

SOME RESULTS ON SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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ABSTRACT. An edge-magic total labeling of a graph G is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$, where there exists a constant k such that $f(u) + f(uv) + f(v) = k$, for every edge $uv \in E(G)$. Moreover, if the vertices are labeled with the numbers $1, 2, \dots, |V(G)|$ such a labeling is called a super edge-magic total labeling. The super edge-magic deficiency of a graph G , denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling or is defined to be ∞ if there exists no such n .

In this paper we study the super edge-magic deficiencies of two types of snake graph and a prism graph D_n for $n \equiv 0 \pmod{4}$. We also give an exact value of super edge-magic deficiency for a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of the ladder, for n odd, and an exact value of super edge-magic deficiency for a square of a path P_n for $n \geq 3$.

1. INTRODUCTION

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. Let $|V(G)| = p$ and $|E(G)| = q$.

An *edge-magic total labeling* of a graph G is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p + q\}$, where there exists a constant k such that

$$f(u) + f(uv) + f(v) = k,$$

for every edge $uv \in E(G)$. The constant k is called a *magic constant*. An edge-magic total labeling f is called *super edge-magic total* if the vertices are labeled with the

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smallest possible labels, i.e., with the numbers $1, 2, \dots, p$. A graph that admits a (super) edge-magic total labeling is called (*super*) *edge-magic total*.

The concept of edge-magic total labeling was given by Kotzig and Rosa [8]. Super edge-magic total labelings were originally defined by Enomoto et al. in [3]. However Acharya and Hegde had introduced in [1] the concept of strongly indexable graphs that is equivalent to the one of super edge-magic total labeling.

Kotzig and Rosa [8] proved that for any graph G there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n . This fact leads to the concept of *edge-magic deficiency* of a graph G , which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic total and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic total}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge-magic deficiency of a graph G with n vertices,

$$\mu(G) \leq F_{n+2} - 2 - n - \frac{n(n-1)}{2},$$

where F_n is the n th Fibonacci number.

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno, Ichishima and Muntaner-Batle [5] defined a similar concept for the super edge-magic total labelings. The *super edge-magic deficiency* of a graph G , denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic total labeling, or is defined to be ∞ if there exists no such n . More precisely, if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic total graph}\},$$

then

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset, \\ \infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is easy to see that for every graph G it holds

$$\mu(G) \leq \mu_s(G).$$

In [5, 7] Figueroa-Centeno, Ichishima and Muntaner-Batle found the exact values of the super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In particular, they proved that

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

In [10] Ngurah, Simanjuntak and Baskoro gave some upper bounds for the super edge-magic deficiency of fans, double fans and wheels. In [6] Figueroa-Centeno,

Ichishima and Muntaner-Batle proved

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd or } m = 3 \text{ and } n \not\equiv 0 \pmod{3}, \\ 0, & \text{otherwise.} \end{cases}$$

In the same paper, they showed that

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } m \text{ is a multiple of } n + 1 \text{ or } n \text{ is a multiple of } m + 1, \\ 1, & \text{otherwise.} \end{cases}$$

They also conjectured that every forest with two components has super edge-magic deficiency less than or equal to 1. Baig, Baskoro and Semaničová-Feňovčíková [2] have determined the super edge magic deficiency of a star forest. Santhosh and Singh [11] studied the corona product of K_2 and C_n and they showed that $\mu_s(K_2 \odot C_n) \leq \frac{n-3}{2}$, for n odd at least 3.

In this paper we study the super edge-magic deficiencies for several classes of graphs. We give upper bounds for the super edge-magic deficiencies of two types of snake graph and for prism graph D_n for $n \equiv 0 \pmod{4}$. We also give an exact value of super edge-magic deficiency for a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of the ladder, for n odd, and an exact value of super edge-magic deficiency for a square of a path P_n for every positive integer n , $n \geq 3$.

