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ON GENERALIZED BISHOP FRAME OF NULL CARTAN CURVE IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we define generalized Bishop frame of a null Cartan curve in Minkowski 3-space by using its Bishop's frame vector fields. We obtain the Cartan equations according to the generalized Bishop frame and give the relations between the generalized Bishop curvatures and Bishop curvatures. We show that among all null Cartan curves in $\mathbb{E}^3_1$, only the null Cartan cubic has two generalized Bishop frames, one of which coincides with its Bishop frame. We also show that there exists a null Cartan curve whose generalized Bishop curvatures and Bishop curvatures are equal, but whose generalized Bishop frame and Bishop frame do not coincide. As an application, we characterize a k-type null Cartan slant helices for $k \in \{0, 1, 2\}$ according to their generalized Bishop frame and give some examples.

1. Introduction

Relatively parallel adapted frame of a regular curve α in Euclidean space $\mathbb{E}^3$ is introduced by R. L. Bishop in [1]. It contains the tangential vector field T and two normal vector fields N_1 and N_2 obtained by rotating the principal normal and the binormal vector fields N and B in the normal plane $T^{\perp} = \text{Span}\{N, B\}$ of α for an angle $\theta(s) = \int \tau(s)\, ds$, where $\tau(s)$ is the torsion (the second Frenet curvature) of α. The Bishop's frame equations read

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix},$$

where $\kappa_1(s) = \kappa(s)\cos\theta(s)$ is the *first Bishop curvature*, $\kappa_2(s) = \kappa(s)\sin\theta(s)$ is the *second Bishop curvature* and $\kappa(s)$ is the first Frenet curvature of α. Since the vector

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fields N_1' and N_2' are collinear with the tangential vector field T, they make minimal rotations in the planes $N_1^{\perp} = \text{Span}\{T, N_2\}$ and $N_2^{\perp} = \text{Span}\{T, N_1\}$, respectively. For this reason, the Bishop frame is also known as the frame with the *minimal rotation property*. The normal vector fields N_1 and N_2 are called *relatively parallel* vector fields. A new version of Bishop frame in Euclidean space $\mathbb{E}^3$ is introduced in [17]. Some applications of N-Bishop frame are given in [10]. The Bishop frame is also generalized in higher dimensional Euclidean spaces (see [7]).

In Minkowski space $\mathbb{E}^3_1$, the Bishop frame is firstly obtained for the timelike and the spacelike curves with non-null Frenet vector fields in their frames ([13]). Recently, the Bishop frames of pseudo null and null Cartan curves in $\mathbb{E}^3_1$ are introduced and applied in [8]. In Minkowski space-time $\mathbb{E}^4_1$, parallel frame (Bishop frame) of a non-null curve is defined in [5] by using a similar approach as in euclidean case. In the same space, the Bishop frame of a null Cartan curve is obtained in [6]. Some applications of the Bishop frame can be found in [3, 9, 11, 14–16].

In this paper, we show that there exists a new frame for a null Cartan curve in $\mathbb{E}^3_1$ having minimal rotation property. We called such frame the *generalized Bishop frame*. The Bishop frame $\{T_1, N_1, N_2\}$ of a null Cartan curve α in $\mathbb{E}^3_1$ is defined as positively oriented pseudo-orthonormal frame containing the tangent vector field $T_1 = \alpha'$ and two relatively parallel vector fields N_1 and N_2 that make minimal rotations in the planes $N_1^{\perp} = \text{Span}\{T_1, N_1\}$ and $N_2^{\perp} = \text{Span}\{N_1, N_2\}$, respectively ([8]). Namely, the normal components of N_1' and N_2' are assumed to be zero implying that N_1' and N_2' are parallel with N_2. As a consequence of introduced assumption, the Bishop curvatures of a null Cartan curve have the form $k_1(s) = 1$ and $k_2(s)$ is the solution of Riccati differential equation $k_2'(s) = -\frac{1}{2}k_2^2(s) - \tau(s)$, where $\tau(s)$ is the torsion (the second Cartan curvature) of α. It is also proved in [8] that among all non-geodesic null Cartan in $\mathbb{E}^3_1$, only the null Cartan cubic has two Bishop frames (one of them coincides with its Cartan frame).

In this paper, we define generalized Bishop frame of a null Cartan curve in Minkowski 3-space by using its Bishop's frame vector fields. We obtain the Cartan equations according to the generalized Bishop frame and give relations between the generalized Bishop curvatures and Bishop curvatures. In particular, we show that among all null Cartan curves in $\mathbb{E}^3_1$, only the null Cartan cubic has two generalized Bishop frames, one of which coincides with its Bishop frame. We also show that there exists a null Cartan curve whose generalized Bishop curvatures and Bishop curvatures are equal, but whose generalized Bishop frame and Bishop frame do not coincide. As an application, we characterize a k-type null Cartan slant helices for $k \in \{0, 1, 2\}$ according to the generalized Bishop frame in terms of their generalized Bishop curvatures and give some examples.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{2.1}$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \neq 0$ in $\mathbb{E}_1^3$ can have one of three causal characters: it can be *spacelike*, *timelike* or *null* (*lightlike*), if $\langle x, x \rangle$ is positive, negative or zero, respectively. In particular, the vector $x = 0$ is a spacelike. The *norm* (length) of a vector $x \in \mathbb{E}_1^3$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. An arbitrary curve $\alpha : I \to \mathbb{E}_1^3$ can locally be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively (see [12]).

