Kragujevac Journal of Mathematics
Volume 43(4) (2019), Pages 503–512.

# BASIC PROPERTIES OF AN EIGENPARAMETER-DEPENDENT $q$-BOUNDARY VALUE PROBLEM

F. AYCA CETINKAYA[1]

Abstract. This paper is devoted to study a $q$-fractional boundary value problem that includes $q$-Jackson derivative in the differential equation and an eigenvalue parameter in the boundary condition. We introduced a modified Hilbert space and a symmetric operator. We illustrated the examined boundary value problem as a spectral problem for this operator. Properties of the eigenvalues and eigenfunctions are investigated and the Green's function is constructed.

## 1. Introduction and Preliminaries

In this paper, we study the $q$-fractional boundary value problem (qFBVP) which consists the differential equation

$$l(u) := -\frac{1}{q} D_{q^{-1}} D_q u(x) + v(x)u(x) = \mu u(x), \quad x \in [0,\pi], \tag{1.1}$$

and the boundary conditions

$$V_1(u) := u(0) = 0, \tag{1.2}$$

$$V_2(u) := \alpha_1 u(\pi) + \alpha_2 D_{q^{-1}} u(\pi) + \mu \left[\alpha_3 u(\pi) + \alpha_4 D_{q^{-1}} u(\pi)\right] = 0, \tag{1.3}$$

where $q \in (0,1)$ is fixed, $v(\cdot)$ is a real valued function defined on $[0,\pi]$ and continuous at zero, $\mu$ is a spectral parameter and $\alpha_i \neq 0$, $i = 1,2,3,4$, are any given real numbers.

There has recently been a considerable attention on $q$-calculus and many papers subject to the boundary value problems consisting a $q$-Jackson derivative in the differential equation have appeared. In [8,10] the authors studied a $q$-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint $q$-difference operator in a Hilbert

*Key words and phrases.* $q$-Jackson derivative, Sturm-Liouville operator, eigenvalues and eigenfunctions.

2010 *Mathematics Subject Classification.* Primary: 34B08. Secondary: 34L05.

*Received*: July 06, 2018.

*Accepted*: August 21, 2018.

space. Their results are applied and developed in different aspects. In [1, 7], for instance, sampling theory associated with $q$-difference equations of the Sturm-Liouville type is considered. In [6, 26] a regular $q$-fractional Sturm-Liouville problem which includes the left-sided Riemann-Liouville and right-sided Caputo $q$-fractional derivatives of the same order is formulated and the properties of eigenvalues and eigenfunctions are investigated. In [5] a Parseval equality and an expansion formula in eigenfunctions for a singular $q$-Sturm-Liouville operator on the whole line are established. In [3] the eigenvalues and the spectral singularities of non-selfadjoint $q$-difference equations of second order are investigated. In [14] a boundary value problem consisting of a second-order $q$-difference equation together with Dirichlet boundary conditions is reduced to an eigenvalue problem for a second-order Euler $q$-difference equation by separation of variables and in [17] a $q$-Sturm-Liouville boundary value problem with a spectral parameter in the boundary condition is considered.

For further studies related to the spectral analysis of $q$-differential equations, the readers are directed to [9, 12, 22] and the references therein. Applicable problems involving mathematical pyhsical problems are extensively studied in [2, 4, 15, 16, 20, 24, 28, 29].

Now we introduce some of the $q$-notations which will be used throughout the paper. We use the standart notations found in [8] and [11]. A set $S \subseteq \mathbb{R}$ is called $q$-geometric if, for every $x \in S$, $qx \in S$. Let $u$ be a real or complex valued function defined on a $q$-geometric set $S$. The $q$-difference operator is defined by

$$D_q u(x) := \frac{u(x) - u(qx)}{x(1-q)}, \quad x \neq 0.$$

If $0 \in S$, the $q$-derivative at zero is defined as

$$D_q u(0) := \lim_{n\to\infty} \frac{u(xq^n) - u(0)}{xq^n}, \quad x \in S,$$

if the limit exists and does not depend on $x$. Since the formulation of self-adjoint eigenvalue problems requires $D_{q^{-1}}$, we define it for $x \in S$ to be