To prove the results presented in this paper, we frequently use the following lemma.

Lemma 1.1. [4] *A graph G with p vertices and q edges is super edge-magic total if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the set $\{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic total labeling of G .*

2. UPPER BOUNDS

In graph theory a *block graph* is a graph in which every bi-connected component (block) is a clique. Block graphs are sometimes erroneously said to be "Husimi trees", but that name more properly refers to cactus graphs, graphs in which every nontrivial bi-connected component is a cycle. In graph theory block graphs may be described as the intersection graphs of the blocks of arbitrary undirected graphs.

Let G be a graph and u and v are two fixed vertices in G . The G^n -snake is a graph obtained from n copies of G by identifying the vertex corresponding to the vertex v in the i th copy of G with the vertex corresponding to the vertex u in the $(i + 1)$ th copy of G , for $i = 1, 2, \dots, n - 1$. The wheel W_k , $k \geq 3$ is a graph obtained by joining every vertex of a cycle C_k with a new vertex.

In the following theorem we will deal with the super edge-magic deficiency of W_4^n -snake. Let us denote the vertex set and the edge set of W_4^n -snake such that

$$\begin{aligned} V(W_4^n\text{-snake}) = & \{x_i : i = 1, 2, \dots, 2n\} \cup \{y_i : i = 1, 2, \dots, n\} \\ & \cup \{z_i : i = 1, 2, \dots, n + 1\}, \end{aligned}$$

$$\begin{aligned}
E(W_4^n\text{-snake}) = & \{x_i x_{n+i} : i = 1, 2, \dots, n\} \cup \{z_i z_{i+1} : i = 1, 2, \dots, n\} \\
& \cup \{x_i z_i, x_{n+i} z_{i+1} : i = 1, 2, \dots, n\} \\
& \cup \{y_i x_i, y_i x_{n+i} : i = 1, 2, \dots, n\} \\
& \cup \{y_i z_i, y_i z_{i+1} : i = 1, 2, \dots, n\}.
\end{aligned}$$

Theorem 2.1. *The graph W_4^n -snake has super edge-magic deficiency at most 1.*

Proof. Let us denote the vertices and edges of $G \cong W_4^n \cup K_1$ such that $V(G) = V(W_4^n\text{-snake}) \cup \{v\}$ and $E(G) = E(W_4^n\text{-snake})$. The graph G has $4n + 2$ vertices and $8n$ edges.

We define the vertex labeling f of G in the following way

$$\begin{aligned}
f(x_i) &= 4i - 3, & \text{if } i = 1, 2, \dots, n, \\
f(x_{n+i}) &= 4i - 1, & \text{if } i = 1, 2, \dots, n, \\
f(y_i) &= 4i, & \text{if } i = 1, 2, \dots, n, \\
f(z_i) &= 4i - 2, & \text{if } i = 1, 2, \dots, n + 1, \\
f(v) &= 4n + 1.
\end{aligned}$$

It is easy to see that the vertices of G are labeled with the numbers $1, 2, 3, \dots, 4n + 2$ as the sets of vertex labels are

$$\begin{aligned}
\{f(x_i) : i = 1, 2, 3, \dots, n\} &= \{1, 5, 9, \dots, 4n - 3\}, \\
\{f(z_i) : i = 1, 2, 3, \dots, n, n + 1\} &= \{2, 6, 10, \dots, 4n - 2, 4n + 2\}, \\
\{f(x_i) : i = n + 1, n + 2, n + 3, \dots, 2n\} &= \{3, 7, 11, \dots, 4n - 1\}, \\
\{f(y_i) : i = 1, 2, 3, \dots, n\} &= \{4, 8, 12, \dots, 4n\}, \\
f(v) &= 4n + 1.
\end{aligned}$$

