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STUDY OF (σ,τ)-GENERALIZED DERIVATIONS WITH THEIR COMPOSITION OF SEMIPRIME RINGS

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Abstract. The main purpose of this paper is to study and investigate certain results concerning the (σ,τ)-generalized derivation D associated with the (σ,τ)-derivation d of semiprime and prime rings $\mathbb{R}$, where σ and τ act as two automorphism mappings of $\mathbb{R}$. We focus on the composition of (σ,τ)-generalized derivations of the Leibniz's formula, where we introduce the general formula to compute the composition of the (σ,τ)-generalized derivation D of $\mathbb{R}$.

1. Introduction and Preliminaries

Ring hypothesis is a masterpiece of scientific unification, bringing together a several branches of the subject and offering a capable mechanism with which to consider issues of impressive verifiable and numerical significance. Rings with derivations are not the kind of subject that experiences colossal transformations or great leaps in methodology. Be that as it may, this subject has been subject to the scrutiny of numerous creators in the last 70 years, in particular in terms of the connections between derivations and the structures of rings. The hypothesis of derivations and automorphisms of affiliated rings are a particular milestone in the advancement of classical Galois Hypothesis (cf. Suzuki [35]) and the hypothesis of invariants.

Commutative ring theory is significant insofar as it is one of the foundations of algebraic geometry and complex analytic geometry. The study of generalized derivations of partially ordered set has its roots in the study of the Krull dimension of rings and modules, where the concept of Krull dimension of commutative rings was originally developed by E. Noether and W. Krull in the 1920s.

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Following [29], the fundamental relations between the operations of derivation and those of the addition and multiplication of functions had been recognized for some considerable time because of the perception of the derivative. These relations were used when it was a construct that the operation of differentiation of mappings on the smooth diversity with considerations to a given tangent field not just has the regular properties of differentiation save also conversely, the operation can define the tangent field as fully characterized. Subsequently, the tangent bundle in terms of sheaves of functions could also be reliably defined. It is extremely active the perception of the ring with derivation and has a highly significant role in the integration of analysis, algebraic geometry and algebra. From the rule of common experience, the idea of a Picard-Vessiot approach to the Galois concept of linear regular differential equations (see van der Placed and Singer [36] for information). Moreover, that the concept covered derivation in its inventory of tools. The classical operation of differentiation of monotony on sorts gave us the perception of differentiation of singular chains on types which is a fundamental notion of the topological and algebraic principle of homology. Differential algebra, which represented a new branch of algebra, was first introduced in the 1950s through the work of Ritt [33] and, in 1973, Kharchenko [23], who wrote the classical texts on differential algebra. The search for derivations in rings actually began many decades ago, but it was only after Posner [35], who in 1957 established two very striking results relating to derivations of prime rings, that this field of study gained a particular impetus. A common question is that of why should derivations should be studied. Initially, we used derivations of rings to help us gain a better understanding of rings, and in particular a description of the structure of rings. For instance, a ring is *commutative* if, and only if, the only *inner derivation* of the ring is zero. Also, derivations can be helpful in relating a ring to the set of matrices with entries in the ring (see [30]). Additionally, derivations can play a significant role in determining whether a ring is commutative or otherwise (see [1, 2, 5–7, 10] and [19]).

Derivations can be useful in other fields. For example, they play a significant role in the calculation of matrix eigenvalues (see [9]), which is important in mathematics and other sciences, business, engineering and quantum physics (see [32]). Derivations can be added and subtracted and still produce a derivation, but when we compose a derivation with itself we do not necessarily get a derivation.

Let $\mathbb{R}$ be an associative ring. A map $d : \mathbb{R} \to \mathbb{R}$ is a *derivation* of $\mathbb{R}$ if d is additive and satisfies the *Leibnitz' rule*: $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathbb{R}$. A simple example is, of course, the usual derivative of various algebras consisting of differentiable functions. Basic examples in noncommutative rings are, however, quite different.

Note that $[a, xy] = [a, x]y + x[a, y]$ for all $a, x, y \in \mathbb{R}$. For a fixed $a \in \mathbb{R}$, we define d: $\mathbb{R} \to \mathbb{R}$ by $d(x) = [x, a]$ for all $x \in \mathbb{R}$. Function d is additive and $d(xy) = [xy, a] = x[y, a] + [x, a]y = xd(y) + d(x)y$ for all $x, y \in \mathbb{R}$. Thus, d is referred to as an *inner derivation* of $\mathbb{R}$ associated with a, and is generally denoted I_a. It is obvious that every inner derivation on a ring is a derivation; however, one can find plenty of examples of derivations which are not inner.

Throughout the current paper, $\mathbb{R}$ will denote an associative ring with a *center* $Z(\mathbb{R})$. Any $x, y \in \mathbb{R}$ the bracket symbol, $[x, y]$, represents the *commutator* $xy - yx$ and for a non-empty subset S of $\mathbb{R}$, whilst the set of *all commutators* of elements of S will be written $[S, S]$; a similar convention is adopted for $xoy = xy + yx$. We will always use the commutator formulae $[x, yz] = y[a, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]b$ for $x, y, z \in \mathbb{R}$. Also, let σ, τ be any two endomorphisms of $\mathbb{R}$. For any $x, y \in \mathbb{R}$, we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(xoy)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$ for all $x, y \in \mathbb{R}$.

Recall that the ring $\mathbb{R}$ is *prime* if $x\mathbb{R}y = 0$ implies $x = 0$ or $y = 0$, and *semiprime* if $x\mathbb{R}x = 0$ implies $x = 0$. In fact, a prime ring is semiprime, but the converse is not true in general. Given an integer $n > 1$, the ring $\mathbb{R}$ is said to be (*n-torsion free*) if for $x \in \mathbb{R}$ $nx = 0$, which implies $x = 0$. An additive mapping d: $\mathbb{R} \to \mathbb{R}$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathbb{R}$. Let σ and τ be two automorphisms of $\mathbb{R}$. An additive mapping d: $\mathbb{R} \to \mathbb{R}$ is called a *(σ,τ)-derivation* if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$. Of course, a *$(1,1)$-derivation* where 1 is the identity map of $\mathbb{R}$ is a derivation. An additive mapping $D : \mathbb{R} \to \mathbb{R}$ is called a *generalized derivation* if there exists a derivation $d : \mathbb{R} \to \mathbb{R}$ such that $D(xy) = D(x)y + xd(y)$ holds for all $x, y \in \mathbb{R}$. Obviously, every derivation is a generalized derivation of $\mathbb{R}$, but the converse is not true in general. A significant example is a map of the form $D(x) = ax + xb$, where for some $a, b \in \mathbb{R}$ such generalized derivations are called *inner*. Generalized derivations have been primarily studied in operator algebras; therefore, any investigation from an algebraic point of view might be interesting in itself (see for example [19] and [25]). In this way, generalized derivation covers both the concepts of derivation and the left multiplier of $\mathbb{R}$.

Inspired by the definition of (σ,τ)-derivation, the notion of generalized derivation was extended as follows: let σ, τ be two automorphisms of $\mathbb{R}$. An additive mapping $D :$ $\mathbb{R} \to \mathbb{R}$ is called a *generalized* *(σ,τ)-derivation* of $\mathbb{R}$ if there exists a (σ,τ)-derivation $d : \mathbb{R} \to \mathbb{R}$ such that $D(xy) = D(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in \mathbb{R}$. Of course, a generalized (1,1)-derivation is a generalized derivation on $\mathbb{R}$, where 1 is the identity mapping on $\mathbb{R}$. A mapping $d : \mathbb{R} \to \mathbb{R}$ is said to be *centralizing* if $[d(x), x] \in Z(\mathbb{R})$ holds for all $x \in \mathbb{R}$. In the special case when $[d(x), x] = 0$, the mapping d is said to be *commuting* on $\mathbb{R}$. Furthermore, a mapping d: $\mathbb{R} \to \mathbb{R}$ is said to be (σ,τ)-*centralizing* (resp. *(σ,τ)-commuting*) if $[d(x), x]_{\sigma,\tau} \in Z(\mathbb{R})$ (resp.$[d(x), x]_{\sigma,\tau} = 0$) holds for all $x \in \mathbb{R}$. Of course, a $(1,1)$-centralizing (resp. $(1,1)$-commuting) mapping is a *centralizing* (resp. *commuting*) on $\mathbb{R}$.

In fact, there are some applications of (σ,τ)-derivations which can help to develop an approach to deformations of Lie algebras, and which have various applications in modelling quantum phenomena and in the analysis of complex systems. The map has been extensively investigated in pure algebra. Recently, it has been treated for Banach algebra theory.

There are several results in the existing literature that deal with centralizing and commuting mappings in rings. The study of centralizing mappings was first introduced by E. C. Posner [31], who stated that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (referred to as Posner's Second Theorem). In an attempt to generalize the above result, J. Vukman [38] confirmed that if $\mathbb{R}$ is a 2-torsion free prime ring and d: $\mathbb{R} \to \mathbb{R}$ is a nonzero derivation such that the map $x \rightsquigarrow [d(x), x]$ is commuting on $\mathbb{R}$, then $\mathbb{R}$ is commutative.