$$D_{q^{-1}} u(x) := \begin{cases} \dfrac{u(x) - u(q^{-1}x)}{x(1-q^{-1})}, & x \neq 0, \\ D_q u(0), & x = 0, \end{cases}$$

provided that $D_q u(0)$ exists. A right inverse, $q$-integration of the $q$-difference operator $D_q$ is defined by Jackson [23] as

$$\int_0^x u(t) d_q t := x(1-q) \sum_{n=0}^{\infty} q^n u(xq^n), \quad x \in S,$$

provided that the series converges. In general the below equation is valid:

$$\int_a^b u(t) d_q t := \int_0^b u(t) d_q t - \int_0^a u(t) d_q t, \quad a, b \in S.$$

There is no unique canonical choice for the $q$-integration over $[0,\infty)$. Hahn [21] defined the $q$-integration for a function $u$ over $[0,\infty)$ by

$$\int_0^\infty u(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n u(q^n),$$

while Matsuo [27] defined $q$-integration on the interval $[0,\infty)$ with

$$\int_0^\infty u(t)d_qt := b(1-q)\sum_{-\infty}^{\infty} q^n u(bq^n), \quad b>0,$$

provided that the series converges.

Consequently, the $q$-integration of a function $u$ defined on $\mathbb{R}$ can be defined as

$$\int_{-\infty/b}^{\infty/b} u(t)d_qt = \frac{1-q}{b}\sum_{-\infty}^{\infty} q^n \left(u(q^n/b)+u(-q^n/b)\right), \quad b>0$$

provided that the series converges absolutely.

**Definition 1.1.** Let $u$ be a function defined on a $q$-geometric set $S$. We say that $u$ is $q$-integrable on $S$ if and only if $\int_0^x u(t)d_qt$ exists for all $x\in S$.

For a detailed analysis of classical Sturm-Liouville problems with eigenparameter-dependent boundary conditions one can refer to [13,18,19,30] and the references cited there. The purpose of this paper is to extend some results obtained in [25] to the case of $q$-fractional boundary value problem (1.1)-(1.3).

The structure of the paper is as follows. In Section 2, we establish an operator-theoretic formulation for the qFBVP (1.1)–(1.3) in the Hilbert space $L_q^2(0,\pi)\oplus\mathbb{C}$ and we give some of the virtues of eigenvalues and eigenfunctions and Section 3 is devoted to construct the Green's function for the inhomogeneous $q$-fractional boundary value problem corresponding to the qFBVP (1.1)–(1.3) and to mention some of its properties.

## 2. Properties of the Eigenvalues and Eigenfunctions

In this section, we give the operator-theoretic formulation for the qFBVP (1.1)–(1.3) in the Hilbert space $L_q^2(0,\pi)\oplus\mathbb{C}$. We formulate a symmetric $q$-difference operator in this Hilbert space and we discuss some characteristics of eigenvalues and eigenfunctions.

In the Hilbert space $H := L_q^2(0,\pi)\oplus\mathbb{C}$ an inner product is defined by

$$(f,g) := \int_0^\pi f_1(x)\overline{g_1(x)}d_qx + \frac{f_2\overline{g_2}}{\chi},$$

where

$$f = \begin{pmatrix} f_1(x) \\ f_2 \end{pmatrix} \in H, \quad g = \begin{pmatrix} g_1(x) \\ g_2 \end{pmatrix} \in H, \quad \chi = \alpha_1\alpha_4 - \alpha_2\alpha_3 > 0.$$

Let us define the operator $A$

$$A(f) := \begin{pmatrix} -\frac{1}{q} D_{q^{-1}} D_q f_1(x) + v(x) f_1(x) \\ \alpha_1 f_1(\pi) + \alpha_2 D_{q^{-1}} f_1(\pi) \end{pmatrix},$$

with the domain $D(A)$ which consists all the functions $u(x) \in H$ that satisfy (1.2), (1.3) such that $D_q u(x)$ is $q$-regular at zero and $D_q^2 u(x)$ lies in $L_q^2(0,\pi)$. Thus $A$ is the operator generated by the differential expression $l(u) = \mu u$ and the boundary conditions (1.2), (1.3).

