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PROXIMATE GROUPS OF HIGHER ORDER

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Abstract. Using the intrinsic definition of shape based on proximate sequences for compact and all topological spaces based on proximate nets indexed by open coverings in the paper Shekutkovski [5] we define proximate fundamental group. In this paper will be introduced proximate groups of higher order and it will be shown that these groups are invariants of pointed intrinsic shape.

1. Introduction

The intrinsic definition of a shape of compact metric spaces by using near continuous functions given in [2] is generalized to paracompact spaces in [1]. More detailed description of this approach for pointed shape, is given in [6]. Using the concept of continuity over a covering introduced in [2], which was more developed in [1], a new approach to the intrinsic definition of a shape of compacta was presented in [3] by introducing the proximate sequences and corresponding homotopy. In this paper we will combine that approach and continuity over a covering to make generalization of intrinsic definition of pointed shape category on paracompact topological spaces. After determining one dimensional proximate fundamental group in a paper [5] in this paper we achieve to define proximate group of higher order and furthermore we prove that it is an invariant of pointed shape of a space.

First in the paper are introduced pointed continuous functions over a covering and homotopy over a covering that connects these functions. We define pointed proximate nets, pointed over a covering homotopy and category of intrinsic shape in Section 2. In the Section 3 we determine higher order proximate loop at a point over a covering, homotopy relatively endpoints that connects higher order proximate loops

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at a point and prove that the set of all homotopy classes of higher order proximate loops at a point is a commutative group for all orders greater than two. In the last Section 4 we associate induced function to a pointed proximate net and prove that it is a homomorphism between correspondent proximate groups of higher order. Furthermore, associating proximate group of higher order to a pointed paracompact topological space and associating to a pointed proximate net an induced function we obtain a functor from category of pointed intrinsic shape to category of groups. Therefore, we show that if two pointed paracompact topological spaces have same pointed intrinsic shape, then their proximate groups of higher order are isomorphic.

2. Pointed Homotopy Over a Covering

Let consider two topological spaces X and Y, $x_0 \in X$, $y_0 \in Y$. All coverings of the spaces will be open. The coverings of the topological spaces will be chosen from a fixed subset of coverings, which is cofinal in the set of all coverings, denoted as $\operatorname{Cov} X$, for topological space X. If X and Y are paracompact, it is enough to take the coverings to be locally finite coverings, since locally finite coverings are cofinal in the set of all coverings. So, let consider two paracompact topological spaces X and Y and $\mathcal{V} \in \operatorname{Cov} Y$.

First we will recall the definition of $\mathcal{V}$-continuous function and homotopy over a covering (see [3] and [4]).

Definition 2.1. Let $\mathcal{V}$ be a covering of Y. A function $f : X \to Y$ is **$\mathcal{V}$-continuous at point** $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$.

A function $f : X \to Y$ is **$\mathcal{V}$-continuous on** X if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x form a covering $\mathcal{U}$ of X. By this, $f : X \to Y$ is $\mathcal{V}$-continuous on X if there exists a covering $\mathcal{U}$ of X such that for any $x \in X$, there exists a neighborhood $U_x \in \mathcal{U}$ of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$. We denote: there exists $\mathcal{U} \in \operatorname{Cov} X$, such that $f(\mathcal{U}) \prec \mathcal{V}$.

Definition 2.2. Let $\mathcal{V}$ be a covering of Y. A star of $M \subseteq Y$ over the covering $\mathcal{V}$, denoted by $\operatorname{st}(M)$ is the union of all $W \in \mathcal{V}$ such that $M \cap W \neq \emptyset$, i.e.,

$$\operatorname{st}(M) = \cup\{W \in \mathcal{V} \mid M \cap W \neq \emptyset\}.$$

Let X and Y be topological spaces, $x_0 \in X$, $y_0 \in Y$ and $\mathcal{V}$ is a covering of Y, i.e., $\mathcal{V} \in \operatorname{Cov} Y$. We form new covering of Y as $\operatorname{st}(\mathcal{V}) = \{\operatorname{st}(V) \mid V \in \mathcal{V}\}$. Now, we will define pointed $\mathcal{V}$-homotopy.

Definition 2.3. Let $f, g : (X, x_0) \to (Y, y_0)$ be $\mathcal{V}$-continuous functions on paracompact spaces X and Y and $f(x_0) = g(x_0) = y_0$. The functions f and g are **pointed $\mathcal{V}$-homotopic functions** if there exists a function $F : (X \times I, x_0 \times I) \to (Y, y_0)$ such that:

(a) F is $\operatorname{st}(\mathcal{V})$-continuous, which is $\mathcal{V}$-continuous on $X \times \partial I$, $\partial I = \{0, 1\}$;
(b) $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$;

(c) $F(x_0, s) = f(x_0) = g(x_0) = y_0$ for all points $s \in I$.

When two $\mathcal{V}$-continuous functions f and g are pointed $\mathcal{V}$-homotopic we denote as $f \underset{\mathcal{V}}{\sim} g \,(rel\,\{x_0\})$.

Theorem 2.1. *The relation of pointed $\mathcal{V}$-homotopy $f \underset{\mathcal{V}}{\sim} g \,(rel\,\{x_0\})$ of $\mathcal{V}$-continuous functions is an equivalence relation.*

Proof. The proof is similar as the proof of the Proposition 2.4 in [3] about unpointed homotopy. □

Theorem 2.2. *Let X, Y, Z be paracompact topological spaces, $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$ and $g : (Y, y_0) \to (Z, z_0)$ is $\mathcal{W}$-continuous function. If $\mathcal{V}$ is a covering such that $g(\mathcal{V}) \prec \mathcal{W}$, then for any two $\mathcal{V}$-continuous functions $f_1, f_2 : (X, x_0) \to (Y, y_0)$ such that $f_1 \underset{\mathcal{V}}{\sim} f_2 \,(rel\,\{x_0\})$, it follows that $g \circ f_1 \underset{\mathcal{W}}{\sim} g \circ f_2 \,(rel\,\{x_0\})$.*

Proof. By the conditions of the proposition, it follows that the compositions $g \circ f_1$, $g \circ f_2$ are also $\mathcal{W}$-continuous functions.

Since $f_1, f_2 : (X, x_0) \to (Y, y_0)$ are pointed $\mathcal{V}$-homotopic, then there exists a function $F : (X \times I, x_0 \times I) \to (Y, y_0)$ such that:

(a) F is $\mathrm{st}(\mathcal{V})$-continuous, which is $\mathcal{V}$-continuous on $X \times \partial I$;
(b) $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for all points $x \in X$;
(c) $F(x_0, s) = f_1(x_0) = f_2(x_0) = y_0$ for all points $s \in I$.

Let consider function $K : (X \times I, x_0 \times I) \to (Z, z_0)$ defined by $K(x, s) = (g \circ F)(x, s)$. Since $g(\mathcal{V}) \prec \mathcal{W}$ implies $g(\mathrm{st}(\mathcal{V})) \prec \mathrm{st}(\mathcal{W})$ and F is $\mathrm{st}(\mathcal{V})$-continuous there exists an open covering $\mathcal{U}$, such that $F(\mathcal{U}) \prec \mathrm{st}(\mathcal{V})$, we conclude that

$$(g \circ F)(\mathcal{U}) = g(F(\mathcal{U})) \prec g(\mathrm{st}(\mathcal{V})) \prec \mathrm{st}(\mathcal{W}).$$

Therefore, the function K is $\mathrm{st}(\mathcal{W})$-continuous.

Since F is $\mathcal{V}$-continuous on $X \times \partial I$, $g(\mathcal{V}) \prec \mathcal{W}$ and g is $\mathcal{W}$-continuous function then it follows that $K = g \circ F$ is $\mathcal{W}$-continuous on $X \times \partial I$.