M. J. Atteya [8] gave the proof that if $\mathbb{R}$ is a 2-torsion free semiprime ring and $d : \mathbb{R} \to \mathbb{R}$ is a derivation of $\mathbb{R}$ such that $d^n(x \circ y) \pm (x \circ y) \in Z(\mathbb{R})$ for all $x, y \in \mathbb{R}$, then there exists C and an additive mapping ξ: $\mathbb{R} \to C$ such that $d(x) = \lambda x + \xi(x)$ for all $x \in \mathbb{R}$, where n is a fixed positive integer. Ö. Gölbaşı and E. Koç [18] proved that (f, d) is a generalized (σ, τ)-derivation of a prime ring $\mathbb{R}$ with char$(\mathbb{R}) \neq 2$. If $af(x) = 0$ for all $x \in \mathbb{R}$, then $a = 0$ or $d = 0$. M. Ashraf, A. Khan and C. Haetinger [4] showed that under certain conditions for the prime ring $\mathbb{R}$, every Jordan (σ, τ)-higher derivation of $\mathbb{R}$ is a (σ, τ)-higher derivation of $\mathbb{R}$. B. Dhara and A. Pattanayak [30] proved that if $\mathbb{R}$ is a semiprime ring, I a nonzero ideal of $\mathbb{R}$, and σ and τ are two epimorphisms of $\mathbb{R}$, an additive mapping $F : \mathbb{R} \to \mathbb{R}$ is a *generalized* (σ, τ)*-derivation* of $\mathbb{R}$ if there exists a (σ, τ)-derivation $d : \mathbb{R} \to \mathbb{R}$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$. If $\tau(I)d(I) \neq 0$, then $\mathbb{R}$ contains a nonzero central ideal of $\mathbb{R}$ if the condition $F[x, y] = \pm(xoy)_{\sigma,\tau}$ holds.

Moreover, the results determined by Ajda Fošner in [15] concentrated on the assumption that I was a separated set of an M-bimodule contained in the algebra generated by all idempotents in A, and let α, β be endomorphisms of A such that $\alpha(I) = I$, $\beta(I) = I$. Then, every local generalized (α, β)-derivation (local (α, β)-derivation, resp.) from an algebra A into an A-bimodule M is a generalized (α, β)-derivation ((α, β)-derivation, resp.), while in [16] the Hyers-Ulam-Rassias stability of $(m, n)_{(\alpha,\beta)}$-derivations on normed algebras, where m and n are non-negative integers, were studied.

Conversely, Marubayashi et al. [27] stated numerous results connecting derivations, (σ, τ)-derivations and generalized derivations to the generalized (σ, τ)-derivation of $\mathbb{R}$. More precisely, the authors studied the commutativity of a prime ring $\mathbb{R}$ admitting a generalized (σ, τ)-derivation F, satisfying certain conditions such as $[F(x), x]_{\sigma,\tau} = 0$ for all x, y in an appropriate subset of $\mathbb{R}$, where σ, τ are automorphisms of $\mathbb{R}$. Recently, Ajda Fošner and M. J. Atteya in [17] introduced the concept of semigeneralized semiderivations of semiprime rings, alongside some of the associated results.

Throughout the present paper, we shall use, without explicitly mentioning, the following basic identities:

$$\begin{aligned}
[xy, z]_{\sigma,\tau} =& x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y, \\
[x, yz]_{\sigma,\tau} =& \sigma(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z), \\
(xo(yz))_{\sigma,\tau} =& (xoy)_{\sigma,\tau}\sigma(z) - \tau(y)(xoz)_{\sigma,\tau} = \tau(y)(xoz)_{\sigma,\tau} + (xoy)_{\sigma,\tau}\sigma(z).
\end{aligned}$$

In the present paper, we establish a number of results concerning the (σ,τ) - generalized derivation D associated with the derivation d of the semiprime ring and prime ring $\mathbb{R}$, in addition to presenting the general formula for the composition of a (σ,τ)-generalized derivation D, and some example applications of such.

2. The Basic Facts

We begin with the following known results, on which our derivation subsequently depends.

Lemma 2.1. ([40, Lemma 3]). *Let $\mathbb{R}$ be a semiprime ring and let $d:\mathbb{R}\to\mathbb{R}$ be an additive mapping. If either $d(x)x=0$ or $xd(x)=0$ holds for all $x\in\mathbb{R}$, then $d=0$.*

Lemma 2.2. ([28, Problem 14, p. 9]). *A seminear-ring $\mathbb{R}$ has no non-zero nilpotent elements if and only if $x^2=0$ implies $x=0$ for all $x\in\mathbb{R}$.*

Lemma 2.3. ([42, Lemma 3.1]). *Let $\mathbb{R}$ be a semiprime ring and let $a\in\mathbb{R}$. If $a[x,y]=0$ for all $x,y\in\mathbb{R}$, then there exists an ideal I of $\mathbb{R}$ such that $a\in I\subset Z(\mathbb{R})$.*

Lemma 2.4. ([41, Lemma 2.4]). *Let $\mathbb{R}$ be a semiprime ring and let $a\in\mathbb{R}$. Then $[a,[a,x]]=0$ holds for all $x\in\mathbb{R}$ if and only if $a^2, 2a\in Z(\mathbb{R})$.*

Lemma 2.5. ([11, Lemma 2]). *Let $\mathbb{R}$ be a prime ring. If $a,b,c\in\mathbb{R}$ are such that $axb=cxa$ for all $x\in\mathbb{R}$, then either $a=0$ or $c=b$.*

Lemma 2.6. ([20, Lemma 1.1.8]). *Let $\mathbb{R}$ be a semiprime ring and let $a\in\mathbb{R}$. If $[a,[x,y]]\in Z(\mathbb{R})$ for all $x,y\in\mathbb{R}$, then $a\in Z(\mathbb{R})$.*

3. The Main Results

Theorem 3.1. *Let $\mathbb{R}$ be a 3-torsion free semiprime ring and let σ, τ be automorphism mappings of $\mathbb{R}$. If D is a generalized (σ,τ)-derivation such that $D^2(\mathbb{R})=0$, then $d=0$.*

Proof. For any $x\in\mathbb{R}$, we have $D^2(x)=0$. Replacing x with xy, we get

$$\begin{aligned} D(D(xy)) =& D(D(x)\sigma(y)+\tau(x)d(y))=0, \quad x,y\in\mathbb{R},\\ D(D(xy)) =& D(D(x)\sigma(y))+D(\tau(x)d(y))=0, \quad x,y\in\mathbb{R}, \end{aligned}$$

and

$$D^2(x)\sigma(y)+\tau(D(x))d(\sigma(y))+D(\tau(x))\sigma(d(y))+\tau(x)d^2(y)=0.$$

In view of the mappings, σ and τ act as automorphism mappings of $\mathbb{R}$ and we obtain

$$D^2(x)y+2D(x)d(y)+xd^2(y)=0. \tag{3.1}$$

According to the relation $D^2(\mathbb{R})=0$, the first item of equation (3.1) becomes zero, which produces

$$2D(x)d(y)+xd^2(y)=0. \tag{3.2}$$

In relation (3.2), we reconstitute y by ty, $t \in \mathbb{R}$, we gain

$$2D(x)(d(t)\sigma(y) + \tau(t)d(y)) + xd(d(t)\sigma(y) + \tau(t)d(y)) = 0.$$

Then, after simple calculation, from the left side of the above relation we get the following

$$\begin{aligned} 2D(x)d(t)\sigma(y) + 2D(x)\tau(t)d(y)) + xd^2(t)\sigma^2(y) \\ + x\tau(d(t))d(y) + xd(\tau(t))\sigma(d(y)) + x\tau^2(t)d^2(y) = 0. \end{aligned} \tag{3.3}$$

In agreement with the relation (3.2), the first item of relation (3.3) becomes $-xd^2(t)y$, which is cancelled with the item $xd^2(t)\sigma^2(y)$ where, based on the fact that the mappings σ and τ act as automorphism mappings of $\mathbb{R}$, relation (3.3) becomes

$$2D(x)\tau(t)d(y)) + x\tau(d(t))d(y) + xd(\tau(t))\sigma(d(y)) + x\tau^2(t)d^2(y) = 0. \tag{3.4}$$

Conforming the fact that σ and τ act as automorphism mappings of $\mathbb{R}$, the relation (3.4) produces

$$2D(x)td(y)) + 2xd(t)d(y) + xtd^2(y) = 0. \tag{3.5}$$

Again, relation (3.2) modifies relation (3.5) to

$$2D(x)td(y)) + 2xd(t)d(y) - 2xD(t)d(y) = 0. \tag{3.6}$$

Restitution of x by $D(x)$ and application of the relation of hypothesis $D^2(\mathbb{R}) = 0$ reduces relation (3.6) to

$$2D(x)d(t)d(y) - 2D(x)D(t)d(y) = 0. \tag{3.7}$$

We rewrite relation (3.7) in agreement with relation (3.2) as

$$-xd^2(t)d(y) + D(x)td^2(y) = 0.$$

The above relation replacing x with $D(x)$ is dependent on the fact that $D^2(\mathbb{R}) = 0$, from which we achieve

$$D(x)d^2(t)d(y) = 0. \tag{3.8}$$

Replacing y with ys, $s \in \mathbb{R}$, in relation (3.8) and applying the result, we propose $D(x)d^2(t)yd(s) = 0$. Replacing y with $yD(x)$ and s with $d(t)$ for the reason that $\mathbb{R}$ is a semiprime ring, we have

$$D(x)d^2(t) = 0. \tag{3.9}$$

Again, in relation (3.2) we set $y = d(y)$, and we retain $2D(x)d^2(y) + xd^3(y) = 0$. Obviously, according to relation (3.9) with consideration for the semiprime nature of $\mathbb{R}$, we can say $d^3(y) = 0$. In the above result, replacing y by xy through application of the relation $d^3(y) = 0$, we achieve $3d^2(x)d(y) + 3d(x)d^2(y) = 0$, for all $x, y \in \mathbb{R}$. Substitution of x with $d(x)$ will depend on the hypothesis that $\mathbb{R}$ is a 3-torsion free semiprime ring, in which the above relation reduces to $d^2(x)d^2(y) = 0$. Right-multiplying the above relation by $rd^2(x)$, $r \in \mathbb{R}$, and left-multiplying by $d^2(x)r$, $r \in \mathbb{R}$, with applying Lemma 2.2, we obtain $d^2(x) = 0$.