A curve $\beta : I \to \mathbb{E}_1^3$ is called a *null curve*, if its tangent vector $\beta' = T$ is null. A null curve $\beta = \beta(s)$ is called a *null Cartan curve*, if it is parameterized by the pseudo-arc function s defined by (see [2])

$$s(t) = \int_0^t \sqrt{||\beta''(u)||}\, du. \tag{2.2}$$

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve β satisfying the Cartan equations (see [4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & \kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where the first Cartan curvature $\kappa(s) = 1$ and the second Cartan curvature (torsion) $\tau(s)$ is an arbitrary function in pseudo-arc parameter s. If $\tau(s) = 0$, the null Cartan curve is called a *null Cartan cubic*. The Cartan's frame vector fields satisfy the relations

$$\begin{aligned} \langle T, T \rangle = &\langle B, B \rangle = 0, \quad \langle N, N \rangle = 1, \\ \langle T, N \rangle = &\langle N, B \rangle = 0, \quad \langle T, B \rangle = -1, \end{aligned} \tag{2.4}$$

$$T \times N = -T, \quad N \times B = -B, \quad B \times T = N. \tag{2.5}$$

The Cartan frame $\{T, N, B\}$ is *positively oriented*, if $\det(T, N, B) = [T, N, B] = 1$. The Bishop frame of a null Cartan curve in $\mathbb{E}_1^3$ is introduced in [8] as follows.

Definition 2.1. The Bishop frame $\{T_1, N_1, N_2\}$ of a non-geodesic null Cartan curve in $\mathbb{E}_1^3$ is positively oriented pseudo-orthonormal frame consisting of the tangential vector field T_1, relatively parallel spacelike normal vector field N_1 and relatively parallel lightlike transversal vector field N_2.

The spacelike normal vector field N_1 and the lightlike transversal vector field N_2 along a null Cartan curve are said to be *relatively parallel*, if the normal component

$T_1^\perp = \text{Span}\{T_1, N_1\}$ of their derivatives N_1' and N_2' is zero. Thus N_1' and N_2' are collinear with N_2 at each point of the curve and hence they make minimal rotations in the planes $N_1^\perp$ and $N_2^\perp$.

The Bishop's frame vector fields satisfy the relations

$$\begin{aligned} \langle T_1, T_1\rangle =& \langle N_2, N_2\rangle = 0, \quad \langle N_1, N_1\rangle = 1,\\ \langle T_1, N_1\rangle =& \langle N_1, N_2\rangle = 0, \quad \langle T_1, N_2\rangle = -1, \end{aligned} \tag{2.6}$$

$$T_1 \times N_1 = -T_1, \quad N_1 \times N_2 = -N_2, \quad N_2 \times T_1 = N_1. \tag{2.7}$$

The Cartan equations of a null Cartan curve according to Bishop frame are given in [8] as follows.

Theorem 2.1. *Let α be a null Cartan curve in $\mathbb{E}_1^3$ parameterized by pseudo-arc s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T, N, B\}$ of α are related by:*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\kappa_2 & 1 & 0 \\ \frac{\kappa_2^2}{2} & -\kappa_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.8}$$

and the Cartan equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.9}$$

where the first Bishop curvature $\kappa_1(s) = 1$ and the second Bishop curvature satisfies Riccati differential equation

$$\kappa_2'(s) = -\frac{1}{2}\kappa_2^2(s) - \tau(s). \tag{2.10}$$

Throughout the next sections let $\mathbb{R}_0$ denotes $\mathbb{R}\backslash\{0\}$.

3. Generalized Bishop Frame of a Null Cartan Curve

In this section, we introduce generalized Bishop frame of a null Cartan curve $\alpha(s)$ in Minkowski 3-space. Let $\{T_1, N_1, N_2\}$ be the Bishop frame and $\kappa_1(s) = 1$ and $\kappa_2(s)$ the first and the second Bishop curvature of α. Consider a vector field W_0 along α given by

$$W_0 = T_1 + \kappa_2 N_1 + \frac{\kappa_2^2}{2} N_2. \tag{3.1}$$

Relations (2.6) and (3.1) imply $\langle W_0, W_0\rangle = 0$, which means that W_0 is a null vector field. Note that in general case when $\kappa_2 \neq 0$, the vector field W_0 is not tangent to α. Also, consider vector the field W_1 along α of the form

$$W_1 = N_1 + \kappa_2 N_2. \tag{3.2}$$

By using relations (2.6), (3.1) and (3.2), it follows that W_1 satisfies the relations

$$\langle W_1, W_1\rangle = 1, \quad \langle W_0, W_1\rangle = 0. \tag{3.3}$$

Hence W_1 is the unit spacelike vector field orthogonal to W_0. For a given vector fields W_0 and W_1, there exist a unique lightlike vector field W_2 satisfying the conditions

$$\langle W_2, W_2\rangle = 0, \quad \langle W_0, W_2\rangle = -1, \quad \langle W_1, W_2\rangle = 0. \tag{3.4}$$