Next we will count the edge sums of the edges in the blocks. For $i = 1, 2, \dots, n$ it holds

$$\begin{aligned}
f(x_i) + f(z_i) &= (4i - 3) + (4i - 2) = 8i - 5, \\
f(x_i) + f(x_{i+n}) &= (4i - 3) + (4i - 1) = 8i - 4, \\
f(x_i) + f(y_i) &= (4i - 3) + 4i = 8i - 3, \\
f(y_i) + f(z_i) &= 4i + (4i - 2) = 8i - 2, \\
f(x_{i+n}) + f(y_i) &= (4i - 1) + 4i = 8i - 1, \\
f(z_i) + f(z_{i+1}) &= (4i - 2) + (4(i + 1) - 2) = 8i, \\
f(x_{i+n}) + f(z_{i+1}) &= (4i - 1) + (4(i + 1) - 2) = 8i + 1, \\
f(y_i) + f(z_{i+1}) &= 4i + (4(i + 1) - 2) = 8i + 2.
\end{aligned}$$

It means that the edge sums are consecutive integers $3, 4, \dots, 8n + 2$. According to Lemma 1.1 the labeling f can be extended to a super edge-magic total labeling of G with magic constant $12n + 5$. \square

A graph is called a *cactus graph* if every block is either a cycle or a complete graph K_2 . Next we will deal with a special type of a cactus graph called an alternate quadrilateral snake. An *alternate quadrilateral snake* $A(C_4^n)$ is obtained from a path $x_1x_2\dots x_n$ by joining the vertices x_i, x_{i+1} , for every odd i , to new vertices y_i, y_{i+1} , respectively and then joining y_i and y_{i+1} . That is every alternate edge of the path is replaced by a cycle C_4 . More precisely, the vertex set and the edge set of $A(C_4^n)$ are the following

$$V(A(C_4^n)) = \{x_i, y_i : i = 1, 2, \dots, n\}$$

and

$$E(A(C_4^n)) = \{x_i x_{i+1} : i = 1, 2, \dots, n-1\} \cup \{x_i y_i : i = 1, 2, \dots, n\} \\ \cup \{y_i y_{i+1} : i = 1, 3, \dots, n-1\}.$$

Theorem 2.2. *For every even positive integer $n, n \geq 4$, for super edge-magic deficiency of the alternate quadrilateral snake $A(C_4^n)$ we have*

$$\mu_s(A(C_4^n)) \leq \frac{n}{2}.$$

Proof. Let n be an even positive integer. Let us denote the vertex set and the edge set of $G \cong A(C_4^n) \cup \frac{n}{2}K_1$ as follows $V(G) = V(A(C_4^n)) \cup \{v_i : i = 1, 2, \dots, \frac{n}{2}\}$ and $E(G) = E(A(C_4^n))$.

We define the vertex labeling of the graph G in the following way

$$f(x_i) = \begin{cases} i, & \text{if } i = 1, 3, \dots, n-1, \\ n + \frac{3i}{2}, & \text{if } i = 2, 4, \dots, n, \end{cases}$$

$$f(y_i) = \begin{cases} n + \frac{3i-1}{2}, & \text{if } i = 1, 3, \dots, n-1, \\ i, & \text{if } i = 2, 4, \dots, n. \end{cases}$$

The remaining $\frac{n}{2}$ numbers $n+2, n+5, \dots, \frac{5n}{2}-1$ are used to label the isolated vertices $v_1, v_2, \dots, v_{\frac{n}{2}}$ of the graph G arbitrary.

It is easy to see that f is a bijection from the vertex set of G onto the set of integers $1, 2, \dots, \frac{5n}{2}$.