If $x \in X$ is an arbitrary point, then

$$\begin{aligned} K(x, 0) &= (g \circ F)(x, 0) = g(F(x, 0)) = g(f_1(x)) = (g \circ f_1)(x), \\ K(x, 1) &= (g \circ F)(x, 1) = g(F(x, 1)) = g(f_2(x)) = (g \circ f_2)(x). \end{aligned}$$

Let $s \in I$ be an arbitrary point, then

$$\begin{aligned} K(x_0, s) &= (g \circ F)(x_0, s) = g(F(x_0, s)) = g(f_1(x_0)) = (g \circ f_1)(x_0) \\ &= z_0 = (g \circ f_2)(x_0). \end{aligned}$$

Therefore, we showed that the functions $g \circ f_1$, $g \circ f_2$ are pointed $\mathcal{W}$-homotopic, i.e., $g \circ f_1 \underset{\mathcal{W}}{\sim} g \circ f_2 \,(rel\,\{x_0\})$. □

Theorem 2.3. *Let $G : (Y \times I, y_0 \times I) \to (Z, z_0)$ be a $\mathrm{st}(\mathcal{W})$-continuous function, $\mathcal{W}$-continuous on $Y \times \partial I$.*

Then there exists a covering $\mathcal{V}$ of Y such that for each $\mathcal{V}$-continuous function $f : (X, x_0) \to (Y, y_0)$ the function $G\,(f \times id) : (X \times I, x_0 \times I) \to (Z, z_0)$ is $\mathrm{st}(\mathcal{W})$-*continuous and the function $G\,(f \times id)$ is $\mathcal{W}$-continuous on $X \times \partial I$.*

Proof. The unpointed version of this theorem is proved for compact metric case, in [3, Theorem 3.0.5] and in paracompact case in more general situation is proved in [4, Theorem 2.2]. □

3. Pointed Proximate Nets. Pointed Over a Covering Homotopy

Let consider two paracompact topological spaces X and Y, $x_0 \in X$, $y_0 \in Y$. Now, we will define pointed proximate net from (X, x_0) to (Y, y_0).

Definition 3.1. A pointed proximate net from (X, x_0) to (Y, y_0) is a family $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in \mathrm{Cov}\, Y)$ of $\mathcal{V}$-continuous functions $f_{\mathcal{V}} : (X, x_0) \to (Y, y_0)$ such that $f_{\mathcal{W}} \underset{\mathcal{V}}{\sim} f_{\mathcal{V}}\,(rel\,\{x_0\})$ whenever $\mathcal{W} \prec \mathcal{V}$.

Definition 3.2. Two pointed proximate nets $\underline{f}$ and $\underline{g}$ from the pair (X, x_0) to the pair (Y, y_0) are pointed homotopic if $f_{\mathcal{V}} \underset{\mathcal{V}}{\sim} g_{\mathcal{V}}\,(rel\,\{x_0\})$ for all coverings $\mathcal{V} \in \mathrm{Cov}\, Y$. We denote by $\underline{f} \sim \underline{g}\,(rel\,\{x_0\})$.

Theorem 3.1. *The relation of pointed homotopy of pointed proximate nets is an equivalence relation.*

The pointed homotopy class of proximate net $\underline{f}$ from the pair (X, x_0) to the pair (Y, y_0) we will denote by $\left[\underline{f}\right]_{x_0}$.

Now, let introduce a notion of composition of pointed proximate nets $\underline{f} : (X, x_0) \to (Y, y_0)$ and $\underline{g} : (Y, y_0) \to (Z, z_0)$.

Let $\underline{f} = \{f_{\mathcal{V}} \mid \mathcal{V} \in \mathrm{Cov}\, Y\}$ and $\underline{g} = \{g_{\mathcal{W}} \mid \mathcal{W} \in \mathrm{Cov}\, Z\}$.

Because $g_{\mathcal{W}}$ is $\mathcal{W}$-continuous, then, by the definition, there exists an open covering $\mathcal{V}$ of Y such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$.

We define $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : (X, x_0) \to (Z, z_0)$. This function is $\mathcal{W}$-continuous. Although the definition depends on the choice of $\mathcal{V}$, the next lemma shows that for two coverings $\mathcal{V}, \mathcal{V}' \in \mathrm{Cov}\, Y$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$ is true that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$.

Lemma 3.1. *If $\underline{f}$ is pointed proximate net and $\mathcal{V}, \mathcal{V}' \in \mathrm{Cov}\, Y$ such that*

$$g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}, \quad \mathcal{W} \in \mathrm{Cov}\, Z,$$

then $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$.

Proof. Let $\mathcal{V}'' \in \mathrm{Cov}\, Y$ be a common refinement of $\mathcal{V}$ and $\mathcal{V}'$, i.e., $\mathcal{V}'' \prec \mathcal{V}, \mathcal{V}'$. Since $\underline{f}$ is pointed proximate net, by the definition, follows that $f_{\mathcal{V}''} \underset{\mathcal{V}}{\sim} f_{\mathcal{V}}\,(rel\,\{x_0\})$ and $f_{\mathcal{V}''} \underset{\mathcal{V}'}{\sim} f_{\mathcal{V}'}\,(rel\,\{x_0\})$. By Theorem 2.2 it follows that $g_{\mathcal{W}} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}}\,(rel\,\{x_0\})$ and

$g_{\mathcal{W}} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$. From the transitivity of the pointed homotopy we conclude that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$. □

Now, we will show that the function $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : (X, x_0) \to (Z, z_0)$ from the above discussion generates a pointed proximate net from (X, x_0) to (Z, z_0), $\underline{h} = \{h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} \mid \mathcal{W} \in \operatorname{Cov} Z\}$, i.e., we will show that for all $\mathcal{W}' \prec \mathcal{W}$ is true that $h_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} h_{\mathcal{W}}\,(rel\,\{x_0\})$.

Let $\mathcal{W}' \prec \mathcal{W}$ and since $\underline{g}$ is a pointed proximate net then $g_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}}\,(rel\,\{y_0\})$ by a pointed homotopy G is a st$(\mathcal{W})$- continuous function and $\mathcal{W}$-continuous on $Y \times \partial I$. By Theorem 2.3, there exists a $\mathcal{V}''$ of Y, such that for each $\mathcal{V}''$-continuous function $f_{\mathcal{V}''} : (X, x_0) \to (Y, y_0)$, the function $G\,(f_{\mathcal{V}''} \times id) : (X \times I, x_0 \times I) \to (Z, z_0)$ is st$(\mathcal{W})$-continuous, and the function $G\,(f_{\mathcal{V}''} \times id)$ is $\mathcal{W}$-continuous on $X \times \partial I$. It follows $g_{\mathcal{W}'} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$.

Now, consider $h_{\mathcal{W}'} = g_{\mathcal{W}'} \circ f_{\mathcal{V}'}$ and $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ for some $\mathcal{V}' \in \operatorname{Cov} Y$, $g_{\mathcal{W}'}\,(\mathcal{V}') \prec \mathcal{W}'$ and some $\mathcal{V} \in \operatorname{Cov} Y$, $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$.

By Lemma 3.1, since $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}\,(\mathcal{V}'') \prec \mathcal{W}$, then $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$.

Now, consider $\mathcal{V}_1 \prec \mathcal{V}', \mathcal{V}''$. Since $g_{\mathcal{W}'}\,(\mathcal{V}_1), g_{\mathcal{W}'}\,(\mathcal{V}') \prec \mathcal{W}'$, by Lemma 3.1, follows that $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \underset{\mathcal{W}'}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$. Because $\mathcal{W}' \prec \mathcal{W}$ then $g_{\mathcal{W}'} \circ f_{\mathcal{V}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}_1}\,(rel\,\{x_0\})$. By Theorem 2.2, since $\underline{f}$ is a pointed proximate net, i.e., $f_{\mathcal{V}_1} \underset{\mathcal{V}''}{\sim} f_{\mathcal{V}''}\,(rel\,\{x_0\})$ and $g_{\mathcal{W}'}\,(\mathcal{V}'') \prec \mathcal{W}$, then is true that $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$. Therefore, $g_{\mathcal{W}'} \circ f_{\mathcal{V}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}_1}\,(rel\,\{x_0\}) \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\}) \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$, i.e., we showed that $h_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} h_{\mathcal{W}}\,(rel\,\{x_0\})$.

Now, we will give the following definition.

Definition 3.3. Let $\left[\underline{f}\right]_{x_0}$ and $\left[\underline{g}\right]_{y_0}$ be two pointed homotopy classes of pointed proximate nets. We define a composition of pointed homotopy classes $\left[\underline{f}\right]_{x_0}$ and $\left[\underline{g}\right]_{y_0}$ by $\left[\underline{g}\right]_{y_0} \circ \left[\underline{f}\right]_{x_0} = \left[\underline{g} \circ \underline{f}\right]_{x_0}$.

