \to \infty} \left\{ \sum_{r=0}^{\infty} \left(\sum_{n=v+1}^{\infty} \left| \widetilde{h}_{nr} \right| \right)^{k^*} \right\}^{\frac{1}{k^*}},$$

where $1 \leq \xi \leq 4$.

Proof. $A \in (|F_u|_k, |F_u|)$ if and only if $A_n = (a_{nv})_{v=0}^{\infty} \in \{|F_u|_k\}^{\beta}$ and $A(x) \in |F_u|$ where $x \in |F_u|_k$. By Theorem 3.5, it can be easily seen that $A_n \in \{|F_u|_k\}^{\beta}$ if and only if (3.5) and (3.6) hold. Also, if any matrix $R = (r_{nv}) \in (\ell_k, c)$, then the series $R_n(x) = \sum_{v=0}^{\infty} r_{nv} x_v$ converges uniformly in n. Because, the remaining term of the series tends to zero uniformly in n, since

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left(\sum_{v=m}^{\infty} |r_{nv}|^{k^*} \right)^{\frac{1}{k^*}} \left(\sum_{v=m}^{\infty} |x_v|^k \right)^{\frac{1}{k}} \to 0, \quad m \to \infty$$

and so, it can be written that

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.8}$$

Then, using (3.8), with a few calculations, we get

$$A_n(x) = \lim_m \sum_{k=0}^{m} a_{nk} x_k = \lim_m \sum_{r=0}^{m} u_r^{-1/k^*} b_{mr}^{(n)} z_r = \sum_{r=0}^{\infty} \bar{h}_{nr} z_r.$$

Since $|F_u|_k \cong \ell_k$ for $1 \leq k < \infty$, by the Theorem 3.4, then, $A(x) \in |F_u|$ for every $x \in |F_u|_k$ if and only if $\bar{H}(z) \in |F_u|$, i.e., $\widetilde{H}(z) = E^{(1)} \circ T \circ \bar{H}(z) \in \ell$ for every $z \in \ell_k$, where $z = E^{(k)} \circ T(x)$. This means that $\widetilde{H} \in (\ell_k, \ell)$. Thus applying Lemma 1.2 to the matrix $\widetilde{H}$, we get (3.7). This completes the proof of first part.

Since $|F_u|_k$ is BK-spaces for every $k \geq 1$, A is a bounded operator by Theorem 4.2.8 of [37].

Additionally, as Theorem 3.4, it can be written that $A = \widetilde{T} \circ \widetilde{E}^{(1)} \circ \widetilde{H} \circ E^{(k)} \circ T$ and so,

$$\begin{aligned}\|A\|_{\left(|F_u|_k,|F_u|\right)} &= \sup_{x\neq 0} \frac{\|A(x)\|_{|F_u|}}{\|x\|_{|F_u|_k}} = \sup_{x\neq 0} \frac{\left\|\widetilde{H} \circ E^{(k)} \circ T(x)\right\|_{\ell}}{\|E^{(k)} \circ T(x)\|_{\ell_k}} \\ &= \sup_{z\neq 0} \frac{\left\|\widetilde{H}(z)\right\|_{\ell}}{\|z\|_{\ell_k}} = \left\|\widetilde{H}\right\|_{(\ell_k,\ell)}.\end{aligned}$$

Finally, let $Q = S_{|F_u|}$. Since $E^{(1)} \circ T \circ AQ = \widetilde{H} \circ E^{(k)} \circ TQ$, it follows by Lemma 2.1, Lemma 2.3 and Lemma 1.2 that

$$\begin{aligned}\|A\|_{\chi} &= \chi\left(AQ\right) = \chi\left(E^{(1)} \circ T \circ AQ\right) = \chi\left(\widetilde{H} \circ E^{(k)} \circ TQ\right) \\ &= \lim_{v\to\infty}\left(\sup_{z\in E^{(k)}(T(Q))} \left\|(I-P_v)(\widetilde{H}(z))\right\|_{\ell_k}\right) \\ &= \frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^{k^*}\right\}^{\frac{1}{k^*}},\end{aligned}$$

which completes the proof. □

Also, the compact operators can immediately be characterized by Lemma 2.2 and Theorem 3.7 as follows.

Corollary 3.2. *Under the conditions of Theorem* 3.7

$$A \in \mathcal{C}\left(|F_u|_k, |F_u|\right) \Leftrightarrow \frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^{k^*}\right\}^{\frac{1}{k^*}} = 0.$$

References

[1] B. Altay and F. Başar, *The fine spectrum and the matrix domain of the difference operator* Δ *on the sequence space* ℓ_p $(0 < p < 1)$, Commun. Math. Anal. **2**(2) (2007), 1–11.

[2] B. Altay and F. Başar, *Some paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math. **30**(5) (2006), 591–608.

[3] B. Altay, F. Başar and M. Mursaleen, *On the Euler sequence spaces which include the spaces* l_p *and* l_∞ *I*, Inform. Sci. **176**(10) (2005), 1450–1462.