For the edge sums we have the following. The edge sum of the edges $x_i y_i, y_i y_{i+1}, x_i x_{i+1}$ and $y_{i+1} x_{i+1}$, for $i = 1, 3, \dots, n-1$, are

$$f(x_i) + f(y_i) = i + \left(n + \frac{3i-1}{2}\right) = n + \frac{5i-1}{2},$$

$$f(y_i) + f(y_{i+1}) = \left(n + \frac{3i-1}{2}\right) + (i+1) = n + \frac{5i-1}{2} + 1,$$

$$f(x_i) + f(x_{i+1}) = i + \left(n + \frac{3(i+1)}{2}\right) = n + \frac{5i-1}{2} + 2,$$

$$f(x_{i+1}) + f(y_{i+1}) = \left(n + \frac{3(i+1)}{2}\right) + (i+1) = n + \frac{5i-1}{2} + 3.$$

The edge sum of the edge $x_{i+1}x_{i+2}$, for $i = 1, 3, \dots, n - 3$, is

$$f(x_{i+1}) + f(x_{i+2}) = \left(n + \frac{3(i+1)}{2} \right) + (i+2) = n + \frac{5i-1}{2} + 4.$$

Moreover, for $i = 1, 3, \dots, n - 3$, we have

$$f(x_{i+2}) + f(y_{i+2}) = (i+2) + \left(n + \frac{3(i+2)-1}{2} \right) = n + \frac{5i-1}{2} + 5.$$

Hence the edge sums are consecutive integers $n + 2, n + 3, \dots, \frac{7n}{2}$. Thus, according to Lemma 1.1, the labeling f can be extended to the super edge-magic total labeling of G with the magic constant $6n + 1$. \square

The graph $A(C_4^2)$ is isomorphic to the cycle C_4 . Figueroa-Centeno, Ichishima and Muntaner-Batle [5] proved that $\mu_s(C_4) = 1$.

A *prism graph* D_n , sometimes also called a circular ladder graph, is a graph corresponding to the skeleton of an n -prism. Prism graphs are both planar and polyhedral. An n -prism graph consist of $2n$ vertices and $3n$ edges, which is equivalent to generalized Petersen graph $P_{n,1}$. The n -prism is isomorphic to circulant graph $Ci_{2n}(2, n)$ for odd n , and can be showed by rotating the inner cycle by 180° , and its radius is equal to that of the outer cycle in the top embedding above. In addition, for odd n , D_n is isomorphic to $Ci_{2n}(4, n), Ci_{2n}(6, n), \dots, Ci_{2n}(n-1, n)$. The prism D_n is isomorphic to the Cartesian product $C_n \times K_2$, where C_n is the cycle on n vertices and K_2 is the complete graph of order 2. The prism graph D_n is equivalent to the Cayley graph of the dihedral group D_n , with respect to the generating set $\{x, x^{-1}, y\}$.

We denote the vertices and edges of D_n such that

$$V(D_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$$

and

$$E(D_n) = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \dots, n-1\} \cup \{x_1 x_n, y_1 y_n\} \cup \{x_i y_i : i = 1, 2, \dots, n\}.$$

The cardinality of the vertex set and the edge set of D_n is $2n$ and $3n$, respectively.

In [4] Figueroa-Centeno, Ichishima and Muntaner-Batle proved that for n odd the graph D_n is super edge-magic total. Ngurah and Baskoro [9] showed that for n even the prism D_n is not edge-magic total. In the following theorem we are dealing with the case when n is divisible by 4.

Theorem 2.3. *Let n be a positive integer, $n \equiv 0 \pmod{4}$. The super edge-magic deficiency of D_n is*

$$\mu_s(D_n) \leq \frac{3n}{2} - 1.$$

Proof. Let n be a positive integer, $n \equiv 0 \pmod{4}$. Let us denote the isolated vertices of $G \cong D_n \cup (\frac{3n}{2} - 1)K_1$ by the symbols $v_1, v_2, \dots, v_{\frac{3n}{2}-1}$.