From the discussion above in order to show that this composition is well defined we have to show that if $\underline{f} \sim \underline{f}'\,(rel\,\{x_0\})$ and $\underline{g} \sim \underline{g}'\,(rel\,\{x_0\})$, then $\underline{h} \sim \underline{h}'\,(rel\,\{x_0\})$, where $\underline{h}$ and $\underline{h}'$ are the compositions of pointed proximate nets $\underline{f}$ and $\underline{g}$, $\underline{f}'$ and $\underline{g}'$, respectively.

Since $\underline{g} \sim \underline{g}'\,(rel\,\{y_0\})$, then for every $\mathcal{W} \in \operatorname{Cov} Z$ is true that $g_{\mathcal{W}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}}'\,(rel\,\{y_0\})$ and, by Theorem 2.2, there exists a covering $\mathcal{U} \in \operatorname{Cov} Y$, $g_{\mathcal{W}}\,(\mathcal{U}) \prec$ W, $g'_{\mathcal{W}}\,(\mathcal{U}) \prec \mathcal{W}$ such that $\mathcal{U}$-continuous function $f_{\mathcal{U}} : (X, x_0) \to (Y, y_0)$ is $g_{\mathcal{W}} \circ f_{\mathcal{U}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}}' \circ f_{\mathcal{U}}\,(rel\,\{x_0\})$.

From the definition of the composition of two pointed proximate nets there exist coverings $\mathcal{V}$ and $\mathcal{V}'$ of Y such $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$ and $g_{\mathcal{W}}'\,(\mathcal{V}') \prec \mathcal{W}$ such $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ and $h_{\mathcal{W}}' = g_{\mathcal{W}}' \circ f_{\mathcal{V}'}'$.

Since $\underline{f} \sim \underline{f}^{'}\,(rel\,\{x_0\})$, then $f_{\mathcal{U}}\underset{\mathcal{U}}{\sim}f_{\mathcal{U}}^{'}\,(rel\,\{x_0\})$, so by this fact, Lemma 3.1 and Theorem 2.2, we can conclude that

$$h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}\underset{\mathcal{W}}{\sim}g_{\mathcal{W}} \circ f_{\mathcal{U}}\,(rel\,\{x_0\})\underset{\mathcal{W}}{\sim}g_{\mathcal{W}}^{'} \circ f_{\mathcal{U}}\,(rel\,\{x_0\})\underset{\mathcal{W}}{\sim}g_{\mathcal{W}}^{'} \circ f_{\mathcal{V}'}^{'}\,(rel\,\{x_0\}) = h_{\mathcal{W}}^{'},$$

i.e., $h_{\mathcal{W}}\underset{\mathcal{W}}{\sim}h_{\mathcal{W}}^{'}\,(rel\,\{x_0\})$ for all $\mathcal{W} \in \operatorname{Cov} Z$.

Therefore, $\underline{h} \sim \underline{h}^{'}\,(rel\,\{x_0\})$.

By the definition of the composition of pointed proximate nets and $\mathcal{U}$-continuous function the following theorem is valid.

Theorem 3.2. *Let* $\left[\underline{f}\right]_{x_0}$, $\left[\underline{g}\right]_{y_0}$ *and* $[\underline{h}]_{z_0}$ *be three pointed homotopy classes of pointed proximate nets. Then*

$$[\underline{h}]_{z_0} \circ \left(\left[\underline{g}\right]_{y_0} \circ \left[\underline{f}\right]_{x_0}\right) = \left([\underline{h}]_{z_0} \circ \left[\underline{g}\right]_{y_0}\right) \circ \left[\underline{f}\right]_{x_0}.$$

In this way we proved that the paracompact topological pointed spaces and pointed homotopy classes of pointed proximate nets form category of pointed intrinsic shape. We say that pointed paracompact topological spaces (X, x_0) and (Y, y_0) has same pointed intrinsic shape if they are isomorphic in this category.

4. Proximate Group of Higher Order

Let X be a paracompact topological space and

$$I^n = [0,1]^n = \{(t_1, t_2, \ldots, t_n) \mid 0 \le t_i \le 1,\ i = 1, 2, \ldots, n\}.$$

In the discussion below we use the following notation: $\partial^0 I^n = I^n$, $\partial^n I^n = (\partial I)^n$ and for all $i = 1, \ldots, n-1$ we consider $\partial^i I^n$ as: ∂I^n is a finite union of copies of I^{n-1}, i.e.,

$$\begin{aligned} \partial I^n = & \bigcup_{i=1}^{n}\{(t_1, t_2, \ldots, t_n) \mid t_i = 0, 0 \le t_j \le 1,\ j \ne i\} \\ & \cup \bigcup_{i=1}^{n}\{(t_1, t_2, \ldots, t_n) \mid t_i = 1, 0 \le t_j \le 1,\ j \ne i\}, \end{aligned}$$

$\partial^2 I^n$ is a finite union of copies of I^{n-2}, and for the others $i = 3, \ldots, n-1$, $\partial^i I^n$ is a finite union of copies of I^{n-i}.

For example, ∂I^3 are all faces of a cube, that is a union of six copies of I^2, $\partial^2 I^3$ are all edges of a cube, that is a union of twelve copies of I and $\partial^3 I^3$ are all vertices of a cube, that is a union of eight points,

$$\operatorname{st}^2(\mathcal{U}) = \operatorname{st}(\operatorname{st}(\mathcal{U})) = \{\operatorname{st}(W) \mid W \in \operatorname{st}(\mathcal{U})\}, \ldots, st^n(\mathcal{U}) = \operatorname{st}(\operatorname{st}^{n-1})(\mathcal{U}),$$

for all integers $n > 1$. Note, $\mathcal{U} \prec \operatorname{st}(\mathcal{U}) \prec \operatorname{st}^2(\mathcal{U}) \prec \cdots \prec \operatorname{st}^n(\mathcal{U}) \prec \cdots$.

Definition 4.1. Let $\mathcal{U}$ be a covering of X and $x_0 \in X$ is a fixed point. The function $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ such that is $\operatorname{st}^{n-i}(\mathcal{U})$-continuous on the set $\partial^i I^n$ for all $i = 0, 1, \ldots, n$, and $k_{\mathcal{U}}(\partial I^n) = x_0$ is called n-dimensional $\mathcal{U}$-loop at x_0.

Note, that a function $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ is $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on the set $\partial^i I^n$ for all $i = 1, \ldots, n$, if there exist open coverings $\mathcal{V}_{n-i}$, $i = 1, \ldots, n$, of the sets $\partial^i I^n$, respectively, such that $k_{\mathcal{U}}(\mathcal{V}_{n-i}) \prec \mathrm{st}^{n-i}(\mathcal{U})$.

Definition 4.2. Let $\mathcal{U} \in \mathrm{Cov}\, X$ and $k_{\mathcal{U}}, l_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be n-dimensional $\mathcal{U}$-loops at x_0. The n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$ are $\mathcal{U}$-homotopic loops relatively endpoints, if there exists a function $F : I^n \times I \to X$ such that

(I) F is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$, and satisfies the usual conditions of homotopy relatively endpoints;

(II) $F(\underline{t}, 0) = k_{\mathcal{U}}(\underline{t})$ and $F(\underline{t}, 1) = l_{\mathcal{U}}(\underline{t})$ for all $\underline{t} \in I^n$;

(III) $F\left(\underline{t}^{'}, s\right) = k_{\mathcal{U}}\left(\underline{t}^{'}\right) = l_{\mathcal{U}}\left(\underline{t}^{'}\right) = x_0$ for all $\underline{t}^{'} \in \partial I^n$ and $s \in I$.

If two n-dimensional $\mathcal{U}$-loops at x_0, $k_{\mathcal{U}}$, $l_{\mathcal{U}}$ are $\mathcal{U}$-homotopic loops relatively endpoints we denote as $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\, (rel\, \partial I^n)$.

Theorem 4.1. *The relation of homotopy relatively endpoints of n-dimensional $\mathcal{U}$-loops at x_0 is an equivalence relation.*

The homotopy relatively endpoints class of n-dimensional $\mathcal{U}$-loops at x_0, $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ we will denote by $[k_{\mathcal{U}}]_{x_0}$.