Using the same argument as in the last part of the proof, we receive the required result. $\square$

Theorem 3.2. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ,τ)-generalized derivation D such that $[D(x),x]_{\sigma,\tau}=0$ for all $x\in\mathbb{R}$, then*

(i) *if the generalized derivation D is the commuting mapping of $\mathbb{R}$ then d is commuting mapping of $\mathbb{R}$;*

(ii) *if the derivation d is the commuting mapping of $\mathbb{R}$ then D is the 2-commuting mapping of $\mathbb{R}$.*

Proof. (i) Let us introduce the mapping $\gamma:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ for all $x,y\in\mathbb{R}$ by the relation

$$\gamma(x,y)=[D(x),y]_{\sigma,\tau}+[D(y),x]_{\sigma,\tau}.$$

We have the fact that γ is symmetric, which means $\gamma(x,y)=\gamma(y,x)$ for all $x,y\in\mathbb{R}$ is additive in both the arguments. Notice that for all $x,y,z\in\mathbb{R}$,

$$\gamma(xy,z)=[D(xy),z]_{\sigma,\tau}+[D(z),xy]_{\sigma,\tau}.$$

Previously, we had

$$\gamma(xy,z)=[D(x)\sigma(y)+\tau(x)d(y),z]_{\sigma,\tau}+[D(z),xy]_{\sigma,\tau}.$$

After that, from the right-hand side of the above relation we gain the following equation

$$\begin{aligned}\gamma(xy,z)=&D(x)[\sigma(y),z]_{\sigma,\tau}+[D(x),z]_{\sigma,\tau}\sigma(y)\\&+\tau(x)[d(y),z]_{\sigma,\tau}+[\tau(x),z]_{\sigma,\tau}d(y)+x[D(z),y]_{\sigma,\tau}+[D(z),x]_{\sigma,\tau}y.\end{aligned}\tag{3.10}$$

On account of σ and τ acting as an automorphism mappings of $\mathbb{R}$, then

$$\begin{aligned}\gamma(xy,z)=&D(x)[y,z]_{\sigma,\tau}+[D(x),z]_{\sigma,\tau}y+x[d(y),z]_{\sigma,\tau}+[x,z]_{\sigma,\tau}d(y)\\&+x[D(z),y]_{\sigma,\tau}+[D(z),x]_{\sigma,\tau}y.\end{aligned}$$

Again we suggest an additive mapping κ from $\mathbb{R}$ onto itself by $\kappa(x)=\gamma(x,x)$. Then, we gain $\kappa(x)=2[D(x),x]_{\sigma,\tau}$. In fact, the mapping κ satisfies the following relation

$$\begin{aligned}\kappa(x+y)=&2[D(x+y),x+y]_{\sigma,\tau}\\=&2([D(x),x]_{\sigma,\tau}+[D(x),y]_{\sigma,\tau}+[D(y),x]_{\sigma,\tau}+[D(y),y]_{\sigma,\tau}).\end{aligned}\tag{3.11}$$

By reason of $\kappa(x)=2[D(x),x]_{\sigma,\tau}$, the relation (3.11) yields that

$$\kappa(x+y)=\kappa(x)+\kappa(y)+2\gamma(x,y).\tag{3.12}$$

According to the fact that κ is an additive mapping, the relation (3.12) becomes $2\gamma(x,y)=0$. In view of $\mathbb{R}$ being a 2-torsion free semiprime ring, we gain $\gamma(x,y)=0$. Now, in the relation $\gamma(x,y)=0$, replacing y with xy, we get

$$\gamma(x,xy)=[D(x),xy]_{\sigma,\tau}+[D(xy),x]_{\sigma,\tau}=0.$$

Then

$$[D(x), xy]_{\sigma,\tau} + [D(x)\sigma(y) + \tau(x)d(y), x]_{\sigma,\tau} = 0.$$

Consequently, from the left-hand side of the above relation, we obtain the following equation

$$\begin{aligned} x[D(x), y]_{\sigma,\tau} &+ [D(x), x]_{\sigma,\tau}y + D(x)[\sigma(y), x]_{\sigma,\tau} \\ &+ [D(x), x]_{\sigma,\tau}\sigma(y) + \tau(x)[d(y), x]_{\sigma,\tau} + [\tau(x), x]_{\sigma,\tau}d(y) = 0. \end{aligned} \tag{3.13}$$

Subsequently, the mappings σ and τ act as automorphism of $\mathbb{R}$, and the relation (3.13) becomes

$$x([D(x), y]_{\sigma,\tau} + [d(y), x]_{\sigma,\tau}) + D(x)[y, x]_{\sigma,\tau} + \kappa(x)\sigma(y) = 0,$$

where the additive mapping D acts as a commuting of $\mathbb{R}$, i.e., $\kappa(x) = 0$, and replacing y with x so the above relation is changed to

$$x[d(x), x]_{\sigma,\tau} = 0. \tag{3.14}$$

Now we left-multiply the relation (3.14) by $[d(x), x]_{\sigma,\tau}$ and right-multiply by x to gain $([d(x), x]_{\sigma,\tau}x)^2 = 0$ for all $x \in \mathbb{R}$. By applying Lemma 2.2 to the above relation and subtracting the result from relation (3.14), we achieve $[[d(x), x]_{\sigma,\tau}, x]_{\sigma,\tau} = 0$ and we get the required result. This completes the proof.

(ii) Let a derivation d acts as commuting of $\mathbb{R}$. In relation (3.10) in part (i), we reconstitute z and y with x by applying the fact that the mappings σ and τ act as automorphisms and d acts as commuting of $\mathbb{R}$, we achieve

$$2[D(x), x]x = -x[D(x), x], \tag{3.15}$$

where we suppose the mapping $\kappa = 2[D(x), x]$, with which relation (3.15) becomes $\kappa(x)x = -x[D(x), x]$ for all $x \in \mathbb{R}$. In above relation, by linearizing x by $x + y$, we gain

$$\begin{aligned} &\kappa(x)x + \kappa(y)x + \kappa(x)y + \kappa(y)y \\ =& - x([D(x), x] + [D(x), y] + [D(y), x] + [D(y), y]) - y([D(x), x] + [D(x), y] \\ &+ [D(y), x] + [D(y), y]), \quad x, y \in \mathbb{R}. \end{aligned}$$

According to the relation $\kappa(x)x = -x[D(x), x]$, the relation should be

$$\begin{aligned} &- x[D(x), x] + \kappa(y)x + \kappa(x)y - y[D(y), y] \\ =& - x([D(x), x] + [D(x), y] + [D(y), x] + [D(y), y]) - y([D(x), x] + [D(x), y] \\ &+ [D(y), x] + [D(y), y]), \quad x, y \in \mathbb{R}. \end{aligned}$$

Again, we depend on the fact that $\kappa(x)x = -x[D(x), x]$, and by replacing y with x, we obtain $-2x[D(x), x]_{\sigma,\tau} = -6x[D(x), x]_{\sigma,\tau}$. In consideration of the fact that $\mathbb{R}$ is a 2-torsion free semiprime ring, we get $x[D(x), x]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$. Then

$$xD(x)x = x^2D(x). \tag{3.16}$$

Substituting the result $x[D(x), x]_{\sigma,\tau} = 0$ into relation (3.15), we gain

$$xD(x)x = D(x)x^2. \tag{3.17}$$

Obviously, the subtraction of equations (3.17) and (3.16) gives $[D(x), x^2]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$. Depending on this relation, we complete the proof. ☐

Proposition 3.1. *Let $\mathbb{R}$ be a ring and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ,τ)-generalized derivation D such that $D(x)[x,y]_{\sigma,\tau} = 0$. Then*

(i) *if $\mathbb{R}$ acts as semiprime ring then (σ,τ)-generalized derivation D is commuting of $\mathbb{R}$;*
(ii) *if $\mathbb{R}$ acts as prime ring then either (σ,τ)-generalized derivation $D = 0$ of $\mathbb{R}$ or $\mathbb{R}$ is commutative ring.*

Proof. (i) In the main relation $D(x)[x,y]_{\sigma,\tau} = 0$ for all $x, y \in \mathbb{R}$. Replacing y by yt, $t \in \mathbb{R}$, we gain

$$D(x)y[x,t]_{\sigma,\tau} + D(x)[x,t]_{\sigma,\tau}t = 0. \tag{3.18}$$

Obviously, the second term of the relation (3.18) depends on the main relation becoming zero, which leads to $D(x)y[x,t]_{\sigma,\tau} = 0$ for all $x, y, t \in \mathbb{R}$. According to the hypothesis, $\mathbb{R}$ is a semiprime ring, so in the above relation we replace y with $x\mathbb{R}$ and t by $D(x)$ to get

$$D(x)x\mathbb{R}[x, D(x)]_{\sigma,\tau} = 0. \tag{3.19}$$

Again, in the previous relation, we substitute y by $\mathbb{R}$ and left-multiply by x, after which we subtract the result to produce relation (3.19), and we obtain $[D(x), x]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$. Straightforwardly, we gain the fact that D is commuting (σ,τ)-generalized derivation of $\mathbb{R}$.