**Lemma 2.1.** *Let $f(\cdot)$ and $g(\cdot)$ be the elements of $H$ which is defined on $[0, q^{-1}\pi]$. Then for $x \in (0,\pi]$ we have*

$$D_q g(xq^{-1}) = D_{q^{-1}} g(x) = D_{q,xq^{-1}} g(xq^{-1}),$$

$$(D_q f, g) = f(\pi) g(\pi q^{-1}) - \lim_{n\to\infty} f(\pi q^n) g(\pi q^{n-1}) + \left(f, -\frac{1}{q} D_{q^{-1}} g\right), \tag{2.1}$$

$$\left(-\frac{1}{q} D_{q^{-1}} f, g\right) = \lim_{n\to\infty} f(\pi q^{n-1}) g(\pi q^n) - f(\pi q^{-1}) g(\pi) + (f, D_q g), \tag{2.2}$$

*Proof.* The proof can be done similar to [10]. □

**Theorem 2.1.** *The operator $A$ is symmetric in the Hilbert space $H$.*

*Proof.* For each $f, g \in D(A)$ we have

$$\begin{aligned}(Af, g) - (f, Ag) &= \int_0^\pi Af_1(x)\overline{g_1(x)} d_q x + \frac{Af_2 \overline{g_2}}{\chi} \\ &\quad - \int_0^\pi f_1(x) \overline{Ag_1(x)} d_q x - \frac{f_2 \overline{Ag_2}}{\chi} \\ &= \int_0^\pi \left(-\frac{1}{q} D_{q^{-1}} D_q f_1(x) + v(x) f_1(x)\right) \overline{g_1(x)} d_q(x) \\ &\quad - \int_0^\pi f_1(x) \overline{\left(-\frac{1}{q} D_{q^{-1}} D_q g_1(x) + v(x) g_1(x)\right)} d_q(x) \\ &\quad + \frac{Af_2 \overline{g_2}}{\chi} - \frac{f_2 \overline{Ag_2}}{\chi}.\end{aligned}$$

Using (2.1) with $f(x) = D_q f_1(x)$, $g(x) = g_1(x)$ to the first integral gives us

$$\begin{aligned}(Af, g) - (f, Ag) &= \lim_{n\to\infty} (D_q f_1)(\pi q^{n-1}) \overline{g_1(\pi q^n)} - (D_q f_1)(\pi q^{-1}) \overline{g_1(\pi)} \\ &\quad + \langle D_q f_1, D_q g_1 \rangle - \int_0^\pi f_1(x) \left(-\frac{1}{q} D_{q^{-1}} D_q g_1(x)\right) d_q x \\ &\quad + \frac{Af_2 \overline{g_2}}{\chi} - \frac{f_2 \overline{Ag_2}}{\chi},\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L_q^2(0, \pi)$. Applying (2.2) with $f(x) = f_1(x)$, $g(x) = D_q g_1(x)$ to the term $\langle D_q f_1, D_q g_1 \rangle$ in the above equation yields

$$(Af, g) - (f, Ag) = [f_1, g_1](\pi) - \lim_{n\to\infty} [f_1, g_1](\pi q^n) + \frac{Af_2\overline{g_2}}{\chi} - \frac{f_2\overline{Ag_2}}{\chi}, \tag{2.3}$$

where

$$[f, g](x) := f(x)\overline{D_{q^{-1}} g(x)} - D_{q^{-1}} f(x)\overline{g(x)}.$$

The definition for the domain of the operator $A$ concludes $\frac{Af_2\overline{g_2}}{\chi} - \frac{f_2\overline{Ag_2}}{\chi} = 0$ and thus equation (2.3) becomes

$$(Af, g) - (f, Ag) = [f_1, g_1](\pi) - \lim_{n\to\infty} [f_1, g_1](\pi q^n).$$

Since $f_1(x),\ g_1(x) \in C_q^2(0)$ satisfy the boundary condition (1.2) we have

$$f_1(0) = 0, \quad g_1(0) = 0. \tag{2.4}$$

The continuity of the functions $f_1(x),\ g_1(x)$ at zero implies

$$\lim_{n\to\infty} [f_1, g_1](\pi q^n) = [f_1, g_1](0),$$

and thus we have

$$(Af, g) - (f, Ag) = [f_1, g_1](\pi) - [f_1, g_1](0).$$

It follows from (2.4) that

$$[f_1, g_1](0) = f_1(0)\overline{D_{q^{-1}} g_1(0)} - D_{q^{-1}} f_1(0)\overline{g_1(0)} = 0.$$