We define the vertex labeling f of G in the following way.

$$f(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i = 1, 3, \dots, n-1, \\ \frac{9n}{4} - 1 + \frac{i}{2}, & \text{if } i = 2, 4, \dots, \frac{n}{2}, \\ \frac{5n}{4} + \frac{i}{2}, & \text{if } i = \frac{n}{2} + 2, \frac{n}{2} + 4, \dots, n, \end{cases}$$

$$f(y_i) = \begin{cases} \frac{11n}{4}, & \text{if } i = 1, \\ n + \frac{i}{2}, & \text{if } i = 2, 4, \dots, n, \\ \frac{13n}{4} + \frac{i-3}{2} & \text{if } i = 3, 5, \dots, \frac{n}{2} + 1, \\ \frac{9n}{4} + \frac{i-1}{2} & \text{if } i = \frac{n}{2} + 3, \frac{n}{2} + 5, \dots, n-1, \end{cases}$$

and the vertices v_i , $i = 1, 2, \dots, \frac{3n}{2} - 1$ are labeled arbitrary with $\frac{3n}{2} - 1$ unused numbers from the set $\{1, 2, \dots, \frac{7n}{2} - 1\}$. It is not difficult to check that the vertices v_i , $i = 1, 2, \dots, \frac{3n}{2} - 1$ are labeled with the numbers $\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, \frac{3n}{2}, \frac{7n}{4} + 1, \frac{7n}{4} + 2, \dots, \frac{9n}{4} - 1, \frac{5n}{2}, \frac{11n}{4} + 1, \frac{11n}{4} + 2, \dots, \frac{13n}{4} - 1$.

Next we prove that the edge sums are consecutive integers. Indeed, we have

$$f(x_1) + f(x_n) = \frac{1+1}{2} + \left(\frac{5n}{4} + \frac{n}{2}\right) = \frac{7n}{4} + 1,$$

$$f\left(x_{\frac{n}{2}+1}\right) + f\left(x_{\frac{n}{2}+2}\right) = \frac{\left(\frac{n}{2}+1\right)+1}{2} + \left(\frac{5n}{4} + \frac{\frac{n}{2}+2}{2}\right) = \frac{7n}{4} + 2,$$

$$f\left(x_{\frac{n}{2}+2}\right) + f\left(x_{\frac{n}{2}+3}\right) = \left(\frac{5n}{4} + \frac{\frac{n}{2}+2}{2}\right) + \frac{\left(\frac{n}{2}+1\right)+3}{2} = \frac{7n}{4} + 3,$$

$$\vdots$$

$$f(x_{n-1}) + f(x_n) = \frac{(n-1)+1}{2} + \left(\frac{5n}{4} + \frac{n}{2}\right) = \frac{9n}{4},$$

$$f(x_1) + f(x_2) = \frac{1+1}{2} + \left(\frac{9n}{4} - 1 + \frac{2}{2}\right) = \frac{9n}{4} + 1,$$

$$f(x_2) + f(x_3) = \left(\frac{9n}{4} - 1 + \frac{2}{2}\right) + \frac{3+1}{2} = \frac{9n}{4} + 2,$$

$$\vdots$$

$$f\left(x_{\frac{n}{2}}\right) + f\left(x_{\frac{n}{2}+1}\right) = \left(\frac{9n}{4} - 1 + \frac{\frac{n}{2}}{2}\right) + \frac{\left(\frac{n}{2}+1\right)+1}{2} = \frac{11n}{4},$$

