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ON GENERALIZED ROTER TYPE MANIFOLDS

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Dedicated to the memory of Professor Witold Roter

Abstract. In the literature of Riemannian geometry there are many conditions for the equivalency of semisymmetric (resp., pseudosymmetric) and Ricci-semisymmetric (resp., Ricci-pseudosymmetric) manifolds. The object of the present paper is to investigate a sufficient condition for the equivalency of semisymmetric (resp., pseudosymmetric) and Ricci-semisymmetric (resp., Ricci-pseudosymmetric) manifolds. It is shown that generalized Roter type condition is a sufficient condition for the equivalency of such structures. Also we obtain alternative proofs of the theorems as given by Deszcz and his coauthors ([5] and [36]) for the equivalency of such structures. Finally the existence of manifolds satisfying generalized Roter type condition is ensured by some non-trivial examples.

1. Introduction

Let M be a smooth and connected semi-Riemannian manifold of dimension $n \geq 3$ with semi-Riemannian metric g (throughout the paper we will consider all the manifolds with such considerations, unless otherwise stated). Let ∇, R, S, κ, C and $C^\infty(M)$ be respectively the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor, the scalar curvature, the Weyl conformal curvature tensor and the ring of all smooth functions on M respectively. In semi-Riemannian geometry, many curvature restricted geometric structures are formed by imposing some restrictions on various curvature tensors. For various curvature restricted geometric structures we refer the reader to see [3,4,10,11,21–24,52,55,59–65,69,70] and also references therein.

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Again a particular restriction imposed on different curvature tensors gives rise to different structures. Recently, in [56] the present authors studied the equivalency of these structures and also obtained their classification. If we impose a certain restriction on R and S then it yields different structures, such as, (i) local symmetry and Ricci-symmetry, (ii) semisymmetry and Ricci-semisymmetry, (iii) pseudosymmetry and Ricci-pseudosymmetry etc. If the restriction operator commutes with contractions, the structure due to R implies the same due to S but not conversely, in general (see [7, 18, 56]). Hence in all the three cases the first one implies the second but not conversely, in general. Consequently it is interesting to investigate the sufficient conditions under which the above pairs of structures are equivalent. In this direction there are many works in the literature, e.g., P. J. Ryan Problem [48], named Ryan's problem [46] (Chapter 12.7), which states the equivalency of semisymmetry and Ricci-semisymmetry on hypersurfaces in Euclidean spaces, as well as semi-Euclidean spaces (see, e.g., [1, 17, 18, 26, 30, 36, 37] and also references therein).

The main objective of this paper is to investigate a sufficient condition for which semisymmetry (resp., pseudosymmetry) and Ricci-semisymmetry (resp., Ricci-pseudosymmetry) are equivalent, and it is proved that if a manifold satisfies generalized Roter type condition then such structures are equivalent. It may be mentioned that under generalized Roter type condition various curvature restricted geometric structures are equivalent, e.g., $C \cdot R = 0$ and $C \cdot S = 0$ are equivalent.

In 1998 Arslan et al. ([5], Theorem 5.2) proved the equivalency of semisymmetric and Ricci-semisymmetric manifolds under certain conditions stated as follows.

Theorem I. Let M, $n \geq 4$, be a semi-Riemannian Ricci-semisymmetric manifold satisfying $R \cdot R = Q(S, R)$. If M is a manifold with pseudosymmetric Weyl tensor then $R \cdot R = 0$ on $U_S = \left\{x \in M : S - \frac{\kappa}{n} g \neq 0 \text{ at } x\right\}$.

Some extensions of the above mentioned result are given in [28] (see Proposition 3.2 and Theorem 3.3, as well as Theorem 4.4 and Corollary 4.5).

Again in 1999 Deszcz et al. ([36], Theorem 4.1) investigated equivalency of pseudosymmetric and Ricci-pseudosymmetric manifolds and proved the following.