Likewise,

$$[f_1, g_1](\pi) = f_1(\pi)\overline{D_{q^{-1}} g_1(\pi)} - D_{q^{-1}} f_1(\pi)\overline{g_1(\pi)} = 0.$$

Hence, the equation $(Af, g) - (f, Ag) = 0$ is satisfied and this completes the proof. □

**Definition 2.1.** A $\mu$ which the qFBVP (1.1)–(1.3) has a nontrivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction. The multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to it. In particular an eigenvalue is simple if and only if it has only one linearly independent solution.

**Corollary 2.1.** *The eigenvalues of the qFBVP* (1.1)–(1.3) *are real.*

The eigenfunctions of the operator $A$ are in the form of

$$\Phi(x, \mu_n) = \Phi_n := \begin{pmatrix} \varphi(x, \mu_n) \\ \alpha_3\varphi(\pi, \mu_n) + \alpha_4 D_{q^{-1}}\varphi(\pi, \mu_n) \end{pmatrix}.$$

**Corollary 2.2.** *Two eigenfunctions $\Phi_1$ and $\Phi_2$ corresponding to the different eigenvalues are orthogonal.*

Now, let us denote

$$\Delta(\mu) := \begin{vmatrix} V_1(\phi_1) & V_1(\phi_2) \\ V_2(\phi_1) & V_2(\phi_2) \end{vmatrix},$$

where $\phi_1(\cdot,\mu)$ and $\phi_2(\cdot,\mu)$ are linearly independent solutions of (1.1) determined by the initial conditions

$$D_q^{j-1}\phi_i(\cdot,\mu) = \delta_{ij}, \quad i,j = 1,2,$$

as $\delta_{ij}$ refers to the Kronecker delta. The function $\Delta(\mu)$ is the characteristic function of the qFBVP (1.1)-(1.3). It is an entire function with respect to $\mu$ and thus the eigenvalues of the qFBVP (1.1)-(1.3) has an at most countable set of $\{\mu_n\}$ with no finite limit points.

In the following theorem, we prove that the eigenvalues of the qFBVP (1.1)–(1.3) are the simple zeros of the characteristic function $\Delta(\mu)$.

**Theorem 2.2.** *The eigenvalues of the qFBVP* (1.1)-(1.3) *coincide with the simple zeros of* $\Delta(\mu)$.

*Proof.* Let us define the functions $\theta_1(\cdot,\mu)$ and $\theta_2(\cdot,\mu)$ as

$$\begin{cases} \theta_1(x,\mu) := U_1(\phi_2)\phi_1(x,\mu) - U_1(\phi_1)\phi_2(x,\mu), \\ \theta_2(x,\mu) := U_2(\phi_2)\phi_1(x,\mu) - U_2(\phi_1)\phi_2(x,\mu). \end{cases} \tag{2.5}$$

The functions $\theta_1(\cdot,\mu)$ and $\theta_2(\cdot,\mu)$ are the two solutions of equation (1.1) which satisfy the conditions

$$\begin{cases} \theta_1(0,\mu) = 0, & D_{q^{-1}}\theta_1(0,\mu) = -1, \\ \theta_2(\pi,\mu) = \alpha_2 + \mu\alpha_4, & D_{q^{-1}}\theta_2(\pi,\mu) = -\left(\alpha_1 + \mu\alpha_3\right). \end{cases} \tag{2.6}$$

It can easily be seen that the below equation holds for the functions $\theta_1(\cdot,\mu)$ and $\theta_2(\cdot,\mu)$:

$$W_q\left(\theta_1(\cdot,\mu),\theta_2(\cdot,\mu)\right) = \Delta(\mu) W_q\left(\phi_1(\cdot,\mu),\phi_2(\cdot,\mu)\right)(x) = \Delta(\mu), \tag{2.7}$$

where the $q$-Wronskian of two functions $y_1(x)$ and $y_2(x)$ is defined as

$$W_q(y_1,y_2)(x) := y_1(x)D_q y_2(x) - y_2(x)D_q y_1(x), \quad x \in [0,\pi]$$

(see [10], pg.60). Now, let $\mu_0$ be an eigenvalue of the qFBVP (1.1)-(1.3). Equation (2.7) leads us to the fact that the functions $\theta_i(x,\mu_0)$, $i = 1,2$, are linearly dependent:

$$\theta_1(x,\mu_0) = k_0\theta_2(x,\mu_0), \quad k_0 \neq 0.$$

Using (2.5) and (2.6) implies

$$\begin{cases} \theta_1(\pi,\mu_0) = k_0\theta_2(\pi,\mu_0) = k_0(\alpha_2 + \mu\alpha_4), \\ D_{q^{-1}}\theta_1(\pi,\mu_0) = k_0 D_{q^{-1}}\theta_2(\pi,\mu_0) = -k_0(\alpha_1 + \mu_0\alpha_3). \end{cases}$$

By applying the $q$-Lagrange identity (see [10], page 81) to the functions $\theta_1(x,\mu)$ and $\theta_1(x,\mu_0)$ we obtain

$$\begin{aligned}(\mu-\mu_0)\int_0^\pi \theta_1(x,\mu)\theta_1(x,\mu_0)d_qx =&\theta_1(\pi,\mu)D_{q^{-1}}\theta_1(\pi,\mu_0)-D_{q^{-1}}\theta_1(\pi,\mu)\theta_1(\pi,\mu)\\ =&k_0\left(\theta_1(\pi,\mu)D_{q^{-1}}\theta_2(\pi,\mu_0)-\theta_2(\pi,\mu_0)D_{q^{-1}}\theta_1(\pi,\mu)\right)\\ =&k_0W_q\left(\theta_1(\cdot,\mu),\theta_2(\cdot,\mu_0)\right)(q^{-1}\pi)\\ =&k_0\Delta(\mu).\end{aligned}$$

Since $\Delta(\mu)$ is an entire function of $\mu$, we have the opportunity to write the expression below:

$$\frac{d}{d\mu}\Delta(\mu)=\lim_{\mu\to\mu_0}\frac{\Delta(\mu)-\Delta_0(\mu)}{\mu-\mu_0}=\frac{1}{k_0}\int_0^\pi \theta_1^2(x,\mu_0)d_qx\neq 0. \tag{2.8}$$

The simplicity of the zeros of the function $\Delta(\lambda)$ is the direct result of (2.8). □

## 3. Construction of the $q$-Type Green's Function

The $q$-type Green's function arises when we pursue a solution of the inhomogeneous qFBVP

$$l(u):=-\frac{1}{q}D_{q^{-1}}D_qu(x)+\{-\mu+v(x)\}u(x)=f(x),\quad x\in[0,\pi], \tag{3.1}$$

$$V_1(u):=u(0)=0, \tag{3.2}$$

$$V_2(u):=\alpha_1u(\pi)+\alpha_2D_{q^{-1}}u(\pi)+\mu\left[\alpha_3u(\pi)+\alpha_4D_{q^{-1}}u(\pi)\right]=f_2, \tag{3.3}$$

as $f(x)\in L_q^2(0,\pi)$.

**Theorem 3.1.** *Assume that $\mu$ is not an eigenvalue of the qFBVP* (1.1)-(1.3)*. Let $\phi(\cdot,\mu)$ satisfy the q-difference equation* (3.1) *and the boundary conditions* (3.2)-(3.3) *where $f(x)\in L_q^2(0,\pi)$. Then*

$$\phi(x,\mu)=\int_0^\pi G(x,t,\mu)f(t)d_qt+\frac{f_2\left(\alpha_3G(0,\cdot,\mu)+\alpha_4D_{q^{-1}}G(0,\cdot,\mu)\right)}{\chi}, \tag{3.4}$$

*where $G(x,t;\mu)$ is the Green's function of the boundary value problem* (1.1)-(1.3) *defined by*

$$G(x,t;\mu)=-\frac{1}{\Delta(\mu)}\begin{cases}\theta_2(x,\mu)\theta_1(t,\mu), & t\le x,\\ \theta_1(x,\mu)\theta_2(t,\mu), & x\le t.\end{cases}$$

*Conversely, the function $\phi(x,\mu)$ defined by* (3.4) *satisfies* (3.1) *and* (3.2), (3.3).

