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NEW STRONG DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS

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Abstract. New strong differential subordination and superordination results are obtained for meromorphic multivalent quasi-convex functions in the punctured unit disk by investigating appropriate classes of admissible functions. Strong differential sandwich results are also obtained.

1. Introduction and Preliminaries

Let Σ_p denote the class of all functions f of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

A function $f \in \Sigma_p$ is meromorphic multivalent starlike if $f(z) \neq 0$ and

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in U^*).$$

Similarly, $f \in \Sigma_p$ is meromorphic multivalent convex if $f'(z) \neq 0$ and

$$-\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in U^*).$$

Moreover, a function $f \in \Sigma_p$ is called meromorphic multivalent quasi-convex function if there exists a meromorphic multivalent convex function g such that $g'(z) \neq 0$

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and

$$-\mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0 \quad (z \in U^*).$$

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $\mathcal{H} = \mathcal{H}[1, 1]$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g, or (equivalently) g is said to be superordinate to f, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalent (see [5])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $G(z, \zeta)$ be analytic in $U \times \bar{U}$ and let $f(z)$ be analytic and univalent in U. Then the function $G(z, \zeta)$ is said to be strongly subordinate to $f(z)$ or $f(z)$ is said to be strongly superordinate to $G(z, \zeta)$, written as $G(z, \zeta) \prec\prec f(z)$, if for $\zeta \in \bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$, $G(z, \zeta)$ as a function of z is subordinate to $f(z)$. We note that

$$G(z, \zeta) \prec\prec f(z) \Leftrightarrow G(0, \zeta) = f(0) \text{ and } G(U \times \bar{U}) \subset f(U).$$

Definition 1.1. [6] Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ and let h be a univalent function in U. If F is analytic in U and satisfies the following (second-order) strong differential subordination:

$$\phi\left(F(z), zF'(z), z^2 F''(z); z, \zeta\right) \prec\prec h(z), \tag{1.1}$$

then F is called a solution of the strong differential subordination (1.1). The univalent function q is called a dominant of the solutions of the strong differential subordination or more simply a dominant if $F(z) \prec q(z)$ for all F satisfying (1.1). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.1) is said to be the best dominant.

Definition 1.2. [7] Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ and let h be analytic function in U. If F and $\phi(F(z), zF'(z), z^2 F''(z); z, \zeta)$ are univalent in U for $\zeta \in \bar{U}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \phi\left(F(z), zF'(z), z^2 F''(z); z, \zeta\right), \tag{1.2}$$

then F is called a solution of the strong differential superordination (1.2). An analytic function q is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant if $q(z) \prec F(z)$ for all F satisfying (1.2). A univalent subordinant $\check{q}$ that satisfies $q(z) \prec \check{q}(z)$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3. [6] Denote by Q the set consisting of all functions q that are analytic and injective on $\bar{U}\backslash E(q)$, where

$$E(q)=\left\{\xi\in\partial U:\lim_{z\to\xi}q(z)=\infty\right\},$$

and are such that $q'(\xi)\neq 0$ for $\xi\in\partial U\backslash E(q)$.

Furthermore, let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$, and $Q(1)\equiv Q_1$.

Definition 1.4. [9] Let Ω be a set in $\mathbb{C}$, $q\in Q$, and $n\in\mathbb{N}$. The class of admissible functions $\Psi_n[\Omega,q]$ consists of those functions $\psi:\mathbb{C}^3\times U\times\bar{U}\to\mathbb{C}$ that satisfy the following admissibility condition: $\psi(r,s,t;z,\zeta)\notin\Omega$, whenever

$$r=q(\xi),\quad s=k\xi q'(\xi)\quad\text{and}\quad \operatorname{Re}\left\{\frac{t}{s}+1\right\}\geq k\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\},$$

$z\in U$, $\xi\in\partial U\backslash E(q)$, $\zeta\in\bar{U}$, and $k\geq n$.

We simply write $\Psi_1[\Omega,q]=\Psi[\Omega,q]$.

Definition 1.5. [8] Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}[a,n]$ with $q'(z)\neq 0$. The class of admissible functions $\Psi'_n[\Omega,q]$ consists of those functions $\psi:\mathbb{C}^3\times U\times\bar{U}\to\mathbb{C}$ that satisfy the following admissibility condition: $\psi(r,s,t;\xi,\zeta)\in\Omega$, whenever

$$r=q(z),\quad s=\frac{zq'(z)}{m}\quad\text{and}\quad \operatorname{Re}\left\{\frac{t}{s}+1\right\}\leq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z\in U$, $\xi\in\partial U$, $\zeta\in\bar{U}$, and $m\geq n\geq 1$.