$$\vdots$$

$$\begin{aligned}
f(x_1) + f(y_1) &= \frac{1+1}{2} + \frac{11n}{4} = \frac{11n}{4} + 1, \\
f\left(x_{\frac{n}{2}+2}\right) + f\left(y_{\frac{n}{2}+2}\right) &= \left(\frac{5n}{4} + \frac{\frac{n}{2}+2}{2}\right) + \left(n + \frac{\frac{n}{2}+2}{2}\right) = \frac{11n}{4} + 2, \\
f\left(x_{\frac{n}{2}+3}\right) + f\left(y_{\frac{n}{2}+3}\right) &= \frac{\left(\frac{n}{2}+1\right)+3}{2} + \left(\frac{9n}{4} + \frac{\left(\frac{n}{2}+3\right)-1}{2}\right) = \frac{11n}{4} + 3, \\
&\vdots \\
f(x_n) + f(y_n) &= \left(\frac{5n}{4} + \frac{n}{2}\right) + \left(n + \frac{n}{2}\right) = \frac{13n}{4}, \\
f(x_2) + f(y_2) &= \left(\frac{9n}{4} - 1 + \frac{2}{2}\right) + \left(n + \frac{2}{2}\right) = \frac{13n}{4} + 1, \\
f(x_3) + f(y_3) &= \frac{3+1}{2} + \left(\frac{13n}{4} + \frac{3-3}{2}\right) = \frac{13n}{4} + 2, \\
&\vdots \\
f\left(x_{\frac{n}{2}+1}\right) + f\left(y_{\frac{n}{2}+1}\right) &= \frac{\left(\frac{n}{2}+1\right)+1}{2} + \left(\frac{13n}{4} + \frac{\left(\frac{n}{2}+1\right)-3}{2}\right) = \frac{15n}{4}, \\
f(y_1) + f(y_2) &= \frac{11n}{4} + \left(n + \frac{2}{2}\right) = \frac{15n}{4} + 1, \\
f\left(y_{\frac{n}{2}+2}\right) + f\left(y_{\frac{n}{2}+3}\right) &= \left(n + \frac{\frac{n}{2}+2}{2}\right) + \left(\frac{9n}{4} + \frac{\left(\frac{n}{2}+3\right)-1}{2}\right) = \frac{15n}{4} + 2, \\
f\left(y_{\frac{n}{2}+3}\right) + f\left(y_{\frac{n}{2}+4}\right) &= \left(\frac{9n}{4} + \frac{\left(\frac{n}{2}+3\right)-1}{2}\right) + \left(n + \frac{\frac{n}{2}+4}{2}\right) = \frac{15n}{4} + 3, \\
&\vdots \\
f(y_1) + f(y_n) &= \frac{11n}{4} + \left(n + \frac{n}{2}\right) = \frac{17n}{4}, \\
f(y_2) + f(y_3) &= \left(n + \frac{2}{2}\right) + \left(\frac{13n}{4} + \frac{3-3}{2}\right) = \frac{17n}{4} + 1, \\
f(y_3) + f(y_4) &= \left(\frac{13n}{4} + \frac{3-3}{2}\right) + \left(n + \frac{4}{2}\right) = \frac{17n}{4} + 2, \\
&\vdots \\
f\left(y_{\frac{n}{2}+1}\right) + f\left(y_{\frac{n}{2}+2}\right) &= \left(\frac{13n}{4} + \frac{\frac{n}{2}+1-3}{2}\right) + \left(n + \frac{\frac{n}{2}+2}{2}\right) = \frac{19n}{4}.
\end{aligned}$$

Hence the edge sums are the numbers $\frac{7n}{4} + 1, \frac{7n}{4} + 2, \dots, \frac{19n}{4}$. According to Lemma 1.1 the labeling f can be extended to the super edge-magic total labeling of G with the magic constant $\frac{33n}{4}$. \square

3. EXACT VALUES

If G has order p , the *corona* of G with H , denoted by $G \odot H$, is the graph obtained by taking one copy of G and p copies of H and joining the i th vertex of G with an edge to every vertex in the i th copy of H .

Let us consider the Cartesian product $P_n \times K_2$, where P_n is the path on n vertices and K_2 is the complete graph of order 2. This graph is also called a ladder. In this section we deal with the super edge-magic deficiency of a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of $P_n \times K_2$, i.e., the corona $(P_n \times K_2) \odot K_1$.