By using the relations (2.6), (3.1), (3.2), (3.3) and (3.4), we get

$$W_2 = N_2. \tag{3.5}$$

In particular, relations (2.7), (3.1), (3.2) and (3.5) imply $\det(W_0, W_1, W_2) = 1$. Hence the frame $\{W_0, W_1, W_2\}$ is a positively oriented.

Definition 3.1. The generalized Bishop frame of a null Cartan curve α in $\mathbb{E}^3_1$ is positively oriented pseudo-orthonormal frame $\{W_0, W_1, W_2\}$ given by

$$\begin{aligned} W_0 =& T_1 + \kappa_2 N_1 + \frac{\kappa_2^2}{2} N_2, \\ W_1 =& N_1 + \kappa_2 N_2, \\ W_2 =& N_2, \end{aligned} \tag{3.6}$$

where $\{T_1, N_1, N_2\}$ is the Bishop frame of α and κ_2 is the second Bishop curvature of α. The generalized Bishop's frame vector fields satisfy the relations

$$\begin{aligned} \langle W_1, W_1\rangle =& 1, \quad \langle W_0, W_2\rangle = -1, \\ \langle W_0, W_0\rangle =& \langle W_2, W_2\rangle = \langle W_0, W_1\rangle = \langle W_1, W_2\rangle = 0. \end{aligned} \tag{3.7}$$

In the first theorem, we obtain the Cartan equations according to the generalized Bishop frame and the relations between Bishop curvatures and generalized Bishop curvatures.

Theorem 3.1. *Let α be a null Cartan curve in $\mathbb{E}^3_1$ parameterized by pseudo-arc s with the Bishop curvatures $\kappa_1(s) = 1$ and $\kappa_2(s)$. The Cartan equations of α according to its generalized Bishop frame read*

$$\begin{bmatrix} W_0' \\ W_1' \\ W_2' \end{bmatrix} = \begin{bmatrix} w_2 & w_1 & 0 \\ 0 & 0 & w_1 \\ 0 & 0 & -w_2 \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \end{bmatrix}, \tag{3.8}$$

where $w_1(s) = \kappa_1(s) + \kappa_2'(s) - \kappa_2^2(s)$ and $w_2(s) = \kappa_2(s)$.

Proof. Differentiating the equations given in (3.6) with respect to s and using (2.9), we obtain

$$\begin{aligned} W_0'(s) =& \kappa_2(s)W_0(s) + (\kappa_1(s) + \kappa_2'(s) - \kappa_2^2(s))W_1(s), \\ W_1'(s) =& (\kappa_1(s) + \kappa_2'(s) - \kappa_2^2(s))W_2(s), \\ W_2'(s) =& -\kappa_2(s)W_2(s). \end{aligned} \tag{3.9}$$

Let us put

$$w_1(s) = \kappa_1(s) + \kappa_2'(s) - \kappa_2^2(s), \quad w_2(s) = \kappa_2(s). \tag{3.10}$$

Substituting (3.10) in (3.9), we get (3.8). □

We call the functions $w_1(s)$ and $w_2(s)$ the *first* and the *second generalized Bishop curvature* of the null Cartan curve α, respectively. According to Theorem 3.1, the vector fields W_1' and W_2' make minimal rotations in the planes $W_1^{\perp} = \text{Span}\{W_0, W_2\}$ and $W_2^{\perp} = \text{Span}\{W_1, W_2\}$, since they are always collinear with the vector field W_2.

In particular, if α is a null Cartan cubic, by Theorem 2.1 its Bishop curvatures are given by $\kappa_1(s) = 1$, $\kappa_2(s) = \frac{2}{s}$, or by $\kappa_1(s) = 1$, $\kappa_2(s) = 0$. Substituting $\kappa_2(s) = \frac{2}{s}$ and $\kappa_2(s) = 0$ in (3.6), we get the next corollary.

Corollary 3.1. *Among all null Cartan curves in $\mathbb{E}_1^3$, only the null Cartan cubic has two generalized Bishop frames, one of which coincides with its Bishop frame.*

Note that there exist null Cartan curves whose generalized Bishop curvatures and Bishop curvatures satisfy $w_1(s) = \kappa_1(s)$ and $w_2(s) = \kappa_2(s)$. This property is given in the next theorem.