(ii) When $\mathbb{R}$ acts as prime ring, we again depend on the following relation

$$D(x)y[x,t]_{\sigma,\tau} = 0, \quad x, y, t \in \mathbb{R}.$$

Restitution of y with $\mathbb{R}$ in the above relation produces $D(x)\mathbb{R}[x,t]_{\sigma,\tau} = 0$. In agreement with $\mathbb{R}$ acting as a prime ring, we gain the following results: either $D(x) = 0$ or $[x,t]_{\sigma,\tau} = 0$. Now we discuss the above options. Let $[x,t]_{\sigma,\tau} \neq 0$. Obviously, we get $D = 0$ for $\mathbb{R}$. Otherwise, if D is nonzero, we get $[x,t]_{\sigma,\tau} = 0$ for all $x, y \in \mathbb{R}$. Conforming to σ and τ being two automorphism mappings of $\mathbb{R}$, we get the result that $\mathbb{R}$ is commutative. ☐

Theorem 3.3. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ,τ)-generalized derivation D such that $D(xy) = D(yx)$ for all $x, y \in \mathbb{R}$, then the derivation d is the commuting mapping of $\mathbb{R}$.*

Proof. Suppose $c \in \mathbb{R}$ is a constant, i.e., an element such that $D(c) = 0$, and let c be an arbitrary element of $\mathbb{R}$. According to the hypothesis, we have $D(xy) = D(yx)$ for all $x, y \in \mathbb{R}$. We replace x with c and y with z, through which we get $D(cz) = D(zc)$ for all $z \in \mathbb{R}$. Then

$$D(c)\sigma(z) + \tau(c)d(z) = D(z)\sigma(c) + \tau(z)d(c). \tag{3.20}$$

Applying the fact that $D(c) = 0$ to the relation (3.20), we have

$$\tau(c)d(z) = \tau(z)d(c). \tag{3.21}$$

Now, for all $x, y \in \mathbb{R}$, the commutator $[x, y]_{\sigma,\tau}$ is a constant and hence from relation (3.21) we obtain

$$\tau([x, y]_{\sigma,\tau})d(z) = \tau(z)d([x, y]_{\sigma,\tau}), \quad \text{for all } x, y, z \in \mathbb{R}.$$

As stated in the hypothesis, let τ be a automorphism mapping of $\mathbb{R}$ and the above equation becomes

$$[x, y]_{\sigma,\tau}d(z) = zd([x, y]_{\sigma,\tau}). \tag{3.22}$$

In equation (3.22), replacing z by zr, we get

$$[x, y]_{\sigma,\tau}d(z)\sigma(r) + [x, y]_{\sigma,\tau}\tau(z)d(r) = zrd([x, y]_{\sigma,\tau}),$$

for all $x, y, z, r \in \mathbb{R}$. In agreement with relation (3.22) the above equation can be modified to $zd([x, y])\sigma(r) + [x, y]_{\sigma,\tau}\tau(z)d(r) = zrd([x, y]_{\sigma,\tau})$. Then

$$z[d([x, y]_{\sigma,\tau}), r] = -[x, y]_{\sigma,\tau}zd(r). \tag{3.23}$$

Substituting r for $d([x, y]_{\sigma,\tau})$, we obtain that $[x, y]_{\sigma,\tau}zd^2([x, y]_{\sigma,\tau}) = 0$ for all $x, y, z \in \mathbb{R}$. Left-multiplying by $d^2([x, y]_{\sigma,\tau})$ and right-multiplying by $[x, y]_{\sigma,\tau}$, we gain

$$d^2([x, y])[x, y]_{\sigma,\tau}\mathbb{R}d^2([x, y]_{\sigma,\tau})[x, y]_{\sigma,\tau} = 0.$$

According to the hypothesis that $\mathbb{R}$ is a semiprime ring, we achieve

$$d^2([x, y]_{\sigma,\tau})[x, y]_{\sigma,\tau} = 0.$$

We establish that $a = d^2([x, y]_{\sigma,\tau})$ in the above relation, and by applying Lemma 2.3, we achieve $a \in Z(\mathbb{R})$, i.e., $d^2([x, y]_{\sigma,\tau}) \in Z(\mathbb{R})$. In the relation (3.22), we substitute z with $d([x, y]_{\sigma,\tau})$, and use the relationship $d^2([x, y]_{\sigma,\tau})[x, y]_{\sigma,\tau} = 0$, where $d^2([x, y]_{\sigma,\tau}) \in Z(\mathbb{R})$, and we thus have $d([x, y]_{\sigma,\tau})^2 = 0$ for all $x, y \in \mathbb{R}$, i.e., we obtain $d([x, y]_{\sigma,\tau})^2 \in Z(\mathbb{R})$ for all $x, y \in \mathbb{R}$. In agreement with Lemma 2.4, we obtain $2[d([x, y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$ for all $x, y, r \in \mathbb{R}$. Since $\mathbb{R}$ is a 2-torsion free semiprime ring in the above relation, we modify this to $[d([x, y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$ for all $x, y, r \in \mathbb{R}$. Moreover, we gain $d([x, y]_{\sigma,\tau}) \in Z(R)$ for all $x, y \in \mathbb{R}$. Now, in relation (3.23) the above fact produces $[x, y]_{\sigma,\tau}zd(r) = 0$ for all $x, y, r, z \in \mathbb{R}$. Left-multiplying by $d(r)$ and right-multiplying by the commutator $[x, y]_{\sigma,\tau}$, and depending on $\mathbb{R}$ being a semiprime ring, we get $d(r)[x, y]_{\sigma,\tau} = 0$ for all $x, y, r \in \mathbb{R}$. In agreement with Lemma 2.3 the above relation gives $d(r) \in Z(\mathbb{R})$, which implies that $[d(r), r]_{\sigma,\tau} = 0$ for all $r \in \mathbb{R}$, which is the desired result. □

Obviously, depending on Lemma 2.3 which used in the proof of Theorem 3.3, this implies the following result.

Corollary 3.1. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring, $\mathbb{U}$ be an ideal and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ, τ)-generalized derivation D such that $D(xy) = D(yx)$ for all $x, y \in \mathbb{R}$, then $\mathbb{R}$ contains a non-zero central ideal.*

Theorem 3.4. *Let $\mathbb{R}$ be a 2-torsion free prime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists a (σ,τ)-generalized derivation D such that $[D(x),D(y)]_{\sigma,\tau}=0$ for all $x,y\in\mathbb{R}$ and that d and D commute, then either D (σ,τ)-commute of $\mathbb{R}$ or $d^2(\mathbb{R})\circ d(\mathbb{R})=0$.*

Proof. In the main relation $[D(x),D(y)]_{\sigma,\tau}=0$ for all $x,y\in\mathbb{R}$. Replacing y with xy, we obtain $[D(x),D(x)\sigma(y)+\tau(x)d(y))]_{\sigma,\tau}=0$ for all $x,y\in\mathbb{R}$. Moreover, we obtain

$$D(x)[D(x),\sigma(y)]_{\sigma,\tau}+\tau(y)[D(x),d(y)]_{\sigma,\tau}+[D(x),\tau(x)]_{\sigma,\tau}d(y)=0. \tag{3.24}$$

In relation (3.24), we substitute y with $D(z)$, $z\in\mathbb{R}$, and since σ and τ act as automorphisms of $\mathbb{R}$ as depending on the main relation $[D(x),D(y)]_{\sigma,\tau}=0$, we achieve

$$x[D(x),d(D(z))]_{\sigma,\tau}=-[D(x),x]d(D(z)),$$

for all $x,z\in\mathbb{R}$. Putting $b=[D(x),d(D(z))]_{\sigma,\tau}$ and $a=-[D(x),x]d(D(z))$, then we have $xb=-a$. Left-multiplying by a and right-multiplying by xa, we again have $ax(bxa)=-a^2xa$. In consonance with Lemma 2.5, we have: either $a=[D(x),x]_{\sigma,\tau}d(D(z))=0$ for all $x,z\in\mathbb{R}$ or $(bxa)=-a^2$, which is guaranteed to be equal to zero where the value of a is zero in the first case. However, we focus on the item $[D(x),x]_{\sigma,\tau}d(D(z))=0$. According to the fact of the hypothesis that d and D commute with each other, we have $[D(x),x]_{\sigma,\tau}D(d(z))=0$ for all $x,z\in\mathbb{R}$. Moreover, after replacing z by zy, we obtain

$$\begin{aligned}&[D(x),x]_{\sigma,\tau}D(d(z)y)+[D(x),x]_{\sigma,\tau}D(zd(y))=0,\\&[D(x),x]_{\sigma,\tau}D(d(z))\sigma(y)+[D(x),x]_{\sigma,\tau}\tau(d(z))d(y)+[D(x),x]_{\sigma,\tau}D(z)\sigma(d(y))\\&\quad+[D(x),x]_{\sigma,\tau}\tau(z)d^2(y)=0.\end{aligned}$$

Again, in agreement with the fact that $[D(x),x]_{\sigma,\tau}d(D(z))=0$ with restitution of z with $d(z)$, we have $[D(x),x]_{\sigma,\tau}d^2(z)d(y)+[D(x),x]_{\sigma,\tau}d(z)d^2(y)=0$. Left-multiplying by $(d^2(z)d(y)+d(z)d^2(y))r$ and right multiplying by $r[D(x),x]_{\sigma,\tau}$ and applying Lemma 2.2, where $r\in\mathbb{R}$, we have $[D(x),x]_{\sigma,\tau}r(d^2(z)d(y)+d(z)d^2(y))=0$. Conforming to the primness of $\mathbb{R}$, we get: either D is (σ,τ)-commuting of $\mathbb{R}$ or $d^2(\mathbb{R})\circ d(\mathbb{R})=0$. □

Remark 3.1. In previous results, we cannot exclude the condition that the mappings σ and τ should be automorphism mappings of $\mathbb{R}$, as shown below.