Theorem 3.1. *For every odd positive integer n the graph $(P_n \times K_2) \odot K_1$ is super edge-magic total, i.e.,*

$$\mu_s((P_n \times K_2) \odot K_1) = 0.$$

Proof. Let n be a positive odd integer. We denote the vertex set and the edge set of $G \cong (P_n \times K_2) \odot K_1$ as follows

$$\begin{aligned} V(G) &= \{x_i, s_i, b_i, d_i : i = 1, 2, \dots, n\}, \\ E(G) &= \{x_i s_i, x_i b_i, s_i d_i : i = 1, 2, \dots, n\} \cup \{x_i x_{i+1}, s_i s_{i+1} : i = 1, 2, \dots, n - 1\}. \end{aligned}$$

The graph G is of order $4n$ and of size $5n - 2$.

For $n \geq 5$ we define the vertex labeling f of G such that

$$\begin{aligned} f(x_i) &= \begin{cases} \frac{4n + 1 + i}{2}, & \text{if } i = 1, 3, \dots, n, \\ \frac{5n + 1 + i}{2}, & \text{if } i = 2, 4, \dots, n - 1, \end{cases} \\ f(s_i) &= \begin{cases} \frac{3n + i}{2}, & \text{if } i = 1, 3, \dots, n, \\ \frac{2n + i}{2}, & \text{if } i = 2, 4, \dots, n - 1, \end{cases} \\ f(b_i) &= \begin{cases} \frac{n - 1}{2}, & \text{if } i = 1, \\ \frac{6n + i}{2}, & \text{if } i = 2, 4, \dots, n - 1, \\ \frac{7n + i}{2}, & \text{if } i = 3, 5, \dots, n - 2, \\ n, & \text{if } i = n, \end{cases} \end{aligned}$$

$$f(d_i) = \begin{cases} n-1, & \text{if } i = 1, \\ \frac{7n+1}{2}, & \text{if } i = 2, \\ 4n, & \text{if } i = 3, \\ \frac{n-3+i}{2}, & \text{if } i = 4, 6, \dots, n-1, \\ \frac{i-3}{2}, & \text{if } i = 5, 7, \dots, n. \end{cases}$$

It is easy to see that the vertices of G are labeled with the numbers $1, 2, \dots, 4n$ as the sets of vertex labels are the following ones.

$$\begin{aligned} \{f(s_i) : i = 1, 2, \dots, n\} &= \{n+1, n+2, \dots, 2n\}, \\ \{f(x_i) : i = 1, 2, \dots, n\} &= \{2n+1, 2n+2, \dots, 3n\}, \\ \{f(b_i) : i = 1, 2, \dots, n\} &= \left\{ \frac{n-1}{2}, n, 3n+1, 3n+2, \dots, \frac{7n-1}{2}, \frac{7n+3}{2}, \frac{7n+5}{2}, \dots, 4n-1 \right\}, \\ \{f(d_i) : i = 1, 2, \dots, n\} &= \left\{ 1, 2, \dots, \frac{n-3}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \dots, n-2, n-1, \frac{7n+1}{2}, 4n \right\}. \end{aligned}$$

Thus f is a bijection.

The edge sums under the labeling f are consecutive integers from the set $\left\{ \frac{3n+5}{2}, \frac{3n+7}{2} + 1, \dots, \frac{13n-1}{2} \right\}$ since we have

$$\begin{aligned} f(s_4d_4) &= \frac{2n+4}{2} + \frac{n-3+4}{2} = \frac{3n+5}{2}, \\ f(s_5d_5) &= \frac{3n+5}{2} + \frac{5-3}{2} = \frac{3n+7}{2}, \\ &\vdots \\ f(s_nd_n) &= \frac{3n+n}{2} + \frac{n-3}{2} = \frac{5n-3}{2}, \\ f(s_1d_1) &= \frac{3n+1}{2} + (n-1) = \frac{5n-1}{2}, \\ f(x_1b_1) &= \frac{4n+1+1}{2} + \frac{n-1}{2} = \frac{5n+1}{2}, \\ f(s_1s_2) &= \frac{3n+1}{2} + \frac{2n+2}{2} = \frac{5n+3}{2}, \\ f(s_2s_3) &= \frac{2n+2}{2} + \frac{3n+3}{2} = \frac{5n+5}{2}, \\ &\vdots \\ f(s_{n-1}s_n) &= \frac{2n+(n-1)}{2} + \frac{3n+n}{2} = \frac{7n-1}{2}, \\ f(x_nb_n) &= \frac{4n+1+n}{2} + n = \frac{7n+1}{2}, \end{aligned}$$