Theorem 3.2. *If the generalized Bishop curvatures and Bishop curvatures of the null Cartan curve α in $\mathbb{E}_1^3$ satisfy*

$$w_1(s) = \kappa_1(s), \quad w_2(s) = \kappa_2(s), \tag{3.11}$$

then α is the null Cartan cubic with parameter equation (*see Figure* 1)

$$\alpha(s) = \left(\frac{s^3}{4} + \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} - \frac{s}{3}\right), \tag{3.12}$$

or the null Cartan curve with parameter equation (*see Figure* 2)

$$\alpha(s) = \left(\frac{s^4}{4} + \frac{1}{16}\ln s, \frac{s^4}{4} - \frac{1}{16}\ln s, \frac{1}{4}s^2\right). \tag{3.13}$$

Proof. If relation (3.11) holds, by using (3.10), we find $\kappa_2'(s) - \kappa_2^2(s) = 0$. The solutions of the previous differential equation are $\kappa_2(s) = 0$ and $\kappa_2(s) = -\frac{1}{s}$. If $\kappa_2(s) = 0$, relation (2.10) implies $\tau(s) = 0$, which means that α is a null Cartan cubic.

If $\kappa_2(s) = -\frac{1}{s}$, relation (2.10) gives $\tau(s) = -\frac{3}{2s^2}$. By using the last equation and Cartan equations (2.3), we get the third order Euler differential equation

$$s^3 T''' - 3sT' + 3T = 0,$$

whose general solution reads

$$T = C_3 s^3 + C_2 s + C_1 \frac{1}{s},$$

where C_1, C_2, C_3 are constant vectors in $\mathbb{E}_1^3$.

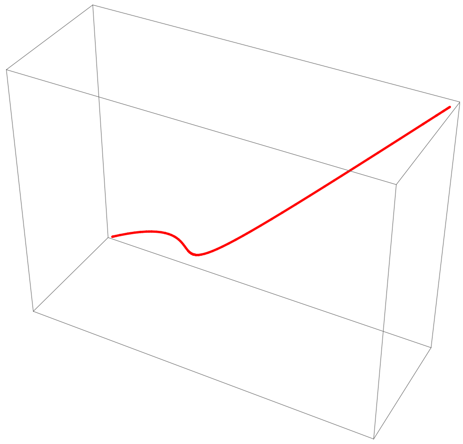

FIGURE 1. Null Cartan cubic

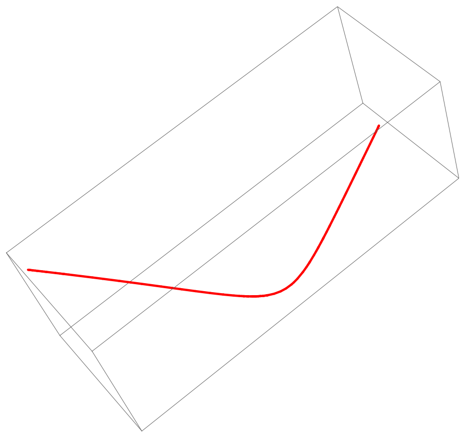

FIGURE 2. Null Cartan curve α

The conditions

$$\langle T, T\rangle = 0, \quad \langle T'', T''\rangle = -\frac{3}{s^2} = 2\tau(s),$$

give

$$T(s) = \left(s^3 + \frac{1}{16s}, s^3 - \frac{1}{16s}, \frac{s}{2}\right).$$

Integrating the last relation, we obtain the null Cartan curve with parameter equation (3.13), which completes the proof. □

Remark 3.1. Note that the null Cartan curve given by (3.13) has equal the corresponding Bishop curvatures and generalized Bishop curvatures, but its Bishop frame and generalized Bishop frame do not coincide.

4. k-Type Null Cartan Slant Helices According to Generalized Bishop Frame

In this section, we define a k-type null Cartan slant helices for $k \in \{0, 1, 2\}$ in Minkowski 3-space, according to their generalized Bishop frame. We give the necessary and the sufficient conditions for null Cartan curve to be the k-type null Cartan slant helix in terms of its generalized Bishop curvatures. We also find the relationship between 0-type and 1-type null Cartan slant helices, as well as between 1-type and 2-type null Cartan slant helices and give some examples.

Definition 4.1. A null Cartan curve α in $\mathbb{E}^3_1$ with the generalized Bishop frame $\{W_0, W_1, W_2\}$ is called a k-type null Cartan slant helix for $k \in \{0, 1, 2\}$, if there exists a non-zero fixed direction $U \in E^3_1$ such that holds

$$\langle W_k, U\rangle = c_0, \quad c_0 \in \mathbb{R}.$$

The fixed direction U spans an *axis* of the helix and it can be a non-null or a null direction. In the first theorem, we characterize 0-type null Cartan slant helices.

Theorem 4.1. *Let $\alpha(s)$ be a null Cartan curve in $\mathbb{E}^3_1$ with generalized Bishop curvatures $w_1(s) \neq 0$ and $w_2(s) \neq 0$. Then α is a 0-type null Cartan slant helix whose axis is not orthogonal to generalized Bishop vector $W_0(s)$ if and only if*

$$c_0 w_2(s) - c_1 w_1(s) \int w_1(s) e^{-\int w_2(s)ds} ds = 0, \tag{4.1}$$

where $c_0 \in \mathbb{R}_0$, $c_1 \in \mathbb{R}^+$.