Proof. Reflexive property. Let $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be a n-dimensional $\mathcal{U}$-loop at x_0. We consider the following function $F : I^n \times I \to X$ defined by

$$F(\underline{t}, s) = k_{\mathcal{U}}(\underline{t}), \quad \text{for all } (\underline{t}, s) \in I^n \times I.$$

Since $k_{\mathcal{U}}$ is $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on the set $\partial^i I^n$ for all $i = 0, 1, \ldots, n$, so there exist open coverings $\mathcal{V}_{n-i}$, $i = 1, \ldots, n$, of the sets $\partial^i I^n$, respectively, such that $k_{\mathcal{U}}(\mathcal{V}_{n-i}) \prec \mathrm{st}^{n-i}(\mathcal{U})$. Let $\mathcal{V}$ be a covering of I, so $\mathcal{V}_{n-i} \times \mathcal{V}$ are coverings for $\partial^i I^{n+1}$, $i = 0, \ldots, n$, and $F(\mathcal{V}_{n-i} \times \mathcal{V}) = k_{\mathcal{U}}(\mathcal{V}_{n-i}) \prec \mathrm{st}^{n-i}(\mathcal{U}) \prec \mathrm{st}^{n+1-i}(\mathcal{U})$.

Now, let $i = n+1$. Then $\partial^i I^{n+1} = \bigcup_{j=1}^{n} \{(t_1, t_2, \ldots, t_n) \mid t_j = 0, 1,\ j = 1, \ldots, n+1\}$, i.e., $\partial^i I^{n+1} = \partial^n I^n \times \{0, 1\}$. We consider a covering $\mathcal{V}_0 \times \{0, 1\}$ of $\partial^i I^{n+1}$, where $\mathcal{V}_0$ is a covering of $\partial^n I^n$ such that $k_{\mathcal{U}}(\mathcal{V}_0) \prec \mathcal{U}$ since $k_{\mathcal{U}}$ is $\mathrm{st}^{n-n}(\mathcal{U}) = \mathcal{U}$-continuous on the set $\partial^n I^n$. So, $F(\mathcal{V}_0 \times \{0, 1\}) = k_{\mathcal{U}}(\mathcal{V}_0) \prec \mathcal{U} = \mathrm{st}^{n+1-(n+1)}(\mathcal{U})$. Therefore, F is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$, and $F(\underline{t}, 0) = F(\underline{t}, 1) = k_{\mathcal{U}}(\underline{t})$ for all $\underline{t} \in I^n$, $F\left(\underline{t}^{'}, s\right) = k_{\mathcal{U}}\left(\underline{t}^{'}\right) = x_0$ for all $\underline{t}^{'} \in \partial I^n$ and $s \in I$. So, F is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loop at x_0, $k_{\mathcal{U}}$ by itself and therefore, the relation of homotopy relatively endpoints of n-dimensional $\mathcal{U}$-loops at x_0 is reflexive.

Symmetric property. Let two n-dimensional $\mathcal{U}$-loops at x_0 $k_{\mathcal{U}}$, $l_{\mathcal{U}}$ be $\mathcal{U}$-homotopic. So, there exists a homotopy relatively endpoints $F : I^n \times I \to X$ that connects the n-dimensional $\mathcal{U}$-loops at x_0 $k_{\mathcal{U}}$, $l_{\mathcal{U}}$. The function $H : I^n \times I \to X$ defined by

$$H(\underline{t}, s) = F(\underline{t}, 1-s), \quad \text{for all } (\underline{t}, s) \in I^n \times I,$$

is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops at x_0, $l_{\mathcal{U}}$ and $k_{\mathcal{U}}$. Therefore, the relation of homotopy relatively endpoints of n-dimensional $\mathcal{U}$-loops at x_0 is symmetric.

Transitive property. Let $k_{\mathcal{U}}, l_{\mathcal{U}}, m_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be n-dimensional $\mathcal{U}$-loops at x_0 such that $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel \partial I^n)$ and $l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} m_{\mathcal{U}}\,(rel \partial I^n)$. So, there exist homotopies relatively endpoints $F : I^n \times I \to X$ and $K : I^n \times I \to X$ that connects the n-dimensional $\mathcal{U}$-loops at x_0 $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$, $l_{\mathcal{U}}$ and $m_{\mathcal{U}}$, respectively. Let consider a function $H : I^n \times I \to X$ defined for all $\underline{t} \in I^n$ by

$$H(\underline{t}, s) = \begin{cases} F(\underline{t}, 2s), & 0 \leq s \leq \frac{1}{2}, \\ K(\underline{t}, 2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

By [4, Theorem 2.2], the function H is well defined and $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for $i = 0, 1, \dots, n+1$. Since, $H(\underline{t}, 0) = F(\underline{t}, 2 \cdot 0) = k_{\mathcal{U}}(\underline{t})$, $H(\underline{t}, 1) = K(\underline{t}, 2 \cdot 1 - 1) = m_{\mathcal{U}}(\underline{t})$ for all $\underline{t} \in I^n$ and for all $\underline{t}' \in \partial I^n$

$$H(\underline{t}', s) = \begin{cases} F(\underline{t}', 2s) = k_{\mathcal{U}}(\underline{t}') = x_0, & 0 \leq s \leq \frac{1}{2}, \\ K(\underline{t}', 2s-1) = m_{\mathcal{U}}(\underline{t}') = x_0, & \frac{1}{2} \leq s \leq 1, \end{cases}$$

the function H is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops at x_0 $k_{\mathcal{U}}$ and $m_{\mathcal{U}}$. So, the relation of homotopy relatively endpoints of n-dimensional $\mathcal{U}$-loops at x_0 is transitive and equivalence relation. □

Definition 4.3. Let consider two n-dimensional $\mathcal{U}$-loops at x_0, $k_{\mathcal{U}}, l_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$. The juxtaposition of these two n-dimensional $\mathcal{U}$-loops at x_0 is:

$$(k * l)(t_1, t_2, \dots, t_n) = \begin{cases} k(t_1, t_2, \dots, t_{n-1}, 2t_n), & 0 \leq t_n \leq \frac{1}{2}, \\ l(t_1, t_2, \dots, t_{n-1}, 2t_n - 1), & \frac{1}{2} \leq t_n \leq 1. \end{cases}$$

By [4, Theorem 2.2], juxtaposition is well defined and $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on $\partial^i I^n$ for $i = 0, 1, \dots, n$. By the definition, $k_{\mathcal{U}} * l_{\mathcal{U}}(\partial I^n) = x_0$.

Therefore, $k_{\mathcal{U}} * l_{\mathcal{U}}(\partial I^n) = x_0$ is n-dimensional $\mathcal{U}$-loop at x_0.

Theorem 4.2. *Let $k_{\mathcal{U}}, k_{\mathcal{U}}' : (I^n, \partial I^n) \to (X, x_0)$, $l_{\mathcal{U}}, l_{\mathcal{U}}' : (I^n, \partial I^n) \to (X, x_0)$ be n-dimensional $\mathcal{U}$-loops at x_0 such that $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}'\,(rel\, \partial I^n)$, $l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}'\,(rel\, \partial I^n)$ and the juxtapositions $k_{\mathcal{U}} * l_{\mathcal{U}}$ and $k_{\mathcal{U}}' * l_{\mathcal{U}}'$ are defined. Then $k_{\mathcal{U}} * l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}' * l_{\mathcal{U}}'\,(rel\, \partial I^n)$.*

Proof. Since $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}'\,(rel\, \partial I^n)$ and $l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}'\,(rel\, \partial I^n)$, there are homotopies relatively endpoints $K : I^n \times I \to X$ and $L : I^n \times I \to X$ that connect the n-dimensional $\mathcal{U}$-loops at x_0, $k_{\mathcal{U}}$ and $k_{\mathcal{U}}'$ and $l_{\mathcal{U}}$ and $l_{\mathcal{U}}'$, respectively.

We define a function $H : I^n \times I \to X$ by

$$H(t_1, \ldots, t_{n-1}, t_n, s) = \begin{cases} K(t_1, \ldots, t_{n-1}, 2t_n, s), & 0 \le t_n \le \dfrac{1}{2}, \\ L(t_1, \ldots, t_{n-1}, 2t_n - 1, s), & \dfrac{1}{2} \le t_n \le 1. \end{cases}$$

This function is well defined since for $t_n = \frac{1}{2}$ and all $t_1, t_2, \ldots, t_{n-1}, s \in I$ is valid

$$\begin{aligned} K\left(t_1, \ldots, t_{n-1}, 2 \cdot \frac{1}{2}, s\right) =& K(t_1, \ldots, t_{n-1}, 1, s) \overset{(t_1,\ldots,t_{n-1},1)\in\partial I^n}{=} = k_{\mathcal{U}}\left((t_1, \ldots, t_{n-1}, 1)\right) \\ =& x_0 = l_{\mathcal{U}}\left((t_1, \ldots, t_{n-1}, 0)\right) \overset{(t_1,\ldots,t_{n-1},0)\in\partial I^n}{=} L(t_1, \ldots, t_{n-1}, 0, s) \\ =& L\left(t_1, \ldots, t_{n-1}, 2 \cdot \frac{1}{2} - 1, s\right). \end{aligned}$$

From this equation and the fact that the functions K and L are $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$, by [4, Theorem 2.2], the function H is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$.