$$\begin{aligned}
 f(s_1x_1) &= \frac{3n+1}{2} + \frac{4n+1+1}{2} = \frac{7n+3}{2}, \\
 f(s_2x_2) &= \frac{2n+2}{2} + \frac{5n+1+2}{2} = \frac{7n+5}{2}, \\
 &\vdots \\
 f(s_nx_n) &= \frac{3n+n}{2} + \frac{4n+1+n}{2} = \frac{9n+1}{2}, \\
 f(s_2d_2) &= \frac{2n+2}{2} + \frac{7n+1}{2} = \frac{9n+3}{2}, \\
 f(x_1x_2) &= \frac{4n+1+1}{2} + \frac{5n+1+2}{2} = \frac{9n+5}{2}, \\
 f(x_2x_3) &= \frac{5n+1+2}{2} + \frac{4n+1+3}{2} = \frac{9n+7}{2}, \\
 &\vdots \\
 f(x_{n-1}x_n) &= \frac{5n+1+(n-1)}{2} + \frac{4n+1+n}{2} = \frac{11n+1}{2}, \\
 f(s_3d_3) &= \frac{3n+3}{2} + 4n = \frac{11n+3}{2}, \\
 f(x_2b_2) &= \frac{5n+1+2}{2} + \frac{6n+2}{2} = \frac{11n+5}{2}, \\
 f(x_3b_3) &= \frac{4n+1+3}{2} + \frac{7n+3}{2} = \frac{11n+7}{2}, \\
 &\vdots \\
 f(x_{n-1}b_{n-1}) &= \frac{5n+1+(n-1)}{2} + \frac{6n+(n-1)}{2} = \frac{13n-1}{2}.
 \end{aligned}$$

According to Lemma 1.1 the labeling f can be extended to the super edge-magic total labeling of $G \cong (P_n \times K_2) \odot K_1$, for $n \geq 5$ with the magic constant $\frac{21n+1}{2}$.

On Figures 1 and 2 are illustrated super edge-magic total labelings of $(P_1 \times K_2) \odot K_1 \cong P_4$ and $(P_3 \times K_2) \odot K_1$, respectively.

This concludes the proof.

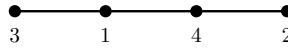


FIGURE 1. A super edge-magic total labeling of $(P_1 \times K_2) \odot K_1 \cong P_4$.

□

4. CONCLUSION

In this paper we have dealt with the problem of finding super edge-magic deficiency of graphs. We were trying to find the exact values of super edge-magic deficiencies

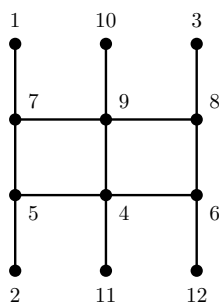


FIGURE 2. A super edge-magic total labeling of $(P_3 \times K_2) \odot K_1$.

of some graphs or to find the upper bound of this parameter for several classes of graphs.

In Theorem 2.3 we described the upper bound of the super edge-magic deficiency of prism D_n for $n \equiv 0 \pmod{4}$. As it is known, see [4], that for n odd the prism D_n is super edge-magic. To conclude the problem of finding the super edge-magic deficiency of prism D_n also for n even, for further investigation we state the following open problem.

Open Problem. Find the super edge-magic deficiency of prism D_n , for $n \equiv 2 \pmod{4}$.

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