Example 3.1. Let $\mathbb{R}=\mathcal{M}_2(\mathbb{F})$ be a ring of 2×2 matrices over a field $\mathbb{F}$, that is: $\mathbb{R}=\mathcal{M}_2(\mathbb{F})=\left\{\begin{pmatrix}a&b\\0&0\end{pmatrix}\middle|\ a,b\in\mathbb{F}\right\}$. Let d be the inner derivation of $\mathbb{R}$ given by

$$d(x)=x\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}x,$$

and the additive mapping D be defined as $D(x)=\begin{pmatrix}0&a\\0&0\end{pmatrix}$ for all $x\in\mathbb{R}$. Now, let $a,b,g,h\in F$. We suppose that $x=\begin{pmatrix}a&b\\0&0\end{pmatrix}$ and $y=\begin{pmatrix}g&h\\0&0\end{pmatrix}$. Then $D(xy)=$

$d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in \mathbb{R}$, where

$$D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right)\sigma\left(\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}\right) + \tau\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right)d\left(\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}\right).$$

Since σ and τ act as automorphism mappings implies the both sides of the above equation to give us $\begin{pmatrix} 0 & ag \\ 0 & 0 \end{pmatrix}$.

In 1984, J. Krempa and J. Matczuk [24] showed that for any associative ring $\mathbb{R}$ and a derivation d such that $d^0 = id$ even $d = 0$. *Leibniz's formula* is $d^r(xy) = \sum_{i=0}^{r} \binom{r}{i} d^i(x)d^{r-i}(y)$ for all $x, y \in \mathbb{R}$, where r and i are positive integers.

The effort of the authors continue to study this idea until to 2007, where M. Samman and N. Alyamani [34] introduced the idea of reverse derivation on a ring $\mathbb{R}$. More precisely, they provided the reverse derivation version of *Leibniz rule* for higher derivations have the formula: if n is odd, $d^n(xy) = \sum_{i=0}^{n} \binom{n}{i} d^{n-i}(y)d^i(x)$; if n is even, $d^n(xy) = \sum_{i=0}^{n} \binom{n}{i} d^{n-i}(x)d^i(y)$ for all $x, y \in \mathbb{R}$ where n and i are positive integers.

Nurcan Argac and Hulya G. Inceboz [3] provided that if we let $\mathbb{R}$ be a prime ring, I be a nonzero ideal of $\mathbb{R}$, d a derivation of $\mathbb{R}$ and n a fixed positive integer, if $(d(x)y + xd(y) + d(y)x)^n = xy + yx$ for all $x, y \in I$, then $\mathbb{R}$ is commutative.

In [22] Hvala initiated the algebraic study of generalized derivations; in particular, the generalized derivations whose products are again generalized derivations were characterized.

Results concerning generalized derivations can also be found in [12, 14, 39] and [37]. Moreover the results in [13] and [26] evidence the relationship between the behavior of generalized derivations in a prime (or semiprime) ring and the structure of the ring.

In this paper we supply the composition of (σ, τ)-generalized derivations of the Leibniz's formula as follows.

Definition 3.1. Let D be an additive mapping which acts as a (σ, τ)-generalized derivation on a set $\mathbb{S}$, and σ and τ be automorphism mappings of $\mathbb{S}$ such that the mappings σ and τ commute with D and d. Then, the composition of D on $\mathbb{S}$ can be defined as $D^n(xy) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y))$ for all $x, y \in \mathbb{S}$, where n and r are positive integers, with important note that the $D^0(x) = id = x$ and $d^0(y) = id = y$.

For motivation and a close view of the composition of (σ, τ)-generalized derivations, we provide the following example.

Example 3.2. Let $\mathbb{S}$ be a set and D be a (σ, τ)-generalized derivation of $\mathbb{S}$, and σ and τ are automorphism mappings of $\mathbb{S}$ such that the mappings σ and τ commute with

D and d. Then, for all $x, y \in \mathbb{S}$ the composition of D satisfies the relation

$$D^n(xy) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)),$$

for all $x, y \in \mathbb{S}$, where n and r are positive integers.

Take $n = 2$, then we have

$$\begin{aligned} D^2(xy) =& \sum_{r=0}^{2} \binom{2}{r} D^{2-r}(\sigma^{2-r}(x))d^r(\tau^r(y)) \\ =& \binom{2}{0} D^{2-0}(\sigma^{2-0}(x))d^0(\tau^0(y)) + \binom{2}{1} D^{2-1}(\sigma^{2-1}(x))d^1(\tau^1(y)) \\ &+ \binom{2}{2} D^0(\sigma^0(x))d^2(\tau^2(y)), \end{aligned}$$

for all $x, y \in \mathbb{S}$. We depend on the fact that $D^0(x) = id = x$ and $d^0(y) = id = y$ when computing the binomials, and from the above relation we obtain

$$D^2(xy) = D^2(\sigma^2(x))y + 2D(\sigma(x))d(\tau(y)) + xd^2(\tau^2(y)).$$

According to the hypothesis that the mappings σ and τ commute with D and d, we have

$$D^2(xy) = \sigma^2(D^2(x)y) + 2D(\sigma(x))d(\tau(y)) + x\tau^2(d^2(y)).$$

Apply that the mappings σ and τ act as automorphisms of $\mathbb{S}$, we get

$$D^2(xy) = D^2(x)\sigma^2(y) + 2D(\sigma(x))d(\tau(y)) + \tau^2(x)d^2(y).$$

On the other hand, we compute the left side

$$\begin{aligned} D^2(xy) =& D(D(x)\sigma(y) + \tau(x)d(y)) \\ =& D(D(x)\sigma(y)) + D(\tau(x)d(y)) \\ =& D^2(x)\sigma^2(y) + \tau(D(x))d(\sigma(y)) + D(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y)). \end{aligned}$$

Furthermore, where the mappings σ and τ commute with D and d, we get

$$D^2(xy) = D^2(x)\sigma^2(y) + 2D(\sigma(x))d(\tau(y)) + \tau^2(x)d^2(y).$$

Remark 3.2. When the values of $\sigma = 1$ and $\tau = 1$ in Definition 3.1 modify the formula of the composition of (σ, τ)-generalized derivations into:

$$D^n(xy) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(x)d^r(y),$$

for all $x, y \in \mathbb{S}$, where n and r are positive integers, with the important note that $D^0(x) = id = x$ and $d^0(y) = id = y$.

We need the following lemma.

Lemma 3.1. ([42, Lemma 1.1, 2.1 (ii)]).

(i) *Let $\mathbb{R}$ be a semiprime ring. If $a, b \in \mathbb{R}$ are such that $axb = 0$ for all $x \in \mathbb{R}$, then $ab = ba = 0$.*
(ii) *Let $\mathbb{R}$ be a semiprime ring, d a derivation of $\mathbb{R}$ and $a \in \mathbb{R}$ some fixed element, then $ax - xa \in Z(\mathbb{R})$, for all $x \in \mathbb{R}$, implies $a \in Z(\mathbb{R})$.*

We begin this section with the following main result.

Theorem 3.5. *Let n and r be a fixed positive integers. Let $\mathbb{R}$ be a 2-torsion free semiprime ring, σ and τ two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d, D a (σ,τ)-generalized derivation with an associated derivation d of $\mathbb{R}$ such that $[D^n(x), x^n]_{\sigma,\tau} = 0$, then*

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(x)d^r(x), x^{2n}] = -[D^n(x), x^{2n}]x \in Z(\mathbb{R}),$$

for all $x \in \mathbb{R}$.