Proof. Assume that α is a 0-type null slant helix parameterized by the pseudo-arc s. According to Definition 4.1, there exists a fixed direction $U \in \mathbb{E}^3_1$ such that

$$\langle W_0, U\rangle = c_0, \quad c_0 \in \mathbb{R}. \tag{4.2}$$

By assumption, U is not orthogonal to W_0, so $c_0 \in \mathbb{R}_0$. With respect to the generalized Bishop frame $\{W_0, W_1, W_2\}$, the fixed direction U can be decomposed as

$$U(s) = a(s)W_0(s) + b(s)W_1(s) - c_0 W_2(s), \tag{4.3}$$

where $a(s)$ and $b(s)$ are some differentiable functions in s. Differentiating the equation (4.3) with respect to s and using (3.8), we obtain the system of differential equations

$$\begin{cases} a' + aw_2 = 0, \\ aw_1 + b' = 0, \\ bw_1 + c_0 w_2 = 0. \end{cases} \tag{4.4}$$

From the first and the second equation of (4.4), we get

$$\begin{cases} a(s) = c_1 e^{-\int w_2(s)ds}, \\ b(s) = -c_1 \int w_1(s) e^{-\int w_2(s)ds}\, ds, \end{cases} \tag{4.5}$$

where $c_1 \in \mathbb{R}^+$. Substituting (4.5) in the third equation of (4.4), we get relation (4.1).

Conversely, assume that the generalized Bishop curvatures $w_1(s) \neq 0$ and $w_2(s) \neq 0$ of α satisfy relation (4.1). Consider the vector $U(s)$ given by

$$U(s) = c_1 e^{-\int w_2(s)ds} W_0(s) - c_1 \int w_1(s) e^{-\int w_2(s)ds}\, ds W_1(s) - c_0 W_2(s),$$

where $c_0 \in \mathbb{R}_0$ and $c_1 \in \mathbb{R}^+$. Differentiating the previous equation with respect to s and using the equations (3.8), we find $U'(s) = 0$. Hence U is a fixed direction. By using (3.7), it can be easily checked that

$$\langle W_0, U \rangle = c_0,$$

$c_0 \in \mathbb{R}_0$. According to Definition 4.1, α is 0-type null Cartan slant helix whose axis is not orthogonal to W_0. □

Corollary 4.1. *An axis of the* 0*-type null Cartan slant helix* α *in* $\mathbb{E}^3_1$ *with generalized Bishop curvatures* $w_1(s) \neq 0$ *and* $w_2(s) \neq 0$ *is spanned by*

$$U(s) = c_1 e^{-\int w_2(s)ds} W_0(s) - c_1 \int w_1(s) e^{-\int w_2(s)ds}\, ds W_1(s) - c_0 W_2(s), \tag{4.6}$$

where $c_0 \in \mathbb{R}_0$ *and* $c_1 \in \mathbb{R}^+$.

In particular, if an axis of the 0-type null Cartan slant helix is orthogonal to vector field W_0, putting $c_0 = 0$ in (4.4) we get the next theorem.

Theorem 4.2. *Let* α *be a null Cartan curve in* $\mathbb{E}^3_1$ *with generalized Bishop curvatures* $w_1(s)$ *and* $w_2(s)$*. Then* α *is* 0*-type null Cartan slant helix whose axis is orthogonal to vector field* W_0 *if and only if* $w_1(s) = 0$*.*

Corollary 4.2. *If an axis of the* 0*-type null Cartan slant helix is orthogonal to* $W_0(s)$*, then it is spanned by spacelike vector given by*

$$U(s) = c_1 e^{-\int w_2(s)ds} W_0(s) + b_0 W_1(s),$$

where $c_1 \in \mathbb{R}^+$*,* $b_0 \in \mathbb{R}_0$*, or by lightlike vector given by*

$$U(s) = c_2 e^{-\int w_2(s)ds} W_0(s), \quad c_2 \in \mathbb{R}_0.$$

The following theorem gives the relationship between 0-type and 1-type null Cartan slant helices.

Theorem 4.3. *Every* 0*-type null Cartan slant helix with generalized Bishop curvatures* $w_1(s) = 0$ *and* $w_2(s) \neq 0$*, whose axis is orthogonal to generalized Bishop vector* $W_0(s)$*, is also* 1*-type null Catran slant helix with respect to the same axis.*

Theorem 4.4. *Let* α *be a null Cartan curve in* $\mathbb{E}^3_1$ *with generalized Bishop curvatures* $w_1(s) \neq 0$ *and* $w_2(s) \neq 0$*. Then* α *is* 0*-type null Cartan slant helix if and only if its generalized Bishop curvatures satisfy*

$$\left(c_1 \int w_1(s) e^{-\int w_2(s)ds}\, ds\right)^2 + 2c_0 c_1 e^{-\int w_2(s)ds} = c_2, \tag{4.7}$$

where $c_0 \in \mathbb{R}_0$*,* $c_1 \in \mathbb{R}^+$ *and* $c_2 \in \mathbb{R}$*.*

Proof. Assume that α is a 0-type null Cartan slant helix parameterized by the pseudo-arc s. According to Corollary 4.1, the axis of α is spanned by (4.6). By using (4.6) and the condition$\langle U, U\rangle = c_2$, $c_2 \in \mathbb{R}$, it follows that (4.7) holds.