In particular, we write $\Psi'_1[\Omega,q]=\Psi'[\Omega,q]$.

In our investigations, we will need the following lemmas.

Lemma 1.1. [9] *Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If $F\in\mathcal{H}[a,n]$ satisfies*

$$\psi\left(F(z),zF'(z),z^2F''(z);z,\zeta\right)\in\Omega,$$

then $F(z)\prec q(z)$.

Lemma 1.2. [8] *Let $\psi\in\Psi'_n[\Omega,q]$ with $q(0)=a$. If $F\in Q(a)$ and $\psi\left(F(z),zF'(z),z^2F''(z);z,\zeta\right)$ is univalent in U for $\zeta\in\bar{U}$, then*

$$\Omega\subset\left\{\psi\left(F(z),zF'(z),z^2F''(z);z,\zeta\right):z\in U,\zeta\in\bar{U}\right\}$$

implies $q(z)\prec F(z)$.

In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1–4]. In this present investigation, by making use of the strong differential subordination results and strong differential superordination results of Oros and Oros [8, 9], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and superordination properties of meromorphic multivalent quasi-convex functions.

2. Strong Subordination Results

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{\mathcal{H}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; z, \zeta) \notin \Omega$, whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \quad q(\xi) \neq 0 \quad \text{and} \quad \operatorname{Re}\left\{\frac{w + v^2}{v}\right\} \geq k\operatorname{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U$, $\zeta \in \bar{U}$, $\xi \in \partial U \backslash E(q)$, and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_{\mathcal{H}}[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

(2.1)
$$\left\{\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.\right.$$
$$\left.\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\} \subset \Omega,$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. Let the analytic function F in U be defined by

$$F(z) = -\frac{(z^p f'(z))'}{g'(z)}. \tag{2.2}$$

After some calculation, we have

$$\frac{zF'(z)}{F(z)} = \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}. \tag{2.3}$$

Further computations show that

$$\begin{aligned} \frac{z^2 F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} - \left(\frac{zF'(z)}{F(z)}\right)^2 &= z\left[\frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}\right]' \\ &= \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} \\ &\quad + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right). \end{aligned} \tag{2.4}$$

Define the transforms from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}.$$

Let

$$\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi\left(r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z, \zeta\right). \tag{2.5}$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

$$\begin{aligned} (2.6)\qquad &\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right)\\ =&\phi\left(-\frac{(z^pf'(z))'}{g'(z)}, \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^pf'(z)\right)'''}{(z^pf'(z))'} + \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right.\\ &\left.\times\left(1-\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right) - \frac{z^2g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)}-1\right); z, \zeta\right).\end{aligned}$$

Therefore, (2.1) becomes $\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) \in \Omega$.

To complete the proof, we next show that the admissibility condition for $\phi \in \Phi_{\mathcal{H}}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4. Note that

$$\frac{t}{s}+1=\frac{w+v^2}{v}.$$

Hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, $F(z) \prec q(z)$ or equivalently

$$-\frac{(z^pf'(z))'}{g'(z)} \prec q(z). \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi_{\mathcal{H}}[h(U), q]$ is written as $\Phi_{\mathcal{H}}[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_{\mathcal{H}}[h, q]$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned} (2.7)\qquad &\phi\left(-\frac{(z^pf'(z))'}{g'(z)}, \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^pf'(z)\right)'''}{(z^pf'(z))'} + \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right.\\ &\left.\times\left(1-\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right) - \frac{z^2g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)}-1\right); z, \zeta\right) \prec\prec h(z),\end{aligned}$$

then

$$-\frac{(z^pf'(z))'}{g'(z)} \prec q(z).$$

By taking $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u+\gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 2.2, we state the following corollary.