[4] F. Başar and N. L. Braha, *Euler-Cesàro difference spaces of bounded, convergent and null sequences*, Tamkang J. Math. **47**(4) (2016), 405–420.

[5] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, Istanbul, 2012.

[6] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix map- pings*, (English, Ukrainian summary) Ukrain. Mat. Zh. **55**(1) (2003), 108–118, reprinted in Ukrainian Math. J. **55**(1) (2003), 136–147.

[7] M. Başarır, F. Başar and E. E. Kara, *On the spaces of Fibonacci difference absolutely p-summable, null and convergent sequences*, Sarajevo J. Math. **12**(25) (2016), 167–182.

[8] J. Boos and P. Cass, *Classical and Modern Methods in Summability*, Oxford University Press, New York, 2000.

[9] H. Bor, *On summability factors of infinite series*, Tamkang J. Math. **16**(1) (1985), 13–20.

[10] H. Bor and B. Thorpe, *On some absolute summability methods*, Analysis **7**(2) (1987), 145–152.

[11] T. M. FLett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc. **7** (1957), 113–141.

[12] G. C. Hazar and M. A. Sarıgöl, *Compact and matrix operators on the space* $|C, -1|_k$, J. Comput. Anal. Appl. **25** (2018), 1014–1024.

[13] G. C. Hazar and F. Gökçe, *Characterizations of matrix transformatios on the series spaces derived by absolute factorable summability*, Developments in Science and Engineering (2016), 411–426.

[14] A. M. Jarrah and E. Malkowsky, *Ordinary absolute and strong summability and matrix transformations*, Filomat **17** (2003), 59–78.

[15] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl. **2013**(1) (38) (2013), 16 pages.

[16] E. E. Kara and M. Ilkhan, *On some Banach sequence spaces derived by a new band matrix*, Br. J. Math. Comput. Sci. **9**(2) (2015), 141–159.

[17] E. E. Kara and M. Ilkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear Multilinear Algebra **64**(11) (2016), 2208–2223.

[18] V. Karakaya, A. K. Noman and H. Polat, *On paranormed λ-sequence spaces of non-absolute type*, Math. Comput. Modelling **54**(5) (2011), 1473–1480.

[19] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, 2011.

[20] I. J. Maddox, *Elements of Functinal Analysis*, Cambridge University Press, London, New York, 1970.

[21] E. Malkowsky and V. Rakocević, *On matrix domains of triangles*, Appl. Math. Comput. **189**(2) (2007), 1146–1163.

[22] E. Malkowsky and V. Rakocević, *An introduction into the theory of sequence space and measures of noncompactness*, Zb. Rad. (Beogr.) **9**(17) (2000), 143–234.

[23] M. R. Mehdi, *Summability factors for generalized absolute summability I*, Proc. Lond. Math. Soc. **3**(1) (1960), 180–200.

[24] R. N. Mohapatra and M. A. Sarıgöl, *On matrix operators on the series spaces* $\left|\bar{N}_p^\theta\right|_k$, Ukrain Mat. Zh. **69**(11) (2017), 1524–1533.

[25] M. Mursaleen, F. Başar and B. Altay, *On the Euler sequence spaces which include the spaces l_p and l_∞ II*, Nonlinear Anal. **65**(3) (2006), 707–717.

[26] C. Orhan and M. A. Sarıgöl, *On absolute weighted mean summability*, Rocky Mountain J. Math. **23**(3) (1993), 1091–1097.

[27] V. Rakocević, *Measures of noncompactness and some applications*, Filomat **12**(2) (1998), 87–120.

[28] M. A. Sarıgöl, *Spaces of series summable by absolute Cesàro and matrix operators*, Communications in Mathematics and Applications **7**(1) (2016), 11–22.

[29] M. A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci. **42**(3) (2015), 28–35.

[30] M. A. Sarıgöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar. **48**(3) (2011), 331–341.

[31] M. A. Sarıgöl, *Necessary and sufficient conditions for the equivalence of the summability methods* $\left|\bar{N}, p_n\right|_k$ *and* $|C, 1|_k$, Indian J. Pure Appl. Math. **22**(6) (1991), 483–489.

[32] M. A. Sarıgöl, *On the local properties of factored Fourier series*, Appl. Math. Comput. **216**(11) (2010), 3386–3390.

[33] M. A. Sarıgöl, *On two absolute Riesz summability factors of infinite series*, Proc. Amer. Math. Soc. **118**(2) (1993), 485–488.

[34] M. A. Sarıgöl, *On absolute weighted mean summability methods*, Proc. Amer. Math. Soc. **115**(1) (1992), 157–160.

[35] M. Stieglitz and H. Tietz, *Matrix transformationen von Folgenraumen. Eine Ergebnisübersicht*, Math. Z. **154**(1) (1977), 1–16.
[36] W. T. Sulaiman, *On summability factors of infinite series*, Proc. Amer. Math. Soc. **115** (1992), 313–317.
[37] A. Wilansky, *Summability Through Functional Analysis*, Mathematics Studies **85**, North Holland, Amsterdam, 1984.

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