Theorem II. Let M, $n \geq 4$, be a semi-Riemannian Ricci-pseudosymmetric manifold (i.e., $R \cdot S = L_S Q(g, S)$ on U_S) satisfying $R \cdot R - Q(S, R) = L_1 Q(g, C)$ and $R \cdot C = L_2 Q(S, C)$ for some scalars L_1 and L_2 on $U_C = \{x \in M : C \neq 0 \text{ at } x\}$ and $U_1 = \{x \in M : Q(S, C) \neq 0 \text{ at } x\}$, respectively. Let $x \in U = U_S \cap U_1$. Assume that $L_S = \frac{\kappa}{n} L_2$ at x, $L_S \neq 0$. Then $R \cdot R = L_S Q(g, R)$ holds at x.

The paper is organized as follows. Section 2 deals with various rudimentary facts of different curvature restricted geometric structures. Section 3 is devoted to the study of generalized Roter type condition and proved that under such condition the curvature restricted geometric structures obtained by a commutative $C^\infty(M)$-linear operator imposed on R and S are equivalent. Hence as a particular case, on a Roter type manifold semisymmetry (resp., pseudosymmetry) and Ricci-semisymmetry (resp., Ricci-pseudosymmetry) are equivalent. As the consequence of our result it follows that on a semi-Riemannian manifold satisfying generalized Roter type condition,

(i) $D \cdot R = 0 \Leftrightarrow D \cdot S = 0$ and
(ii) $D \cdot R = LQ(g, R) \Leftrightarrow D \cdot S = LQ(g, S)$,

where L is a smooth function on U_S and D is any one of R, C, W, K. In this section we obtain alternative proofs of Theorem I and Theorem II. We also obtain a sufficient condition for which a generalized Roter type condition turns into a Roter type condition. The last section deals with the proper existence of manifolds satisfying generalized Roter type condition.

2. Preliminaries

Let us consider a connected semi-Riemannian manifold M of dimension $n \geq 3$ with semi-Riemannian metric g. Let $\mathcal{T}^r_k(M)$ be the space of all smooth tensor fields of type (r, k) on M, $r, k \in \mathbb{N} \cup \{0\}$. For $A, E \in \mathcal{T}^0_2(M)$, their Kulkarni-Nomizu product ([26, 30, 34, 41]) $A \wedge E$ is given by

$$\begin{aligned}(A \wedge E)(X_1, X_2, Y_1, Y_2) =& A(X_1, Y_2)E(X_2, Y_1) + A(X_2, Y_1)E(X_1, Y_2)\\ &- A(X_1, Y_1)E(X_2, Y_2) - A(X_2, Y_2)E(X_1, Y_1),\end{aligned}$$

where $X_1, X_2, Y_1, Y_2 \in \chi(M)$, $\chi(M)$ being the Lie algebra of all smooth vector fields on M. Throughout the paper we consider $X, Y, X_i, Y_i \in \chi(M)$, $i = 1, 2, \ldots$.
Again, if $A \in \mathcal{T}^0_2(M)$ and $T \in \mathcal{T}^0_k(M)$, then the generalized Kulkarni-Nomizu product ([6, 27]) is defined as $(0, k+2)$ tensor $A \wedge T$ given by

$$\begin{aligned}&(A \wedge T)(X_1, X_2, Y_1, Y_2, \cdots, Y_k)\\ =& A(X_1, Y_2)T(X_2, Y_1, \cdots, Y_k) + A(X_2, Y_1)T(X_1, Y_2, \ldots, Y_k)\\ &- A(X_1, Y_1)T(X_2, Y_2, \cdots, Y_k) - A(X_2, Y_2)T(X_1, Y_1, \ldots, Y_k).\end{aligned}$$

If $A \in \mathcal{T}^0_2(M)$ and A is symmetric, then a $C^\infty(M)$-linear endomorphism $\mathcal{A}$, called the corresponding endomorphism operator, is defined as

$$g(\mathcal{A}X, Y) = A(X, Y).$$

The k-th level tensor of A, $k \geq 1$, denoted by A^k, of the same order with corresponding endomorphism operator $\mathcal{A}^k$ is defined as follows: $A^1 = A$ and

$$A^k(X, Y) = g(\mathcal{A}^k X, Y) = A(\mathcal{A}^{k-1}X, Y), \quad k \geq 2.$$