The usual conditions (II) and (III) from the Definition 4.2 are true by the definition of the function H.

Therefore, $k^0_{\mathcal{U}} * l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}} * l^1_{\mathcal{U}}\,(rel\ \{0, 1\})$. □

Theorem 4.3. *Let $k_{\mathcal{U}}, l_{\mathcal{U}}, p_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be n-dimensional $\mathcal{U}$-loops in x_0 and the juxtapositions $k_{\mathcal{U}} * l_{\mathcal{U}}$, $l_{\mathcal{U}} * p_{\mathcal{U}}$ are defined. Then*

$$(k_{\mathcal{U}} * l_{\mathcal{U}}) * p_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * (l_{\mathcal{U}} * p_{\mathcal{U}})\,(rel \partial I^n).$$

Proof. First let represent the $n+1$-dimensional cube $I^n \times I$ as union of the sets $I^{n-1} \times A$, $I^{n-1} \times B$ and $I^{n-1} \times C$, such that $A = \left\{(t, s) \mid s \in I, 0 \le t \le \frac{s+1}{4}\right\}$, $B = \left\{(t, s) \mid s \in I, \frac{s+1}{4} \le t \le \frac{s+2}{4}\right\}$ and $C = \left\{(t, s) \mid s \in I, \frac{s+2}{4} \le t \le 1\right\}$.

Let define functions $a : A \to I$, $b : B \to I$ and $c : C \to I$ by $a(t, s) = \frac{4t}{s+1}$, $b(t, s) = 4t - 1 - s$, $c(t, s) = \frac{4t-2-s}{2-s}$ and function $H : I^n \times I \to X$ by

$$H(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, a(t_n, s)), & (t_1, \ldots, t_{n-1}, t_n, s) \in I^{n-1} \times A, \\ l_{\mathcal{U}}(t_1, \ldots, t_{n-1}, b(t_n, s)), & (t_1, \ldots, t_{n-1}, t_n, s) \in I^{n-1} \times B, \\ p_{\mathcal{U}}(t_1, \ldots, t_{n-1}, c(t_n, s)), & (t_1, \ldots, t_{n-1}, t_n, s) \in I^{n-1} \times C. \end{cases}$$

If $(\underline{t}, s) \in (I^{n-1} \times A) \cap (I^{n-1} \times B) = \left\{\left(t_1, \ldots, t_{n-1}, \frac{s+1}{4}, s\right) \mid s \in I\right\}$, then

$$k_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, a\left(\frac{s+1}{4}, s\right)\right) = x_0 = l_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, b\left(\frac{s+1}{4}, s\right)\right).$$

If $(\underline{t}, s) \in (I^{n-1} \times B) \cap (I^{n-1} \times C) = \left\{\left(t_1, \ldots, t_{n-1}, \frac{s+2}{4}, s\right) \mid s \in I\right\}$, then

$$l_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, b\left(\frac{s+2}{4}, s\right)\right) = x_0 = p_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, c\left(\frac{s+2}{4}, s\right)\right).$$

So, the functions are equal on the intersections $(I^{n-1} \times A) \cap (I^{n-1} \times B)$ and $(I^{n-1} \times B) \cap (I^{n-1} \times C)$.

Since the n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}}$, $l_{\mathcal{U}}$ and $p_{\mathcal{U}}$ at x_0 are $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on $\partial^i I^n$ for all $i = 0, 1, \ldots, n$, and the functions a, b and c are continuous on the sets A, B and C, respectively, then, by Theorem 2.3, the functions $k_{\mathcal{U}}(id \times a) : I^{n-1} \times A \to X$, $l_{\mathcal{U}}(id \times b) : I^{n-1} \times B \to X$ and $p_{\mathcal{U}}(id \times c) : I^{n-1} \times C \to X$ are $\mathrm{st}^{n-i}(\mathcal{U})$ and so $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i(I^{n-1} \times A)$, $\partial^i(I^{n-1} \times B)$ and $\partial^i(I^{n-1} \times B)$, respectively, for all $i = 0, 1, \ldots, n$, and $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on $\partial^i(\partial I^{n-1} \times A)$, $\partial^i(\partial I^{n-1} \times B)$ and $\partial^i(\partial I^{n-1} \times B)$, respectively for all $i = 0, 1, \ldots, n$.

The function H is defined by three $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous functions on closed sets for all $i = 0, 1, \ldots, n$, which are equal on their intersection. Therefore, by [4, Theorem 2.2], the function H is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$.

The usual conditions (II) and (III) from the Definition 4.2 are true by the definition of the function H.

So, the function H is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops $(k_{\mathcal{U}} * l_{\mathcal{U}}) * p_{\mathcal{U}}$ and $k_{\mathcal{U}} * (l_{\mathcal{U}} * p_{\mathcal{U}})$ at x_0, as required. ☐

Definition 4.4. Let X be a paracompact topological space and $x_0 \in X$ is fixed point. The function $c_{x_0} : I^n \to X$, defined by $c_{x_0}(\underline{t}) = x_0$, $\underline{t} \in I^n$, is called **constant n-dimensional $\mathcal{U}$-loop at** x_0.

Theorem 4.4. *Let $k_{\mathcal{U}} : I^n \to X$ be a n-dimensional $\mathcal{U}$-loop at x_0. Then $k_{\mathcal{U}} * c_{x_0} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\partial I^n)$ and $c_{x_0} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\partial I^n)$.*

Proof. First we will prove that $k_{\mathcal{U}} * c_{x_0} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\partial I^n)$.

Let represent the $n+1$-dimensional cube $I^n \times I$ as union of the sets $I^{n-1} \times D_1$ and $I^{n-1} \times D_2$, such that $D_1 = \left\{(t,s) \mid s \in I,\ 0 \le t \le \frac{s+1}{2}\right\}$ and $D_2 = \left\{(t,s) \mid s \in I,\ \frac{s+1}{2} \le t \le 1\right\}$.

Let define the function $d : D_1 \to I$ by $d(t,s) = \frac{2t}{s+1}$ for all $(t,s) \in D_1$ and the function $H : I^n \times I \to X$ by

$$H(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, d(t_n, s)), & (t_1, \ldots, t_{n-1}, t_n, s) \in I^{n-1} \times D_1, \\ x_0, & (t_1, \ldots, t_{n-1}, t_n, s) \in I^{n-1} \times D_2. \end{cases}$$

If $(\underline{t}, s) \in (I^{n-1} \times D_1) \cap (I^{n-1} \times D_2) = \left\{\left(t_1, \ldots, t_{n-1}, \frac{s+1}{2}, s\right) \mid s \in I\right\}$, then

$$k_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, d\left(\frac{s+1}{2}, s\right)\right) = k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 1) = x_0.$$

Therefore, the function H is well defined.

Since $k_{\mathcal{U}}$ and c_{x_0} are n-dimensional $\mathcal{U}$-loops at x_0, they are $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on $\partial^i I^n$, $i = 0, 1, \ldots, n$. Since d is continuous on D_1, then, by Theorem 2.3, the function $k_{\mathcal{U}}(id \times d) : I^{n-1} \times D_1 \to X$ is $\mathrm{st}^{n-i}(\mathcal{U})$ and so $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i(I^{n-1} \times D_1)$ for all $i = 0, 1, \ldots, n$, and $\mathrm{st}^{n-i}(\mathcal{U})$-continuous on $\partial^i(\partial I^{n-1} \times D_1)$ for all $i = 0, 1, \ldots, n$.

The function H is defined by two $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous functions on closed sets for all $i = 0, 1, \ldots, n$,which are equal at their intersection. So, by [4, Theorem 2.2], the function H is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$.

The usual conditions (II) and (III) from the Definition 4.2 are true by the definition of the function H.

Therefore, the function H is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}} * c_{x_0}$ and $k_{\mathcal{U}}$ at x_0, as required.