Conversely, assume that (4.7) holds. Differentiating the relation (4.7) with respect to s, we obtain

$$c_0 w_2(s) - c_1 w_1(s) \int w_1(s) e^{-\int w_2(s)ds} ds = 0.$$

According to Theorem 4.1, α is 0-type null Cartan slant helix. □

Next, let us consider 1-type null Cartan slant helices.

Theorem 4.5. *Every null Cartan curve α in $\mathbb{E}^3_1$ with generalized Bishop curvatures $w_1(s) \neq 0$ and $w_2(s) \neq 0$ is 1-type and 2-type null Cartan slant helix with respect to the same axis.*

Proof. It is sufficient to consider the vector $U(s)$ in $\mathbb{E}^3_1$ given by

$$U(s) = c_0 W_1(s) + e^{\int w_2(s)ds}(c_1 - c_0 \int w_1(s) e^{-\int w_2(s)ds} ds) W_2(s), \tag{4.8}$$

where $c_0, c_1 \in \mathbb{R}$ are not both equal to zero. If $c_0 \neq 0$, by using (4.8) and (3.7), it follows $\langle U, W_1\rangle = c_0 \neq 0$ and $\langle U, W_2\rangle = 0$. Differentiating (4.8) with respect to s and using (3.8), we find $U' = 0$. If $c_0 = 0$, substituting $c_0 = 0$ in relation (4.8) we get

$$U(s) = c_1 e^{\int w_2(s)ds} W_2(s), \tag{4.9}$$

where $c_1 \in \mathbb{R}_0$. Differentiating (4.9) with respect to s and using (3.8), we also find $U' = 0$. Therefore, $U(s)$ is a constant vector, so Definition 4.1 implies that α is 1-type and 2-type null Cartan slant helix with respect to the same axis spanned by $U(s)$. □

Corollary 4.3. *An axis of the 1-type and 2-type null Cartan slant helix with generalized Bishop curvatures $w_1(s) \neq 0$ and $w_2(s) \neq 0$ is a spacelike or a lightlike direction.*

In the following theorems, we consider the cases when one of the generalized Bishop curvatures is equal to zero.

Theorem 4.6. *Let α be null Cartan curve in $\mathbb{E}^3_1$ parameterized by pseudo-arc s with generalized Bishop curvatures $w_1(s)$ and $w_2(s)$. Then α is 1-type null Cartan slant helix whose axis is not orthogonal to W_2 if and only if $w_1(s) = 0$.*

Proof. Assume that α is 1-type null Cartan slant helix whose axis is not orthogonal to W_2. According to Definition 4.1, there exists a fixed direction U in $\mathbb{E}^3_1$ such that

$$\langle U, W_1\rangle = c_0, \quad c_0 \in \mathbb{R}. \tag{4.10}$$

With respect to the generalized Bishop frame $\{W_0, W_1, W_2\}$, the fixed direction U can be decomposed as

$$U(s) = a(s)W_0(s) + c_0 W_1(s) + b(s)W_2(s), \tag{4.11}$$

where $a(s)$ and $b(s)$ are some differentiable functions in s. By assumption and relation (3.7), we have $\langle U, W_2\rangle = -a \neq 0$. Differentiating the equation (4.11) with respect to s and using (3.8), we obtain the system of differential equations

$$\begin{cases} a' + aw_2 = 0, \\ aw_1 = 0, \\ c_0 w_1 + b' - bw_2 = 0. \end{cases} \tag{4.12}$$

The second equation of (4.12) implies $w_1(s) = 0$.

Conversely, assume that α has generalized Bishop curvature $w_1(s) = 0$. Consider the vector $U(s)$ given by

$$U(s) = c_1 e^{-\int w_2(s)ds} W_0(s) + c_0 W_1(s) + c_2 e^{\int w_2(s)ds} W_2(s), \tag{4.13}$$

where $c_0, c_2 \in \mathbb{R}$ and $c_1 \in \mathbb{R}_0$. Differentiating the relation (4.13) with respect to s and using (3.8), we find $U'(s) = 0$. Since $\langle U, W_1\rangle = c_0$, α is 1-type null Cartan slant helix whose axis is not orthogonal to the generalized Bishop vector $W_2(s)$. □

Corollary 4.4. *An axis of the* 1*-type null Cartan slant helix* α *in* $\mathbb{E}^3_1$ *with generalized Bishop curvatures* $w_1(s) = 0$ *and* $w_2(s) \neq 0$ *is spanned by*

$$U(s) = c_1 e^{-\int w_2(s)ds} W_0(s) + c_0 W_1(s) + c_2 e^{\int w_2(s)ds} W_2(s), \tag{4.14}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ *are not all equal to zero.*

The next theorem can be proved in a similar way as Theorem 4.4, so we omit its proof.