Corollary 2.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U with $h(0) = 1$ and* $\operatorname{Re}\{\beta h(z) + \gamma\} > 0$. *If $f \in \Sigma_p$ satisfies*

$$-\frac{(z^pf'(z))'}{g'(z)} + \frac{\frac{z(z^pf'(z))''}{(z^pf'(z))'}g'(z) - zg''(z)}{\gamma g'(z) - \beta\left(z^pf'(z)\right)'} \prec\prec h(z),$$

then

$$-\frac{(z^pf'(z))'}{g'(z)} \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.2. *Let* $\Omega \in \mathbb{C}$ *and* q *be univalent in* U *with* $q(0)=1$. *Let* $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ *for some* $\rho \in (0,1)$, *where* $q_\rho(z) = q(\rho z)$. *If* $f \in \Sigma_p$ *satisfies*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.$$
$$\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) \in \Omega,$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. Theorem 2.1 yields $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$. The result is now deduced from the fact that $q_\rho(z) \prec q(z)$. □

Theorem 2.3. *Let* h *and* q *be univalent in* U *with* $q(0)=1$ *and set* $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$. *Let* $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ *satisfy one of the following conditions:*

(1) $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ *for some* $\rho \in (0,1)$;
(2) *there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ *satisfies* (2.7), *then*

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. (1) By applying Theorem 2.1, we obtain $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$, we deduce

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

(2) Let $F(z) = -\frac{(z^p f'(z))'}{g'(z)}$ and $F_\rho(z) = F(\rho z)$. Then

$$\phi\left(F_\rho(z), zF'_\rho(z), z^2 F''_\rho(z); \rho z, \zeta\right) = \phi\left(F(\rho z), zF'(\rho z), z^2 F''(\rho z); \rho z, \zeta\right) \in h_\rho(U).$$

By using Theorem 2.1 and the comment associated with

$$\phi\left(F(z), zF'(z), z^2 F''(z); w(z), \zeta\right) \in \Omega,$$

where w is any function mapping U into U, with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $F(z) \prec q(z)$. Therefore,

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$ □

The next result gives the best dominant of the strong differential subordination (2.7).

Theorem 2.4. *Let h be univalent in U and $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2; z, \zeta\right) = h(z) \tag{2.8}$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

(1) *$q \in Q_1$ and $\phi \in \Phi_{\mathcal{H}}[h, q]$;*
(2) *q is univalent in U and $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ for some $\rho \in (0, 1)$;*
(3) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z)$$

and q is the best dominant.

Proof. By applying Theorem 2.2 and Theorem 2.3, we deduce that q is a dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7) and therefore q will be dominated by all dominants. Hence, q is the best dominant of (2.7). □

In the particular case $q(z) = 1 + Mz, M > 0$ and in view of Definition 2.1, the class of admissible functions $\Phi_{\mathcal{H}}[\Omega, q]$ denoted by $\Phi_{\mathcal{H}}[\Omega, M]$ can be expressed in the following form.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible function $\Phi_{\mathcal{H}}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left(\frac{kM}{M + e^{-i\theta}}\right)^2; z, \zeta\right) \notin \Omega, \tag{2.9}$$

whenever $z \in U$, $\zeta \in \bar{U}$, $\theta \in \mathbb{R}$, $\operatorname{Re}\left\{Le^{-i\theta}\right\} \geq k(k-1)M$, for all θ and $k \geq 1$.

Corollary 2.3. *Let $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \times \left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) \in \Omega,$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

When $\Omega = q(U) = \{w : |w - 1| < M\}$, the class $\Phi_{\mathcal{H}}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{H}}[M]$, then Corollary 2.3 takes the following form.

Corollary 2.4. *Let $\phi \in \Phi_{\mathcal{H}}[M]$. If $f \in \Sigma_p$ satisfies*

$$\left|\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) - 1\right| < M,$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

Example 2.1. If $M > 0$ and $f \in \Sigma_p$ satisfies

$$\left|\frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} - \left(\frac{z(z^p f'(z))''}{(z^p f'(z))'}\right)^2 - \frac{z^2 g'''(z)}{g'(z)} + \left(\frac{zg''(z)}{g'(z)}\right)^2\right| < M,$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

This implication follows from Corollary 2.4 by taking $\phi(u, v, w; z, \zeta) = w - v + 1$.

Example 2.2. If $M > 0$ and $f \in \Sigma_p$ satisfies

$$\left|\frac{z(z^p f'(z))''}{(z^p f'(z))'} - \left(\frac{zg''(z)}{g'(z)} + 1\right)\right| < \frac{M}{M+1},$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

This implication follows from Corollary 2.3 by taking $\phi(u, v, w; z, \zeta) = v$ and $\Omega = h(U)$, where $h(z) = \frac{M}{M+1}z, M > 0$. To apply Corollary 2.3, we need to show that $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$, that is the admissibility condition (2.9) is satisfied follows from

$$\left|\phi\left(1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left(\frac{kM}{M + e^{-i\theta}}\right)^2; z, \zeta\right)\right| = \frac{kM}{M+1} \geq \frac{M}{M+1},$$

for $z \in U$, $\zeta \in \bar{U}$, $\theta \in \mathbb{R}$, and $k \geq 1$.