Similarly, is easy to verified that $c_{x_0} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \,(rel\, \partial I^n)$. □

Definition 4.5. Let X be a paracompact topological space and $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ is n-dimensional $\mathcal{U}$-loop at x_0. **Inverse n-dimensional $\mathcal{U}$-loop at** x_0, $k_{\mathcal{U}}^{-1} : (I^n, \partial I^n) \to (X, x_0)$ is defined by $k_{\mathcal{U}}^{-1}(t_1, \ldots, t_n) = k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 1 - t_n)$ for all elements $\underline{t} \in I^n$. Notice $(k_{\mathcal{U}}^{-1})_{\mathcal{U}}^{-1} = k_{\mathcal{U}}$.

Theorem 4.5. *Let $k_{\mathcal{U}}, l_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be n-dimensional $\mathcal{U}$-loops at x_0. If $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\, \partial I^n)$, then $k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}^{-1} \,(rel\, \partial I^n)$.*

Proof. Since $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\, \partial I^n)$, there is homotopy relatively endpoints $F : I^n \times I \to X$ that connects the n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$ at x_0.

Let define a function $H : I^n \times I \to X$ by $H(t_1, \ldots, t_{n-1}, t_n, s) = F(t_1, \ldots, t_{n-1}, 1 - t_n, s)$.

Since the function H is a composition of continuous function and the function F, by [1, Proposition 1.3 (iv)], the function H is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n, n+1$.

The usual conditions (II) and (III) from the Definition 4.2 are true by the definition of the function H.

Therefore, the function H is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}}^{-1}$ and $l_{\mathcal{U}}^{-1}$ at x_0, as required. □

Theorem 4.6. *Let $k_{\mathcal{U}} : (I^n, \partial I^n) \to (X, x_0)$ be a n-dimensional $\mathcal{U}$-loop at x_0. Then $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} c_{x_0} \,(rel\, \partial I^n)$ and $k_{\mathcal{U}}^{-1} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} c_{x_0} \,(rel\, \partial I^n)$.*

Proof. By the definition of the juxtaposition

$$\left(k_{\mathcal{U}} * k_{\mathcal{U}}^{-1}\right)(\underline{t}) = \begin{cases} k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 2t_n), & 0 \le t_n \le \dfrac{1}{2}, \\ k_{\mathcal{U}}^{-1}(t_1, \ldots, t_{n-1}, 2t_n - 1), & \dfrac{1}{2} \le t_n \le 1, \end{cases}$$

$$= \begin{cases} k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 2t_n), & 0 \le t_n \le \dfrac{1}{2}, \\ k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 2 - 2t_n), & \dfrac{1}{2} \le t_n \le 1. \end{cases}$$

Let define a function $H : I^n \times I \to X$ by

$$H(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, 2t_n(1-s)), & 0 \le t_n \le \frac{1}{2}, \\ k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, (2-2t_n)(1-s)), & \frac{1}{2} \le t_n \le 1. \end{cases}$$

It is well defined since for all $t_n = \frac{1}{2}$ and $s \in I$ is true

$$H(\underline{t}, s) = k_{\mathcal{U}}(t_1, \ldots, t_{n-1}, (1-s)) = k_{\mathcal{U}}\left(t_1, \ldots, t_{n-1}, \left(2 - 2 \cdot \frac{1}{2}\right)(1-s)\right).$$

Since the function H is defined by two $\mathrm{st}^{n-i}(\mathcal{U})$-continuous functions on the sets $\partial^i \left(I^{n-1} \times \left[0, \frac{1}{2}\right] \times I\right)$ and $\partial^i \left(I^{n-1} \times \left[\frac{1}{2}, 1\right] \times I\right)$ for all $i = 0, 1, \ldots, n$, which are equal at their intersection. So, by [4, Theorem 2.2], this function is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, 1, \ldots, n+1$.

The usual conditions (II) and (III) from the Definition 4.2 are true by the definition of the function H.

Therefore, the function H is a homotopy relatively endpoints that connects the n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1}$ and c_{x_0} at x_0, as required.

Similar, we can prove that $k_{\mathcal{U}}^{-1} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} c_{x_0}\,(rel\,\partial I^n)$. □

Definition 4.6. A proximate n-dimensional loop at x_0 (over $\mathrm{Cov}\,X$) is a family $\underline{k} = \{k_{\mathcal{U}} \mid \mathcal{U} \in \mathrm{Cov}\,X\}$ such that $k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\partial I^n)$ for all $\mathcal{V} \prec \mathcal{U}$.

Definition 4.7. Two proximate n-dimensional loops $\underline{k}$ and $\underline{l}$ at x_0 are said to be homotopic over a covering if $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel \partial I^n)$ for all $\mathcal{U} \in \mathrm{Cov}\,X$, we denote that by $\underline{k} \sim \underline{l}\,(rel\,\partial I^n)$.

Theorem 4.7. *Homotopy relatively endpoints of proximate n-dimensional loop at x_0 is an equivalence. The homotopy class of proximate n-dimensional loop $\underline{k}$ at x_0 is denoted by $[\underline{k}]_{x_0}$.*

Proof. Since, from the Theorem 4.1, the relation of homotopy relatively endpoints of n-dimensional $\mathcal{U}$-loops at x_0 is equivalence relation for any arbitrary covering $\mathcal{U} \in \mathrm{Cov}\,X$, then homotopy relatively endpoints of proximate n-dimensional loop at x_0 is an equivalence. □

Definition 4.8. Let $\underline{k}$ and $\underline{l}$ be two proximate n-dimensional loops at x_0. Then their juxtaposition is the proximate n-dimensional $\mathcal{U}$-loop $\underline{k} * \underline{l}$ at x_0 defined by $\underline{k} * \underline{l} = (k_{\mathcal{U}} * l_{\mathcal{U}} \mid \mathcal{U} \in \mathrm{Cov}\,X)$.

In order to justify that the juxtaposition in Definition 4.8 is well defined we will show that $\underline{k} * \underline{l}$ is a proximate n-dimensional at x_0.

By the definition of the n-dimensional loop at x_0, the juxtaposition $k_{\mathcal{U}} * l_{\mathcal{U}}$ is n-dimensional $\mathcal{U}$-loop at x_0 for all coverings $\mathcal{U} \in \mathrm{Cov}\,X$. Let consider an arbitrary covering $\mathcal{V} \prec \mathcal{U}$. Since $\underline{k}$ and $\underline{l}$ are proximate n-dimensional loops at x_0, then

$k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\partial I^n)$ and $l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\,\partial I^n)$ for all coverings $\mathcal{U} \in \operatorname{Cov} X$. By the Theorem 4.2, the following relation $k_{\mathcal{V}} * l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * l_{\mathcal{U}}\,(rel\,\{0,1\})$ is true. Therefore, $\underline{k} * \underline{l}$ is a proximate n-dimensional at x_0.

Now, let consider the following set:

$$\operatorname{prox} \pi_n\,(X, x_0) = \left\{ [\underline{k}]_{x_0} \mid \underline{k} \text{ is a proximate } n\text{-dimensional loop at } x_0 \right\}.$$

In this set we define the operation $*$ by $[\underline{k}]_{x_0} * [\underline{l}]_{x_0} = [\underline{k} * \underline{l}]_{x_0}$, where $\underline{k} * \underline{l}$ is defined by the Definition 4.8.

This operation is well defined. Let $\underline{k}^0, \underline{k}^1 \in [\underline{k}]_{x_0}$ and $\underline{l}^0, \underline{l}^1 \in [\underline{l}]_{x_0}$ be proximate n-dimensional loops at x_0 from the homotopy classes of proximate n-dimensional loops $[\underline{k}]_{x_0}$ and $[\underline{l}]_{x_0}$, respectively. Then $k^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}}\,(rel\,\partial I^n)$ and $l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l^1_{\mathcal{U}}\,(rel\,\partial I^n)$ for all coverings $\mathcal{U} \in \operatorname{Cov} X$. By the Theorem 4.2, $k^0_{\mathcal{U}} * l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}} * l^1_{\mathcal{U}}\,(rel\,\partial I^n)$ for all coverings $\mathcal{U} \in \operatorname{Cov} X$.