*Proof.* We shall search the solution of the boundary value problem (3.1)-(3.3) as

$$\phi(x,\mu)=c_1(x)\theta_1(x,\mu)+c_2(x)\theta_2(x,\mu) \tag{3.5}$$

where the functions $c_1(x)$ and $c_2(x)$ are the solutions of the system of equations

$$\begin{cases}D_{q,x}c_1(x)\theta_1(x,\mu)+D_{q,x}c_2(x)\theta_2(x,\mu)=0,\\ D_{q,x}c_1(x)D_{q,x}\theta_1(x,\mu)+D_{q,x}c_2(x)D_{q,x}\theta_2(x,\mu)=f(x).\end{cases} \tag{3.6}$$

If the functions $D_{q,x}c_i(x)$ $(i=1,2)$ are $q$-integrable on $[0,t]$ then

$$\lim_{n\to\infty} tq^n\theta_i\left(tq^{n+1},\lambda\right)f(tq^{n+1})=0,\quad i=1,2,$$

holds. Now, let us define the $q$-geometric set $S_f$ by

$$S_f:=\left\{x\in[0,\pi]:\lim_{n\to\infty} xq^n\left|f(xq^n)\right|^2=0\right\}.$$

Since $f\in L_q^2(0,\pi)$ the set $S_f$ is a $q$-geometric set containing $\{aq^m : m\in\mathbb{N}_0\}$. Therefore, the functions $D_qc_i(\cdot)$ $(i=1,2)$ are $q$-integrable on $[0,x]$ for all $x\in S_f$ and the solutions of (3.6) are

$$\left\{\begin{aligned} c_1(x)&=\tilde{c_1}+\frac{q}{\Delta(\mu)}\int_0^x\theta_2(qt,\mu)f(qt)d_qt,\\ c_2(x)&=\tilde{c_2}+\frac{q}{\Delta(\mu)}\int_x^\pi\theta_1(qt,\mu)f(qt)d_qt,\end{aligned}\right.\tag{3.7}$$

where $\tilde{c_1}$, $\tilde{c_2}$ are unknown constants and $x\in S_f$. Substituting (3.7) into (3.5) and taking (3.2), (3.3) into consideration leads us to (3.4). Conversely, if $\phi(x,\mu)$ is given by (3.4), then it is a solution of (3.1) which satisfies the boundary conditions (3.2), (3.3) and this completes the proof. ☐

The theorem which is given below lists a number of properties of the Green's function.

**Theorem 3.2.** *Green's function has the following properties:*

(a) $G(x,t,\mu)$ *is continuous at the point* $(0,0)$*;*
(b) $G(x,t,\mu)=G(t,x,\mu)$*;*
(c) *for each fixed* $t\in(0,q\pi]$, $G(x,t,\mu)$ *satisfies the q-difference equation* (3.1) *in the intervals* $[0,t)$, $(t,\pi]$ *and it also satisfies the boundary conditions* (3.2)-(3.3).

*Proof.* The proof can easily be obtained by using a similar procedure to [10]. ☐

**Acknowledgements.** We are immensely grateful to the anonymous reviewers for their valuable comments.

## References

[1] L. Abreu, *Sampling theory associated with q-difference equations of the Sturm-Liouville type*, J. Phys. A **38**(48) (2005), 10311–10319.
[2] H. I. Abdel-Gawad and A. A. Aldailami, *On q-dynamic equations modelling and complexity*, Appl. Math. Model. **34**(3) (2010), 697–709.
[3] M. Adivar and M. Bohner, *Spectral analysis of q-difference equations with spectral singularities*, Math. Comput. Modelling **43**(7-8) (2006), 695–703.
[4] A. Algin, *A comparative study on q-deformed fermion oscillators*, International Journal of Theoretical Physics **50**(5) (2011), 1554–1568.
[5] B. P. Allahverdiev and H. Tuna, *An expansion theorem for q-Sturm-Liouville operators on the whole line*, Turkish J. Math. **42**(3) (2018), 1060–1071.

[6] M. A. Al-Towailb, *A q-fractional approach to the regular Sturm-Lioville problems*, Electron. J. Differential Equations **2017**(88) (2017), 1–13.