Proof. Initially we have the relation $[D^n(x), x^n]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$. Replacing x with yx, we obtain $[D^n(xy), (xy)^n]_{\sigma,\tau} = 0$ for all $x, y \in \mathbb{R}$. Now we apply the formula of Definition 3.1 to above relation, and we get

$$\left[\sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n\right]_{\sigma,\tau} = 0,$$

for all $x, y \in \mathbb{R}$, where n and r are positive integers. Then

$$\Bigg[\binom{n}{0} D^n(\sigma^n(x))d^0(\tau^0(y)) + \binom{n}{1} D^{n-1}(\sigma^{n-1}(x))d(\tau(y))$$
$$+\binom{n}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y)) + \cdots + \binom{n}{n} D^{n-n}(\sigma^{n-n}(x))d^n(\tau^n(y)), (xy)^n\Bigg]_{\sigma,\tau} = 0.$$

Moreover, after simple calculations, with the important note that the $D^0(x) = id = x$ and $d^0(y) = id = y$, from the above relation we get

$$[D^n(\sigma^n(x))y + nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))$$
$$+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} = 0.$$

Then, we divide the above relation into

$$\Bigg[nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))$$
$$+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} + [D^n(\sigma^n(x))y, (xy)^n\Bigg]_{\sigma,\tau} = 0.$$

We rewrite the relation above as

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} + [D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} = 0. \tag{3.25}$$

At the same time, we left-multiply and right-multiply the relation (3.25) by t, where $t \in \mathbb{R}$, we retain

$$\sum_{r=1}^{n} \binom{n}{r} t[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} t + t[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} t = 0.$$

We set $\sum_{r=1}^{n} \binom{n}{r} t[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} = a$ and

$$[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} t = b.$$

The above relation reduces to

$$at + tb = 0. \tag{3.26}$$

Left-multiplying relation (3.26) by s, where $s \in \mathbb{R}$, and we get

$$sat + stb = 0. \tag{3.27}$$

In relation (3.26), we replace t with st, and we obtain $ast + stb = 0$ for all $s, t \in \mathbb{R}$. Subtracting this result from relation (3.27), we achieve

$$[s, a]_{\sigma,\tau} t = 0, \tag{3.28}$$

where we assume that $a = \sum_{r=1}^{n} \binom{n}{r} t[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}$ and after that we replace t by st. Therefore, $a = \sum_{r=1}^{n} \binom{n}{r} st[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}$, so from the relation (3.28), we obtain

$$\sum_{r=1}^{n} \binom{n}{r} [s, st[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}, (xy)^n]_{\sigma,\tau}]_{\sigma,\tau} t = 0.$$

We displace s by t and set $h = \sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}$, thus, the above relation becomes $t^2[t, h]_{\sigma,\tau} t = 0$. Left-multiplying by $t[t, h]_{\sigma,\tau}$, we gain $(t[t, h]_{\sigma,\tau} t)^2 = 0$. Obviously, we gain $(t[t, h]_{\sigma,\tau} t)^2 \in Z(\mathbb{R})$, which implies, with the application of Lemma 2.4 $2(t[t, h]_{\sigma,\tau} t) = 0$. Conforming to the fact that $\mathbb{R}$ is a 2-torsion free semiprime ring, and right-multiplying by $[t, h]_{\sigma,\tau}$, we achieve $(t[t, h]_{\sigma,\tau})^2 = 0$. Repeating the previous technique, we have

$$t[t, h]_{\sigma,\tau} = 0. \tag{3.29}$$

Left-multiplying the relation (3.29) by $[s, r]_{\sigma,\tau}$, we gain $[s, r]_{\sigma,\tau} t[t, h]_{\sigma,\tau} = 0$ for all $s, r \in \mathbb{R}$. Additionally, using Lemma 3.1 (i) with replacing s by t and r by h, we have $[t, h]_{\sigma,\tau}^2 = 0$ for all $t \in \mathbb{R}$. Again, as dependent on Lemma 2.4 and $\mathbb{R}$ being a 2-torsion free semiprime ring,we get $[t, h]_{\sigma,\tau} = 0$ for all $t \in \mathbb{R}$. Clearly, we obtain $h \in Z(\mathbb{R})$, which means

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} \in Z(\mathbb{R}). \tag{3.30}$$

Now we depend on relation (3.30), and from relation (3.26), we obtain

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} = -[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} \in Z(\mathbb{R}).$$

In consideration of the fact that σ and τ act as two automorphism mappings of $\mathbb{R}$ when replacing y with x, we get the required result. This completes the proof. □

Corollary 3.2. *Let n and r be fixed positive integers. Let $\mathbb{R}$ be a 2-torsion free semiprime ring, σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d, D is a (σ, τ)-generalized derivation with an associated derivation d of $\mathbb{R}$ such that $D^n(x) \in Z(\mathbb{R})$ for all $x \in \mathbb{R}$, then*

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(x)d^r(x), x^{2n}] = -[D^n(x), x^{2n}]x \in Z(\mathbb{R}),$$

for all $x \in \mathbb{R}$.

In view of Theorem 3.5 and Lemma 2.4 with 2.2, we immediately get the following corollary.

Corollary 3.3. *Let n and r be fixed positive integers. Let $\mathbb{R}$ be a 2-torsion free semiprime ring, σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d, D is a (σ, τ)-generalized derivation with an associated derivation d of $\mathbb{R}$ such that $[D^n(x), x^n]_{\sigma,\tau} = 0$, then $[D^n(\sigma^n(x)), x^{2n}]_{\sigma,\tau}, x]_{\sigma,\tau}, x]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$.*

Proof. By the same manner as in the previous theorem, we have the relation $[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x, y \in \mathbb{R}$. Furthermore, we obtain

$$[[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau}, r]_{\sigma,\tau} = 0, \quad x, y, r \in \mathbb{R}.$$

Replacing y and r by x, we receive $[[D^n(\sigma^n(x)), x^{2n}]_{\sigma,\tau}, x]_{\sigma,\tau}x = 0$. Now, we left-multiply by x and right-multiply by $[[D^n(\sigma^n(x)), x^{2n}]_{\sigma,\tau}, x]_{\sigma,\tau}$ and apply Lemma 2.2 and subtract the result from the above to complete the proof. □

Theorem 3.6. *Let n and r be fixed positive integers. Let $\mathbb{R}$ be a 2-torsion free semiprime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d, D a (σ, τ)-generalized derivation with an associated derivation d of $\mathbb{R}$ such that $[D^{n+1}(x), x^{n+1}]_{\sigma,\tau} = 0$, then*

$$\sum_{r=1}^{n+1} \binom{n+1}{r} [D^{(n+1)-r}(x)d^r(x), x^{2n+2}] = -[D^{n+1}(x), x^{2n+2}]x \in Z(\mathbb{R}), \quad x \in \mathbb{R}.$$

Proof. As stated in the hypothesis we have the relation $[D^{n+1}(xy), (xy)^{n+1}]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$, after we replace x with xy in the main relation $[D^{n+1}(x), x^{n+1}]_{\sigma,\tau} = 0$ for all $x \in \mathbb{R}$. Now, we apply the formula of Definition 3.1 to the above relation, and afterwards rewrite it to become a suitable power of D, we obtain

$$\sum_{r=0}^{n+1} \binom{n+1}{r} [D^{n-r+1}(\sigma^{n-r+1}(x))d^r(\tau^r(y)), (xy)^{n+1}]_{\sigma,\tau} = 0, \quad x, y \in \mathbb{R},$$

where n and r are positive integers, though with the important note that $D^0(x) = id = x$ and $d^0(y) = id = y$. Additionally, we get

$$\Bigg[\binom{n+1}{0}D^{n+1}(\sigma^{n+1}(x))y + \binom{n+1}{1}D^{n}(\sigma^{n}(x))d(\tau(y)) + \binom{n+1}{2}D^{n-1}(\sigma^{n-1}(x))d^2(\tau^2(y)) + \cdots + \binom{n+1}{n+1}xd^{n+1}(\tau^{n+1}(y)), (xy)^{n+1}\Bigg]_{\sigma,\tau} = 0.$$

After a simple calculation, from above relation we get

$$\Bigg[\frac{(n+1)!}{n!}D^{n}(\sigma^{n}(x))d(\tau(y)) + \frac{(n+1)!}{2!(n-1)!}D^{n-1}(\sigma^{n-1}(x))d^2(\tau^2(y)) + \frac{(n+1)!}{3!(n-2)!}D^{n-2}(\sigma^{n-2}(x))d^3(\tau^3(y)) + \cdots + xd^{n+1}(\tau^{n+1}(y)), (xy)^{n+1}\Bigg]_{\sigma,\tau} + [D^{n+1}(\sigma^{n+1}(x))y, (xy)^{n+1}]_{\sigma,\tau} = 0.$$

We rewrite the above relation as the composition of generalized derivations of *Leibniz's formula*, and we get

$$\sum_{r=1}^{n+1}\binom{n+1}{r}[D^{(n+1)-r}(\sigma^{(n+1)-r}(x))d^r(\tau^r(y)), (xy)^{n+1}]_{\sigma,\tau} + [D^{n+1}(\sigma^{n+1}(x))y, (xy)^{n+1}]_{\sigma,\tau} = 0, \quad x, y \in \mathbb{R}.$$

□

Using a similar approach as above in Theorem 3.5, we can complete the proof.

Analogously, we can prove the following (though we omit the proof for the brevity) where we depend on Lemma 2.4 and 2.2.