Theorem 4.7. *Every null Cartan curve* α *in* $\mathbb{E}^3_1$ *with generalized Bishop curvatures* $w_1(s) = 1$ *and* $w_2(s) = 0$ *is* 1*-type and* 2*-type null Cartan slant helix with respect to the same axis spanned by*

$$U(s) = c_0 W_1(s) + (-c_0 s + c_1) W_2(s), \tag{4.15}$$

where $c_0, c_1 \in \mathbb{R}$ *are not both equal to zero.*

Finally, let us consider 2-type null Cartan slant helices. According to Theorem 4.4 and its proof, every null Cartan curve with generalized Bishop curvatures different from zero is 2-type null Cartan slant helix. The next theorem gives the necessary and the sufficient conditions for a null Cartan curve to be the 2-type null Cartan slant helix in terms of its generalized Bishop curvature $w_2(s)$.

Theorem 4.8. *Let* α *be null Cartan curve in* $\mathbb{E}^3_1$ *parameterized by pseudo-arc* s *with generalized Bishop curvatures* $w_1(s)$ *and* $w_2(s)$*. Then* α *is* 2*-type null Cartan slant helix whose axis is not orthogonal to the generalized Bishop vector* $W_2(s)$ *if and only if* $w_2(s) = 0$*.*

Proof. Assume that α is a 2-type null Cartan slant helix whose axis that is not orthogonal to generalised Bishop vector $W_2(s)$. By Definition 4.1, there exists a fixed direction $U(s)$ in $\mathbb{E}^3_1$ such that holds

$$\langle U, W_2\rangle = c_0, \quad c_0 \in \mathbb{R}_0. \tag{4.16}$$

With respect to the generalized Bishop frame $\{W_0, W_1, W_2\}$, the fixed direction U can be decomposed as

$$U(s) = -c_0 W_0(s) + b(s)W_1(s) + c(s)W_2(s), \tag{4.17}$$

where $b(s)$ and $c(s)$ are some differentiable functions in s. Differentiating the equation (4.17) with respect to s and using (3.8), we obtain the system of differential equations

$$\begin{cases} c_0 w_2 = 0, \\ b' - c_0 w_1 = 0, \\ b w_1 + c' - c w_2 = 0. \end{cases} \tag{4.18}$$

Since $c_0 \in \mathbb{R}_0$, from the first equation of (4.18) we get $w_2(s) = 0$.

Conversely, assume that α has generalized Bishop curvature $w_2(s) = 0$. By using relation (3.10), we get $w_1(s) = 1$. Consider the vector $U(s)$ given by

$$U(s) = -c_0 W_0(s) + (c_0 s + c_1)W_1(s) + \left(-c_0\frac{s^2}{2} - c_1 s + c_2\right) W_2(s), \tag{4.19}$$

where $c_0 \in \mathbb{R}_0$, $c_1, c_2 \in \mathbb{R}$. Differentiating the relation (4.19) with respect to s and using (3.8), we find $U'(s) = 0$. Since $\langle U, W_2\rangle = c_0$, α is 2-type null Cartan slant helix whose axis is not orthogonal to the generalized Bishop vector $W_2(s)$. □

Corollary 4.5. *An axis of the 2-type null Cartan slant helix with generalized Bishop curvatures $w_1(s) = 1$ and $w_2(s) = 0$ is spanned by*

$$U(s) = -c_0 W_0(s) + (c_0 s + c_1)W_1(s) + \left(-c_0\frac{s^2}{2} - c_1 s + c_2\right) W_2(s), \tag{4.20}$$

where $c_0 \in \mathbb{R}_0$ and $c_1, c_2 \in \mathbb{R}$.

5. Some Examples

Example 5.1. Let us consider a null Cartan cubic in $\mathbb{E}^3_1$ given by

$$\alpha(s) = \left(\frac{s^3}{4} + \frac{s}{3}, \frac{s^2}{2}, \frac{s^3}{4} - \frac{s}{3}\right),$$

whose Bishop frame coincides with its Cartan frame. According to Theorem 2.1, its Bishop curvatures read $\kappa_1(s) = 1$, $\kappa_2(s) = 0$, so Theorem 3.1 implies that its generalized Bishop curvatures have the form $w_1(s) = \kappa_1(s) = 1$ and $w_2(s) = \kappa_2(s) = 0$. Moreover, the generalized Bishop frame of α coincides with its Bishop and Cartan frame and read

$$\begin{aligned} W_0(s) =& T_1(s) = T(s) = \left(\frac{3s^2}{4} + \frac{1}{3}, s, \frac{3s^2}{4} - \frac{1}{3}\right), \\ W_1(s) =& N_1(s) = N(s) = \left(\frac{3s}{2}, 1, \frac{3s}{2}\right), \\ W_2(s) =& N_2(s) = B(s) = \left(\frac{3}{2}, 0, \frac{3}{2}\right). \end{aligned}$$

By Theorem 4.8, α is 2-type null Cartan slant helix. The axis of α is given by (4.20) and has the form $U(s) = \left(\frac{3c_2}{2} - \frac{c_0}{3}, c_1, \frac{3c_2}{2} + \frac{c_0}{3}\right)$, where $c_0 \in \mathbb{R}_0$ and $c_1, c_2 \in \mathbb{R}$. It can be easily checked that $\langle U, W_2\rangle = c_0$, $c_0 \in \mathbb{R}_0$.