[7] M. Annaby, J. Bustoz, and M. Ismail, *On sampling theory and basic Sturm-Liouville systems*, J. Comput. Appl. Math. **206** (2007), 73–85.

[8] M. H. Annaby and Z. S. Mansour, *Basic Sturm-Liouville problems*, J. Phys. A **38** (2005), 3775–3797.

[9] M. H. Annaby and Z. S. Mansour, *Asymptotic formulae for eigenvalues and eigenfunctions of q-Sturm-Liouville problems*, Math. Nachr. **284**(4) (2011), 443–470.

[10] M. H. Annaby and Z. S. Mansour, *q-Fractional Calculus and Equations*, Springer, Berlin, Heidelberg, 2012.

[11] M. H. Annaby, Z. S. Mansour, I. A. Soliman, *q-Titchmarsh-Weyl theory: series expansion*, Nagoya Math. J. **205** (2012), 67–118, DOI 10.1215/00277630-1543787.

[12] M. B. Bekker, M. J. Bohner, A. N. Herega and H. Voulov, *Spectral analysis of a q-difference operator*, J. Phys. A **43**(14) (2010), 15 pages.

[13] P. A. Binding, P. J. Browne and K. Seddighi, *Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **37** (1993), 57–72.

[14] M. Bohner and T. Hudson, *Euler-type boundary value problems in quantum calculus*, IJAMAS **9**(J07) (2007), 19–23.

[15] A. Dobrogowska and A. Odzijewicz, *Second order q-difference equations solvable by factorization method*, J. Comput. Appl. Math. **193**(1) (2006), 319–346.

[16] V. V. Eremin and A. A. Meldianov, *The q-deformed harmonic oscillator, coherent states, and the uncertainty relation*, Theoret. and Math. Phys. **147**(2) (2008), 709–715.

[17] A. Eryilmaz, *Spectral analysis of a q-Sturm Liouville problem with the spectral parameter in the boundary condition*, J. Funct. Spaces **2012** (2012) Article ID 736437, 17 pages.

[18] G. Freiling and V. A. Yurko, *Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter*, Inverse Problems **26**(055003) (2010), 17 pages.

[19] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **77** (1977), 293–308.

[20] M. S. Ben Hammouda and A. Nemri, *Polynomial expansions for solutions of higher-order q-Bessel heat equation*, Tamsui Oxf. J. Inf. Math. Sci. **24**(2) (2008), 153–174.

[21] W. Hahn, *Beitrage zur theorie der heineschen reihen*, Math. Nachr. **2** (1949), 340–379.

[22] A. Huseynov and E. Bairamov, *An eigenvalue problem for quadratic pencils of q-difference equations and its applications*, Appl. Math. Lett. **22**(4) (2009), 521–527.

[23] F. H. Jackson, *On q-definite integrals*, The Quarterly Journal of Pure and Applied Mathematics **41** (1910), 193–203.

[24] H. Mabrouk, *q-heat operator and q-Poisson's operator*, Fract. Calc. Appl. Anal. **9**(3) (2006), 265–286.

[25] Kh. R. Mamedov and F. Ayca Cetinkaya, *Inverse problem for a class of Sturm-Liouville operator with spectral parameter in boundary condition*, Bound. Value Probl. **2013**(183) (2013), 16 pages.

[26] Z. S. Mansour, *On fractional q-Sturm-Liouville problems*, J. Fixed Point Theory Appl. **19** (2017), 1591–1612.

[27] A. Matsuo, *Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations*, Comm. Math. Phys. **151** (1993), 263–273.

[28] S. Sergeev, *Quantum curve in q-oscillator model*, Int. J. Math. Math. Sci. **2006** (2006), 1–31, Article ID 92064.

[29] T. Tsuda, *On an integrable system of q-difference equations satisfied by the universal characters: Its lax formalism and an application to q-Painleve equations*, Comm. Math. Phys. **293**(2) (2010), 347–359.

[30] Y. P. Wang, *Inverse problems for a class of Sturm-Liouville operators with the mixed spectral data*, Oper. Matrices **11**(1) (2017), 89–99.

[1]Department of Mathematics,
Science and Letters Faculty,
Mersin University, 33343, Mersin, Turkiye
*Email address*: faycacetinkaya@mersin.edu.tr