Corollary 3.4. *Let n and r be fixed positive integers. Let $\mathbb{R}$ be a 2-torsion free semiprime ring, σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d, D a (σ,τ)-generalized derivation with an associated derivation d of $\mathbb{R}$ such that $[D^{n+1}(x), x^{n+1}]_{\sigma,\tau} = 0$, then*

$$[[D^{n+1}(\sigma^{n+1}(x)), x^{2n+2}]_{\sigma,\tau}, x]_{\sigma,\tau}, x]_{\sigma,\tau} = 0, \quad x \in \mathbb{R}.$$

Theorem 3.7. *Let $\mathbb{R}$ be 2-torsion free prime ring, σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d. Suppose that there exists some (σ,τ)-generalized derivation D such that $D^n(x) \in Z(\mathbb{R})$ for all $x \in \mathbb{R}$, then either $D^n(\sigma^n(\mathbb{R})) = 0$ or $\mathbb{R}$ is commutative, where n is a fixed positive integer.*

Proof. From the hypothesis, we obtain $[D^n(x), r] = 0$ for all $x, r \in \mathbb{R}$. After some calculations that depend on the same technique as given in Theorem 3.5, from the relation (3.30), we have the further relation

$$\sum_{r=1}^{n}\binom{n}{r}[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), r]_{\sigma,\tau} \in Z(\mathbb{R}).$$

Again, by the same manner as in Theorem 3.5, from relation (3.25) we achieve

$$[D^n(\sigma^n(x))y, r]_{\sigma,\tau} \in Z(\mathbb{R}).$$

Moreover, applying the relation $D^n(\sigma^n(x)) \in Z(\mathbb{R})$ to the above commutator, we have $D^n(\sigma^n(x))[y, r]_{\sigma,\tau} \in Z(\mathbb{R})$. Then $[D^n(\sigma^n(x))[y, r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$ for all $x, y, r, s \in \mathbb{R}$. Again, in agreement with the fact that $D^n(\sigma^n(x)) \in Z(\mathbb{R})$, the above relation becomes $D^n(\sigma^n(x))[[y, r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$ for all $x, y, r, s \in \mathbb{R}$. Replacing s with $st, t \in \mathbb{R}$, with apply the primeness of $\mathbb{R}$ and receive: either $D^n(\sigma^n(\mathbb{R})) = 0$ or $[\mathbb{R}, \mathbb{R}] \in Z(\mathbb{R})$, which implies that $\mathbb{R}$ is commutative. ☐

The following results are inspired by the work of Motoshi Hongan [21]. We put $\mathbb{V}_{\mathbb{R}}(\mathbb{S}) = \{x \in \mathbb{R} : [x, s] = 0\}$ for all $x \in \mathbb{S}$, where $\mathbb{S}$ is a subset of $\mathbb{R}$ and $\mathbb{R}$ represents a ring.

Corollary 3.5. *Let $\mathbb{R}$ be a semiprime ring, $\mathbb{U}$ be a non-zero ideal of $\mathbb{R}$, σ and τ be two automorphism mappings of $\mathbb{R}$ such that the mappings σ and τ commute with D and d. Suppose that there exists some (σ, τ)-generalized derivation D such that $D^n([x, y]_{\sigma,\tau}) \mp D^q([x, y]_{\sigma,\tau}) \mp [x, y]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x, y \in \mathbb{U}$, if $D^n(\mathbb{U}) \mp D^q(\mathbb{U}) \subseteq \mathbb{V}_{\mathbb{R}}(\mathbb{U})$, then $\mathbb{U}$ is commutative and $\mathbb{U} \subseteq Z(\mathbb{R})$, where n and q are fixed positive integers.*

Proof. Suppose that $a \in \mathbb{U}$; in agreement with the hypothesis that $D^n([x, y]_{\sigma,\tau}) \mp D^q([x, y]_{\sigma,\tau}) \mp [x, y]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x, y \in \mathbb{U}$, we obtain $[D^n([\mathbb{U}, \mathbb{U}]_{\sigma,\tau}) \mp D^q([\mathbb{U}, \mathbb{U}]_{\sigma,\tau}) \mp [x, y]_{\sigma,\tau}, a]_{\sigma,\tau} = 0$ for all $x, y \in \mathbb{U}$. Hence,

$$[D^n([\mathbb{U}, \mathbb{U}]_{\sigma,\tau}) \mp D^q([\mathbb{U}, \mathbb{U}]_{\sigma,\tau}), a]_{\sigma,\tau} \mp [[x, y], a]_{\sigma,\tau} = 0.$$

In consideration of the hypothesis $(D^n(\mathbb{U}) \mp D^q(\mathbb{U})) \subseteq \mathbb{V}_{\mathbb{R}}(\mathbb{U})$, i.e., $[a, D^n(\mathbb{U}) \mp D^q(\mathbb{U})]_{\sigma,\tau} = 0$. Thus, the above relation reduces to $[[x, y], a]_{\sigma,\tau} = 0$ for all $x, y \in \mathbb{U}$. In agreement with Lemma 2.6, we have $a \in \mathbb{U} \subseteq Z(\mathbb{R})$. ☐

In a similar manner, as dependent on Lemma 2.6, we can prove the following corollary.

Corollary 3.6. *Let $\mathbb{R}$ be a semiprime ring, $\mathbb{U}$ be a non-zero ideal of $\mathbb{R}$, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists some (σ, τ)-generalized derivation D such that $D^n([x, y]_{\sigma,\tau}) \mp [x, y]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x, y \in \mathbb{U}$, and such that the mappings σ and τ commute with D and d. If $D^n(\mathbb{U}) \subseteq \mathbb{V}_{\mathbb{R}}(\mathbb{U})$, then $\mathbb{U}$ is commutative and $\mathbb{U} \subseteq Z(\mathbb{R})$, where n is a fixed positive integer.*

In the following theorem, we exclude the conditions $D^n(\mathbb{U}) \mp D^q(\mathbb{U}) \subseteq \mathbb{V}_{\mathbb{R}}(\mathbb{U})$ and $\mathbb{U}$ as non-zero ideals of $\mathbb{R}$, as per the hypothesis used in previous corollaries.

Theorem 3.8. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists some (σ, τ)-generalized derivation D, and the mappings σ and τ commute with D and d such that $D^q([x, y]_{\sigma,\tau}) \mp D^n([x, y]_{\sigma,\tau}) \mp [x, y]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x, y \in \mathbb{R}$. If*

(i) $D \neq 0$, *then* $D^{q-1}(\sigma^{q-1}(\mathbb{R})) \mp D^{n-1}(\sigma^{n-1}(\mathbb{R})) \in Z(\mathbb{R})$;
(ii) $D = 0$, *then* $[x,y]_{\sigma,\tau} \in Z(\mathbb{R})$,

where n *and* q *are fixed positive integers.*

Proof. (i) Firstly, we suppose that $D \neq 0$, putting $y = [y,z]_{\sigma,\tau}$, for all $x,y,z \in \mathbb{R}$, we have $D^q([x,y]_{\sigma,\tau}) \mp D^n([x,y]_{\sigma,\tau}) \mp [x,y]_{\sigma,\tau} \in Z(\mathbb{R})$. Clearly, after substituting the value of y the above relation becomes

$$D^q([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) \mp D^n([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) \mp [x,[y,z]_{\sigma,\tau}]_{\sigma,\tau} \in Z(\mathbb{R}). \tag{3.31}$$

Now, we simplify the items of the relation (3.31), and we get

$$D^q([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) = D^{q-1}(D(x\sigma([y,z]_{\sigma,\tau})) - D^{q-1}(D(\tau([y,z]_{\sigma,\tau})x)))$$

and

$$D^n([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) = D^{n-1}(D(x\sigma([y,z]_{\sigma,\tau})) - D^{n-1}(D(\tau([y,z]_{\sigma,\tau})x))).$$

Applying Definition 3.1 on the right-hand side, we get

$$\begin{aligned} D^q([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) =& \sum_{r=1}^{q-1}\binom{q-1}{r}(D^{q-r-1}(\sigma^{q-r-1}(x)))d^r(\tau^r([y,z]_{\sigma,\tau})) \\ &- (D^{q-r-1}(\sigma^{q-r-1}([y,z]_{\sigma,\tau})))d^r(\tau^r(x)). \end{aligned}$$

In a similar manner for $D^n([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau})$, we get

$$\begin{aligned} D^n([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) =& \sum_{r=1}^{n-1}\binom{n-1}{r}(D^{n-r-1}(\sigma^{n-r-1}(x)))d^r(\tau^r([y,z]_{\sigma,\tau})) \\ &- (D^{n-r-1}(\sigma^{n-r-1}([y,z]_{\sigma,\tau})))d^r(\tau^r(x)). \end{aligned}$$

Furthermore, when we substitute the previous values in relation (3.31), we obtain

$$\begin{aligned} &D^n([x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}) \\ =&\binom{q-1}{0}(D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau} - (D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau})))(x)) \\ &+\cdots+\binom{q-1}{q-1}xd^{q-1}(\tau^{q-1}([y,z]_{\sigma,\tau}) - [y,z]_{\sigma,\tau}d^{q-1}(\tau^{q-1}(x)) \\ &\mp\binom{n-1}{0}(D^{n-1}(\sigma^{n-1}(x)))[y,z]_{\sigma,\tau} - (D^{n-1}(\sigma^{n-1}([y,z]_{\sigma,\tau})))(x)) \\ &+\cdots+\binom{n-1}{n-1}xd^{n-1}(\tau^{n-1}([y,z]_{\sigma,\tau}) - [y,z]_{\sigma,\tau}d^{n-1}(\tau^{n-1}(x)) \\ &\mp [x,[y,z]_{\sigma,\tau}]_{\sigma,\tau} \in Z(\mathbb{R}). \end{aligned}$$

Additionally, we have

$$\begin{aligned} &[(D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau} - (D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau})))(x)) \\ &+\cdots+(xd^{q-1}(\tau^{q-1}([y,z]_{\sigma,\tau}) - [y,z]_{\sigma,\tau}d^{q-1}(\tau^{q-1}(x)),r]_{\sigma,\tau} \mp [D^{n-1}(\sigma^{n-1}(x)))[y,z]_{\sigma,\tau} \end{aligned}$$

$$- D^{n-1}(\sigma^{n-1}([y,z]_{\sigma,\tau})))(x) + \cdots + xd^{n-1}(\tau^{n-1}([y,z]_{\sigma,\tau}) - [y,z]_{\sigma,\tau}d^{n-1}(\tau^{n-1}(x)), r]_{\sigma,\tau}$$
$$\mp [[x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}, r] = 0, \quad x,y,z,r \in \mathbb{R}.$$