Therefore, proximate n-dimensional loops at x_0, $\underline{k}^0 * \underline{l}^0 = \{k^0_{\mathcal{U}} * l^0_{\mathcal{U}} \mid \mathcal{U} \in \operatorname{Cov} X\}$ and $\underline{k}^1 * \underline{l}^1 = \{k^1_{\mathcal{U}} * l^1_{\mathcal{U}} \mid \mathcal{U} \in \operatorname{Cov} X\}$ are homotopic over a covering, i.e.,

$$\underline{k}^0 * \underline{l}^0,\ \underline{k}^1 * \underline{l}^1 \in [\underline{k} * \underline{l}]_{x_0}.$$

So, the operation $*$ in the set $\operatorname{prox} \pi_n\,(X, x_0)$ is well defined.

Theorem 4.8. *The set* $\operatorname{prox} \pi_n\,(X, x_0)$ *with the operation* $*$ *is a commutative group for all* $n \geq 2$.

Proof. Since proximate n-dimensional loop is a pointed proximate net, then, by Theorem 3.2, for all homotopy classes of proximate n-dimensional loops $[\underline{k}]_{x_0}$, $[\underline{l}]_{x_0}$ and $[\underline{p}]_{x_0}$ at x_0 is valid $\left([\underline{k}]_{x_0} * [\underline{l}]_{x_0}\right) * \left[\underline{p}\right]_{x_0} = [\underline{k}]_{x_0} * \left([\underline{l}]_{x_0} * \left[\underline{p}\right]_{x_0}\right)$.

By Theorem 4.4, identity element is the homotopy class of constant proximate n-dimensional loop $\left[\underline{c_{x_0}}\right]_{x_0}$ at x_0 defined by the constant $\mathcal{U}$-loop c_{x_0} at x_0.

By Theorem 4.6, inverse element of the homotopy class of proximate n-dimensional loop $[\underline{k}]_{x_0}$ at x_0 is the homotopy class of proximate n-dimensional loop $[\underline{k}^{-1}]_{x_0}$ defined by the inverse proximate $\mathcal{U}$-loop $k^{-1}_{\mathcal{U}}$ at x_0.

Therefore, $\operatorname{prox} \pi_n\,(X, x_0)$ is a group. We will show that the commutative law for the group $\operatorname{prox} \pi_n\,(X, x_0)$ is valid for all $n \geq 2$.

We should show that if there are two homotopy classes of proximate n-dimensional loops $[\underline{k}]_{x_0}$ and $[\underline{l}]_{x_0}$ at x_0, then

$$[k]_{x_0} * [\underline{l}]_{x_0} = [\underline{l}]_{x_0} * [k]_{x_0}. \tag{4.1}$$

Since for the left side of the equality (4.1) is true

$$[\underline{k}]_{x_0} * [\underline{l}]_{x_0} = [\underline{k} * \underline{l}]_{x_0}, \tag{4.2}$$

and for the right side of (4.1) is true

$$[\underline{l}]_{x_0} * [\underline{k}]_{x_0} = [\underline{l} * \underline{k}]_{x_0}, \tag{4.3}$$

in order to show that the equality (4.1) is true is enough to show $[\underline{k} * \underline{l}]_{x_0} = [\underline{l} * \underline{k}]_{x_0}$, i.e., that the proximate n-dimensional loops $\underline{k} * \underline{l}$ and $\underline{l} * \underline{k}$ at x_0 are homotopic over a covering. It means that we should show that $k_{\mathcal{U}} * l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} * k_{\mathcal{U}} \, (rel\, \partial I^n)$ for all coverings $\mathcal{U} \in \operatorname{Cov} X$.

Let $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$ be n-dimensional $\mathcal{U}$-loops at x_0 for all coverings $\mathcal{U} \in \operatorname{Cov} X$.

We define functions $F, G, H : I^n \times I \to X$ such that $F(\underline{t}, 1) = G(\underline{t}, 0)$, $G(\underline{t}, 1) = H(\underline{t}, 0)$ and $F(\partial I^n \times I) = G(\partial I^n \times I) = H(\partial I^n \times I) = \{x_0\}$.

Then we will show that the function $K : I^n \times I \to X$ defined by

$$K(\underline{t}, s) = \begin{cases} F(\underline{t}, s), & 0 \le s \le \dfrac{1}{3}, \\ G(\underline{t}, 3s-1), & \dfrac{1}{3} \le s \le \dfrac{2}{3}, \\ H(\underline{t}, 3s-2), & \dfrac{2}{3} \le s \le 1, \end{cases}$$

is a homotopy relatively endpoints that connects n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}} * l_{\mathcal{U}}$ and $l_{\mathcal{U}} * k_{\mathcal{U}}$ at x_0.

We define a function $F : I^n \times I \to X$ by

$$F(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}\left(\dfrac{2t_1}{2-s}, t_2, \ldots, t_{n-1}, 2t_n\right), & 0 \le t_n \le \dfrac{1}{2},\ 0 \le t_1 \le \dfrac{2-s}{2}, \\ l_{\mathcal{U}}\left(\dfrac{2t_1 - s}{2-s}, t_2, \ldots, t_{n-1}, 2t_n - 1\right), & \dfrac{1}{2} \le t_n \le 1,\ \dfrac{s}{2} \le t_1 \le 1, \\ x_0, & \text{otherwise.} \end{cases}$$

The function is well defined since for $t_1 = \frac{2-s}{2}$ is $k_{\mathcal{U}}(1, t_2, \ldots, t_{n-1}, 2t_n) = x_0$ for $t_1 = \frac{s}{2}$ is $l_{\mathcal{U}}(0, t_2, \ldots, t_{n-1}, 2t_n - 1) = x_0$ and for $t_n = \frac{1}{2}$ is

$$k_{\mathcal{U}}\left(\frac{2t_1}{2-s}, t_2, \ldots, t_{n-1}, 2 \cdot \frac{1}{2}\right) = x_0 = k_{\mathcal{U}}\left(\frac{2t_1 - s}{2-s}, t_2, \ldots, t_{n-1}, 2 \cdot \frac{1}{2} - 1\right).$$

By the definition of the function for all $\underline{t} \in I^n$ is $F(\underline{t}, 0) = (k_{\mathcal{U}} * l_{\mathcal{U}})(\underline{t})$.

We define a function $G : I^n \times I \to X$ by

$$G(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}(2t_1, t_2, \ldots, t_{n-1}, 2t_n - s), & \dfrac{s}{2} \le t_n \le \dfrac{s+1}{2},\ 0 \le t_1 \le \dfrac{1}{2}, \\ l_{\mathcal{U}}(2t_1 - 1, t_2, \ldots, t_{n-1}, 2t_n - 1 + s), & \dfrac{1-s}{2} \le t_n \le \dfrac{2-s}{2},\ \dfrac{1}{2} \le t_1 \le 1, \\ x_0, & \text{otherwise.} \end{cases}$$

The function is well defined since for $t_1 = \frac{1}{2}$ is

$$k_{\mathcal{U}}(1, t_2, \ldots, t_{n-1}, 2t_n - s) = x_0 = l_{\mathcal{U}}(0, t_2, \ldots, t_{n-1}, 2t_n - 1 + s),$$

for $t_n = \frac{s}{2}$ is $k_{\mathcal{U}}(t_1, t_2, \ldots, t_{n-1}, 0) = x_0$, for $t_n = \frac{s+1}{2}$ is $k_{\mathcal{U}}(t_1, t_2, \ldots, t_{n-1}, 1) = x_0$ and analogues for all other points from the edge.

We define a function $H : I^n \times I \to X$ by

$$H(\underline{t}, s) = \begin{cases} k_{\mathcal{U}}\left(\dfrac{2t_1}{1+s}, t_2, \ldots, t_{n-1}, 2t_n - 1\right), & \dfrac{1}{2} \le t_n \le 1,\ 0 \le t_1 \le \dfrac{1+s}{2}, \\ l_{\mathcal{U}}\left(\dfrac{2t_1 - 1 + s}{1+s}, t_2, \ldots, t_{n-1}, 2t_n\right), & 0 \le t_n \le \dfrac{1}{2},\ \dfrac{1-s}{2} \le t_1 \le 1, \\ x_0, & \text{otherwise}, \end{cases}$$

The function is well defined since for $t_n = \frac{1}{2}$ is

$$k_{\mathcal{U}}\left(\frac{2t_1}{1+s}, t_2, \ldots, t_{n-1}, 0\right) = x_0 = l_{\mathcal{U}}\left(\frac{2t_1 - 1 + s}{1+s}, t_2, \ldots, t_{n-1}, 1\right),$$

for $t_1 = \frac{s+1}{2}$ is $k_{\mathcal{U}}(1, t_2, \ldots, t_{n-1}, 2t_n - 1) = x_0$, and for $t_1 = \frac{1-s}{2}$ is

$$l_{\mathcal{U}}(0, t_2, \ldots, t_{n-1}, 2t_n) = x_0.$$

By the definition of the function, for all $\underline{t} \in I^n$ is $H(\underline{t}, 1) = (l_{\mathcal{U}} * k_{\mathcal{U}})(\underline{t})$.