3. Strong Superordination Results

In this section, we obtain strong differential superordination. For this purpose the class of admissible functions given in the following definition will be required.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{\mathcal{H}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; \xi, \zeta) \in \Omega$, whenever

$$u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0 \quad \text{and} \quad \mathrm{Re}\left\{\frac{w + v^2}{v}\right\} \leq \frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in U$, $\zeta \in \bar{U}$, $\xi \in \partial U$, and $m \geq 1$.

Theorem 3.1. *Let $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$, and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U, then

(3.1)

$$\Omega \subset \left\{\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \times \left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\}$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

Proof. Let F defined by (2.2) and $\psi(F(z), zF'(z), z^2F''(z); z, \zeta)$ defined by (2.6). Since $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$, from (2.6) and (3.1), we have

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence $\psi \in \Psi'[\Omega, q]$ and by Lemma 1.2, $q(z) \prec F(z)$ or equivalently

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi'_{\mathcal{H}}[h(U), q]$ is written as $\Phi'_{\mathcal{H}}[h, q]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $\phi \in \Phi'_{\mathcal{H}}[h, q]$, $q \in \mathcal{H}$, and h be analytic in U. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$, and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U*, then*

(3.2)
$$h(z) \prec\prec \phi\left(-\frac{\left(z^p f'(z)\right)'}{g'(z)}, \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'} - \frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^p f'(z)\right)'''}{\left(z^p f'(z)\right)'} \right.$$
$$\left. + \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'}\left(1 - \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

implies

$$q(z) \prec -\frac{\left(z^p f'(z)\right)'}{g'(z)}.$$

By taking $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 3.2, we state the following corollary.

Corollary 3.1. *Let* $\beta, \gamma \in \mathbb{C}$ *and let* h *be convex in* U *with* $h(0) = 1$*. Suppose that the differential equation* $q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$ *has a univalent solution* q *that satisfies* $q(0) = 1$ *and* $q(z) \prec h(z)$*. If* $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$*, and*

$$-\frac{\left(z^p f'(z)\right)'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - zg''(z)}{\gamma g'(z) - \beta\left(z^p f'(z)\right)'}$$

is univalent in U*, then*

$$h(z) \prec\prec -\frac{\left(z^p f'(z)\right)'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - zg''(z)}{\gamma g'(z) - \beta\left(z^p f'(z)\right)'}$$

implies

$$q(z) \prec -\frac{\left(z^p f'(z)\right)'}{g'(z)}.$$

The next result gives the best subordinant of the strong differential superordination (3.2).

Theorem 3.3. *Let* h *be analytic in* U *and* $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$*. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, \frac{z^2 q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2; z, \zeta\right) = h(z)$$

has a solution $q \in Q_1$*. If* $\phi \in \Phi'_{\mathcal{H}}[h, q]$, $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$*, and*

$$\phi\left(-\frac{\left(z^p f'(z)\right)'}{g'(z)}, \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'} - \frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^p f'(z)\right)'''}{\left(z^p f'(z)\right)'} \right.$$
$$\left. + \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'}\left(1 - \frac{z\left(z^p f'(z)\right)''}{\left(z^p f'(z)\right)'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U, then

$$h(z) \prec\prec \phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

and q is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4 and is omitted. □

4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich theorem.

Theorem 4.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{\mathcal{H}}[h_2, q_2] \cap \Phi'_{\mathcal{H}}[h_1, q_1]$. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$ and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U, then

$$h_1(z) \prec\prec \phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich corollary.

Corollary 4.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h_1, h_2 be convex in U with $h_1(0) = h_2(0) = 1$. Suppose that the differential equations $q_1(z) + \frac{zq_1'(z)}{\beta q_1(z)+\gamma} = h_1(z)$, $q_2(z) + \frac{zq_2'(z)}{\beta q_2(z)+\gamma} = h_2(z)$*

have a univalent solutions q_1 *and* q_2*, respectively, that satisfy* $q_1(0) = q_2(0) = 1$ *and* $q_1(z) \prec h_1(z)$*,* $q_2(z) \prec h_2(z)$*. If* $f \in \Sigma_p$*,* $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$*, and*

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

is univalent in U*, then*

$$h_1(z) \prec\prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'} \prec\prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

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