We rewrite the above relation as per below

$$[(D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau} - (D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau})))(x)), r]_{\sigma,\tau} + [(D^{q-2}(\sigma^{q-2}(x)))[y,z]_{\sigma,\tau}$$
$$- (D^{q-2}(\sigma^{q-2}([y,z]_{\sigma,\tau})))(x)) + \cdots + (xd^{q-1}(\tau^{q-1}([y,z]_{\sigma,\tau}) - [y,z]_{\sigma,\tau}d^{q-1}(\tau^{q-1}(x))$$
$$\mp ([D^{n-1}(\sigma^{n-1}(x)))[y,z]_{\sigma,\tau} - D^{n-1}(\sigma^{n-1}([y,z]_{\sigma,\tau})))(x) + \cdots + xd^{n-1}(\tau^{n-1}([y,z]_{\sigma,\tau})$$
$$- [y,z]_{\sigma,\tau}d^{n-1}(\tau^{n-1}(x)) \mp [x,[y,z]_{\sigma,\tau}]_{\sigma,\tau}, r]_{\sigma,\tau} = 0, \quad x,y,z,r \in \mathbb{R}.$$

In the same manner as Theorem 3.5

$$[(D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau} - (D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau})))x), r]_{\sigma,\tau} \in Z(\mathbb{R}),$$

for all $x,y,z,r \in \mathbb{R}$. Obviously, according to Lemma 3.1 (ii), we obtain

$$D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau} - (D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau})))x \in Z(\mathbb{R}),$$

for all $x,y,z \in \mathbb{R}$. Then

$$[D^{q-1}(\sigma^{q-1}(x)))[y,z]_{\sigma,\tau}, r]_{\sigma,\tau} - [D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau}))x, r]_{\sigma,\tau} = 0,$$

for all $x,y,z,r \in \mathbb{R}$. Again, as dependent on the same technique as used in Theorem 3.5, we get

$$[D^{q-1}(\sigma^{q-1}([y,z]_{\sigma,\tau}))x, r]_{\sigma,\tau} \in Z(\mathbb{R}),$$

which implies to $[D^{q-1}(\sigma^{q-1}(x))[y,z]_{\sigma,\tau}, r]_{\sigma,\tau} \in Z(\mathbb{R})$. In agreement with Lemma 3.1 (ii) on the second relation above, we achieve $D^{q-1}(\sigma^{q-1}(x))[y,z]_{\sigma,\tau} \in Z(\mathbb{R})$. Apparently, we get $[D^{q-1}(\sigma^{q-1}(x))[y,z]_{\sigma,\tau}, r]_{\sigma,\tau} = 0$, for all $x,y,z,r \in \mathbb{R}$. Possibly, we can rewrite the above relation as

$$D^{q-1}(\sigma^{q-1}(x))[[y,z]_{\sigma,\tau}, r]_{\sigma,\tau} + [D^{q-1}(\sigma^{q-1}(x)), r]_{\sigma,\tau}[y,z]_{\sigma,\tau} = 0.$$

We put $a = [D^{q-1}(\sigma^{q-1}(x)), r]_{\sigma,\tau}[y,z]_{\sigma,\tau}$ and $b = D^{q-1}(\sigma^{q-1}(x))[[y,z]_{\sigma,\tau}, r]_{\sigma,\tau}$, and by applying the same method as in Theorem 3.5 to above relation, we obtain

$$[D^{q-1}(\sigma^{q-1}(x)), r]_{\sigma,\tau}[y,z]_{\sigma,\tau} \in Z(\mathbb{R}).$$

Replacing r with z and y with $D^{q-1}(\sigma^{q-1}(x))$, as dependent on Lemma 2.4, we gain $2[D^{q-1}(\sigma^{q-1}(x)), z]_{\sigma,\tau} = 0$ for all $x,z \in \mathbb{R}$. Conforming to the fact that $\mathbb{R}$ is a 2-torsion free semiprime ring throug the use of Lemma 3.1(ii), we have that $D^{q-1}(\sigma^{q-1}(x)) \in Z(\mathbb{R})$, for all $x,z \in \mathbb{R}$. In other words, we achieve $D^{q-1}(\sigma^{q-1}(\mathbb{R})) \in Z(\mathbb{R})$. Also, in a similar way we gain $D^{n-1}(\sigma^{n-1}(\mathbb{R})) \in Z(\mathbb{R})$, which completes the proof of branch (i) of the theorem.

The proof of branch (ii) is straightforward in manner. □

Proposition 3.2. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists some (σ,τ)-generalized derivation D, and the mappings σ and τ commute with D and d such that $D^q([x,y]_{\sigma,\tau}) \mp D^n([x,y]_{\sigma,\tau}) \mp (xoy)_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x,y \in \mathbb{R}$, and the square of each element of $\mathbb{R}$ lies in $Z(\mathbb{R})$, where n and q are fixed positive integers.*

Proof. Conforming to the hypothesis, we have that $D^q([x,y]_{\sigma,\tau}) \mp D^n([x,y]_{\sigma,\tau}) \mp (xoy)_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x,y \in \mathbb{R}$. When we replace y by x with depend on the fact that $\mathbb{R}$ be a 2-torsion free semiprime ring, and σ and τ act as automorphism mappings of $\mathbb{R}$, and we obtain $x^2 \in Z(\mathbb{R})$ for all $x \in \mathbb{R}$, the result we desire. ☐

Using a similar approach as above, we can prove the following.

Proposition 3.3. *Let $\mathbb{R}$ be a 2-torsion free semiprime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists some (σ,τ)-generalized derivation D, and the mappings σ and τ commute with D and d such that $D^q(xoy)_{\sigma,\tau} \mp D^n(xoy)_{\sigma,\tau} \mp [x,y]_{\sigma,\tau} \in Z(\mathbb{R})$ for all $x,y \in \mathbb{R}$, then $D^q(x^2) \mp D^n(x^2) \in Z(\mathbb{R})$, where n and q are fixed positive integers.*

We close the paper with the following theorem.

Theorem 3.9. *Let $\mathbb{R}$ be a semiprime ring, and σ and τ be two automorphism mappings of $\mathbb{R}$. Suppose that there exists some (σ,τ)-generalized derivation D, and the mappings σ and τ commute with D and d such that $D^2(\mathbb{R}) = 0$, then $\prod_{i=0}^{n+1} d^i(\mathbb{R}) = 0$, where n is a fixed non-negative integer.*

Proof. As per the situation in the hypothesis, we have $D^2(\mathbb{R}) = 0$. Then, for all $x,y \in \mathbb{R}$, we have $D(D(x)\sigma(y) + \tau(x)d(y)) = 0$. After that,

$$D(x)^2\sigma^2(y) + \tau(D(x))d(\sigma(y)) + D(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0.$$

According to the main relation $D^2(\mathbb{R}) = 0$, the above relation reduces to

$$\tau(D(x))d(\sigma(y)) + D(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0. \tag{3.32}$$

Since we suppose that the mappings σ and τ commute with D and d, relation (3.32) becomes $D(\tau(x))d(\sigma(y)) + D(\tau(x))d(\sigma(y)) + \tau^2(x)d^2(y) = 0$. Obviously, we gain

$$2D(\tau(x))d(\sigma(y)) + \tau^2(x)d^2(y) = 0. \tag{3.33}$$

By replacing x by $D(x)$ with depend on the facts that $D^2(\mathbb{R}) = 0$ and the mappings σ and τ acts as automorphism of $\mathbb{R}$. From relation (3.33), we get $D(x)d^2(y) = 0$. Replacing x by $xr, r \in \mathbb{R}$, using the fact that the mappings σ and τ act as automorphism of $\mathbb{R}$ in the above relation, we get that $D(x)rd^2(y) + xd(r)d^2(y) = 0$. Replacing r with $d^2(y)$, as dependent on the result $D(x)d^2(y) = 0$, we obtain $xd(d^2(y))d^2(y) = 0$. Right-multiplying by $d(y)y$ and left-multiplying by $(d^{n+1}(y)d^n(y)d^{n-1}(y)\cdots)$ and applying the fact that $\mathbb{R}$ is a semiprime ring, we get $\prod_{i=0}^{n+1} d^i(\mathbb{R}) = 0$. This is the result which is required. ☐

Remark 3.3. In previous results, we can not exclude the condition "n-torsion free", as below.

Example 3.3. Let $\mathbb{R}$=$\mathcal{M}_2(\mathbb{F})$ be a ring of 2×2 matrices over a field $\mathbb{F}$, that is $\mathbb{R} = \mathcal{M}_2(\mathbb{F}) = \left\{ \left(\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) \middle|\ a,b \in \mathbb{F} \right\}$, with char $\mathbb{R} = n$. For all $x \in \mathbb{R}$, let the additive map $D(x) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $d(x) = [x,a] = x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Also, we define the mappings σ and τ as follows $\sigma(x) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $\tau(x) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Take $x, y \in \mathbb{R}$ such that $x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix}$, where we know that the additive mapping D is a (σ,τ)-generalized derivation which has the formula $D(xy) = D(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in \mathbb{R}$. The left side produces $D(xy) = D\begin{pmatrix} ag & ah \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ag & 0 \\ 0 & 0 \end{pmatrix}$. Moreover, the right side gives

$$D(x)\sigma(y) + \tau(x)d(y) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & h \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ag & 2ah \\ 0 & 0 \end{pmatrix}.$$

By reason of $\mathbb{R}$ having char $\mathbb{R} = n$, then the above matrix can be modified to $= \begin{pmatrix} ag & 0 \\ 0 & 0 \end{pmatrix}$. Obviously, the sides are equal to each other, i.e., the above formula is satisfied.

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