By the definitions of the functions F, G, H and [4, Theorem 2.2], the function K is $\mathrm{st}^{n+1-i}(\mathcal{U})$-continuous on $\partial^i I^{n+1}$ for all $i = 0, \ldots, n+1$, i.e., it is a homotopy relatively endpoints that connects the proximate n-dimensional $\mathcal{U}$-loops $k_{\mathcal{U}} * l_{\mathcal{U}}$ and $l_{\mathcal{U}} * k_{\mathcal{U}}$ at x_0 for all coverings $\mathcal{U} \in \operatorname{Cov} X$, as required.

Therefore, the set $\operatorname{prox} \pi_n(X, x_0)$ with the operation $*$ is a commutative group for all $n \ge 2$. □

5. Induced Function

Let X and Y be paracompact topological spaces, and $\underline{f} = \{f_{\mathcal{V}} \mid \mathcal{V} \in \operatorname{Cov} Y\}$ be a pointed proximate net from (X, x_0) to (Y, y_0).

Definition 5.1. An **induced function** $f_{\text{prox}} : \operatorname{prox} \pi_n(X, x_0) \to prox\pi_n(Y, y_0)$ associated to a pointed proximate net $\underline{f}$ is defined as follows.

Let $[\underline{k}]_{x_0} \in \operatorname{prox} \pi_n(X, x_0)$, where $\underline{k} = \{k_{\mathcal{U}} \mid \mathcal{U} \in \operatorname{Cov} X\}$ be a proximate n-dimensional loop at x_0. Then $\underline{p} = \underline{f} \circ \underline{k} = \{p_{\mathcal{V}} = f_{\mathcal{V}} \circ k_{\mathcal{U}} \mid \mathcal{V} \in \operatorname{Cov} Y\}$ is a proximate n-dimensional loop at y_0 and

$$f_{prox}([\underline{k}]_{x_0}) = [\underline{p}]_{y_0}.$$

We will show that this function is well defined.

Let $\underline{k^0}$ and $\underline{k^1}$ be proximate n-dimensional loops at x_0 from the same homotopy class of proximate loop $[\underline{k}]_{x_0}$. So there exists a homotopy $\underline{K}$ between the proximate n-dimensional loops $\underline{k^0}$ and $\underline{k^1}$. Then the proximate n-dimensional loops $\underline{f} \circ \underline{k^0}$ and $\underline{f} \circ \underline{k^1}$ are homotopic by a homotopy $\underline{f} \circ \underline{K}$. Therefore, the induced function f_{prox} is well defined.

Theorem 5.1. *Let X and Y be paracompact topological spaces, $\underline{f} = \{f_{\mathcal{V}} \mid \mathcal{V} \in \operatorname{Cov} Y\}$ is a pointed proximate net from (X, x_0) to (Y, y_0). Then the induced function f_{prox} : $\operatorname{prox} \pi_1(X, x_0) \to \operatorname{prox} \pi_1(Y, f(x_0))$ is homomorphism.*

Proof. Let $[\underline{k}]_{x_0}, [\underline{l}]_{x_0} \in \operatorname{prox}\pi_n(X, x_0)$. We should show that

$$f_{\text{prox}}\left([\underline{k}]_{x_0} * [\underline{l}]_{x_0}\right) = f_{\text{prox}}\left([\underline{k}]_{x_0}\right) * f_{\text{prox}}\left([\underline{l}]_{x_0}\right).$$

Because

$$\begin{aligned} f_{\text{prox}}\left([\underline{k}]_{x_0} * [\underline{l}]_{x_0}\right) &= f_{\text{prox}}\left([\underline{k} * \underline{l}]_{x_0}\right) = f_{\text{prox}}\left[(k_{\mathcal{U}} * l_{\mathcal{U}})_{\mathcal{U}\in\operatorname{Cov}X}\right]_{x_0} \\ &= \left[(f_{\mathcal{V}}(k_{\mathcal{U}} * l_{\mathcal{U}}))_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0} \end{aligned}$$

and

$$\begin{aligned} f_{\text{prox}}\left([\underline{k}]_{x_0}\right) * f_{\text{prox}}\left([\underline{l}]_{x_0}\right) &= \left[(f_{\mathcal{V}} \circ k_{\mathcal{U}})_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0} * \left[(f_{\mathcal{V}} \circ l_{\mathcal{U}})_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0} \\ &= \left[((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0}, \end{aligned}$$

we should show that

$$\left[(f_{\mathcal{V}}(k_{\mathcal{U}} * l_{\mathcal{U}}))_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0} = \left[((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))_{\mathcal{V}\in\operatorname{Cov}Y}\right]_{y_0}.$$

The equality follows since

$$\begin{aligned} ((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))(t) &= \begin{cases} (f_{\mathcal{V}} \circ k_{\mathcal{U}})(2t), & 0 \le t \le \frac{1}{2}, \\ (f_{\mathcal{V}} \circ l_{\mathcal{U}})(2t-1), & \frac{1}{2} \le t \le 1, \end{cases} \\ &= \begin{cases} f_{\mathcal{V}}(k_{\mathcal{U}}(2t)), & 0 \le t \le \frac{1}{2}, \\ f_{\mathcal{V}}(l_{\mathcal{U}}(2t-1)), & \frac{1}{2} \le t \le 1, \end{cases} \\ &= f_{\mathcal{V}}((k_{\mathcal{U}} * l_{\mathcal{U}})(t)). \end{aligned}$$

□

Since the proximate n-dimensional loop is a pointed proximate net, by Theorem 3.2, the following theorem is valid.

Theorem 5.2. *Let* $\underline{f} = \{f_{\mathcal{V}} \mid \mathcal{V} \in \operatorname{Cov}Y\}$ *be a pointed proximate net from* (X, x_0) *to* (Y, y_0) *and* $\underline{g} = \{g_{\mathcal{W}} \mid \mathcal{W} \in \operatorname{Cov}Z\}$*is a pointed proximate net from* (Y, y_0) *to* (Z, z_0). *For any* $[\underline{k}]_{x_0} \in \operatorname{prox}\pi_n(X, x_0)$ *is true that*

$$\left(\underline{g} \circ \underline{f}\right)_{prox}([\underline{k}]_{x_0}) = g_{prox}(f_{prox}([\underline{k}]_{x_0})).$$

Theorem 5.3. *Let* $\underline{f} = \{f_{\mathcal{V}} \mid \mathcal{V} \in \operatorname{Cov}Y\}$ *and* $\underline{f'} = \{f'_{\mathcal{V}} \mid \mathcal{V} \in \operatorname{Cov}Y\}$ *be two pointed proximate nets from* (X, x_0) *to* (Y, y_0). *For any proximate* n*-dimensional loop* $\underline{k}$ *at* x_0 *if* $\underline{f}$ *and* $\underline{f'}$ *are homotopic, then proximate* n*-dimensional loop* $\underline{f} \circ \underline{k}$ *and* $\underline{f'} \circ \underline{k}$ *at* y_0 *are homotopic.*

Proof. The proof of this theorem is analogues to the proof of one dimensional case given in [6]. □

By Theorem 5.1, 5.2 and 5.3, the following result is obtained.

Theorem 5.4. *Associating* $\operatorname{prox}\pi_n(X, x_0)$ *to a pointed paracompact topological space* (X, x_0) *and associating to a proximate net* $[\underline{f}]_{x_0}$ *the homomorphism*

$$f_{prox} : \operatorname{prox}\pi_n(X, x_0) \to \operatorname{prox}\pi_n(Y, f(x_0))$$

we obtain a functor from category of pointed intrinsic shape to category of groups.

By this theorem is proved that the proximate group of higher order $\operatorname{prox}\pi_n(X, x_0)$ is an invariant of pointed intrinsic shape of a pointed paracompact space (X, x_0) and if (X, x_0) and (Y, y_0) have same pointed intrinsic shape, then their proximate groups of higher order are isomorphic.

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