Example 5.2. Let us consider a null Cartan curve in $\mathbb{E}^3_1$ with parameter equation

$$\alpha(s) = \left(\int \frac{e^{4s} - e^{2s} + 1}{2e^s(e^{2s}+1)} ds, \int \frac{e^{4s} - 3e^{2s} + 1}{2e^s(e^{2s}+1)} ds, \int \frac{e^{2s} - 1}{e^{2s}+1} ds\right).$$

The Cartan frame of α is given by

$$\begin{aligned} T(s) =& \left(\frac{e^{4s} - e^{2s} + 1}{2e^s(e^{2s}+1)}, \frac{e^{4s} - 3e^{2s} + 1}{2e^s(e^{2s}+1)}, \frac{e^{2s} - 1}{e^{2s}+1}\right), \\ N(s) =& \left(\frac{e^{7s} + 4e^{5s} - 4e^{3s} - e^s}{2e^{2s}(1+e^{2s})^2}, \frac{e^{7s} + 6e^{5s} - 6e^{3s} - e^s}{2e^{2s}(1+e^{2s})^2}, \frac{4e^{2s}}{(1+e^{2s})^2}\right), \\ B(s) =& \left(\frac{e^{9s} + 13e^{7s} + 36e^{5s} + 13e^{3s} + e^s}{4e^{2s}(1+e^{2s})^3}, \frac{-e^{9s} + 11e^{7s} + 40e^{5s} + 11e^{3s} + e^s}{4e^{2s}(1+e^{2s})^3}, \right. \\ & \left. \frac{-e^{6s} - 5e^{4s} + 5e^{2s} + 1}{2(1+e^{2s})^3}\right). \end{aligned}$$

In particular, the Cartan curvatures of α read

$$\kappa(s) = 1, \quad \tau(s) = \frac{-e^{4s} + 10e^{2s} - 1}{2(1+e^{2s})^2}. \tag{5.1}$$

By using (2.10) and (5.1), it follows that the Bishop curvatures of α are given by

$$\kappa_1(s) = 1, \quad \kappa_2(s) = \frac{1 - e^{2s}}{1 + e^{2s}}. \tag{5.2}$$

According to Theorem 2.1, the Bishop frame of α has the form

$$\begin{aligned} T_1(s) =& \left(\frac{e^{4s} - e^{2s} + 1}{2e^s(e^{2s}+1)}, \frac{e^{4s} - 3e^{2s} + 1}{2e^s(e^{2s}+1)}, \frac{e^{2s} - 1}{e^{2s}+1}\right), \\ N_1(s) =& \left(\frac{e^{2s} - 1}{e^s}, \frac{e^{2s} - 1}{e^s}, 1\right), \\ N_2(s) =& \left(\frac{1 + e^{2s}}{e^s}, \frac{1 + e^{2s}}{e^s}, 0\right). \end{aligned} \tag{5.3}$$

Substituting (5.2) in (3.10), we obtain that generalized Bishop curvatures of α read

$$w_1(s) = 0, \quad w_2(s) = \frac{1 - e^{2s}}{1 + e^{2s}}. \tag{5.4}$$

Since $w_1(s) = 0$, by Theorem 4.2 the curve α is 0-type null Cartan slant helix whose axis is orthogonal to W_0. By using (3.6), (5.2) and (5.3), it follows that the generalized Bishop frame of α is given by

$$\begin{aligned} W_0(s) =& \left(\frac{e^s}{2(1 + e^{2s})}, -\frac{e^s}{2(1 + e^{2s})}, 0 \right), \\ W_1(s) =& (0, 0, 1), \\ W_2(s) =& \left(\frac{1 + e^{2s}}{e^s}, \frac{1 + e^{2s}}{e^s}, 0 \right). \end{aligned} \tag{5.5}$$

By Corollary 4.2, the spacelike axis of α is spanned by

$$U(s) = c_1 e^{-\int w_2(s) ds} W_0(s) + b_0 W_1(s), \tag{5.6}$$

where $c_1 \in \mathbb{R}^+$, $b_0 \in \mathbb{R}_0$. Substituting (5.4) and (5.5) in (5.6), we get $U(s) = (\frac{c_1}{2}, -\frac{c_1}{2}, b_0)$. It can be easily checked that $\langle U, W_0 \rangle = 0$.

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