We also have $\mathcal{A}^0$ =identity and $A^0 = g$. Thus the second, third and fourth level Ricci tensor S^2, S^3, S^4 are respectively given as follows:

$$\begin{aligned}S(X, Y) = g(\mathcal{S}X, Y), \quad S^2(X, Y) &= g(\mathcal{S}^2 X, Y) = S(\mathcal{S}X, Y),\\ S^3(X, Y) &= g(\mathcal{S}^3 X, Y) = S(\mathcal{S}^2 X, Y),\\ S^4(X, Y) &= g(\mathcal{S}^4 X, Y) = S(\mathcal{S}^3 X, Y).\end{aligned}$$

A tensor D of type $(1, 3)$ on M is said to be a generalized curvature tensor ([26,30,34]), if

(i) $D(X_1, X_2)X_3 + D(X_2, X_3)X_1 + D(X_3, X_1)X_2 = 0$;
(ii) $D(X_1, X_2)X_3 + D(X_2, X_1)X_3 = 0$;

(iii) $D(X_1, X_2, X_3, X_4) = D(X_3, X_4, X_1, X_2)$;

where $D(X_1, X_2, X_3, X_4) = g(D(X_1, X_2)X_3, X_4)$ for all $X_1, X_2, X_3, X_4 \in \chi(M)$. Here we use the same symbol D for the generalized curvature tensor of type $(1, 3)$ and $(0, 4)$. Moreover if D satisfies the second Bianchi like identity, i.e.,

$$(\nabla_{X_1} D)(X_2, X_3)X_4 + (\nabla_{X_2} D)(X_3, X_1)X_4 + (\nabla_{X_3} D)(X_1, X_2)X_4 = 0,$$

then D is called a proper generalized curvature tensor. Some most useful generalized curvature tensors are the Gaussian curvature tensor G, the Weyl conformal curvature tensor C, the concircular curvature tensor W and the conharmonic curvature tensor K, which are respectively given by

$$\begin{aligned} G =& \frac{1}{2}(g \wedge g), \\ C =& R - \frac{1}{(n-2)}(g \wedge S) + \frac{\kappa}{2(n-1)(n-2)}(g \wedge g), \\ W =& R - \frac{\kappa}{2n(n-1)}(g \wedge g), \\ K =& R - \frac{1}{(n-2)}(g \wedge S). \end{aligned}$$

We can easily operate an $C^\infty(M)$-linear endomorphism $\mathcal{H}$ over $\chi(M)$, on a $(0, k)$-tensor T, $k \geq 1$, and get the tensor $\mathcal{H}T$, given by ([26, 30, 34])

$$(\mathcal{H}T)(X_1, X_2, \dots, X_k) = -T(\mathcal{H}X_1, X_2, \dots, X_k) - \cdots - T(X_1, X_2, \dots, \mathcal{H}X_k).$$

Now for $D \in \mathcal{T}^0_4(M)$ and given two vector fields $X, Y \in \chi(M)$ one can define an endomorphism $\mathcal{D}(X, Y)$ by

$$\mathcal{D}(X, Y)(X_1) = D(X, Y)X_1, \quad \text{for all } X_1 \in \chi(M).$$

Again if $X, Y \in \chi(M)$ then for a symmetric $(0, 2)$-tensor A one can define an endomorphism $X \wedge_A Y$, by

$$(X \wedge_A Y)X_1 = A(Y, X_1)X - A(X, X_1)Y, \quad \text{for all } X_1 \in \chi(M).$$

It is clear that the endomorphisms $\mathcal{D}(X, Y)$ and $X \wedge_A Y$ both are $C^\infty(M)$-linear.

For $T \in \mathcal{T}^0_k(M)$, $k \geq 2$, and $D \in \mathcal{T}^0_4(M)$, one can define a tensor $D \cdot T \in \mathcal{T}^0_{k+2}(M)$ given by ([26, 30, 34, 56])

$$\begin{aligned} & D \cdot T(X_1, X_2, \dots, X_k; X, Y) = (\mathcal{D}(X, Y) \cdot T)(X_1, X_2, \dots, X_k) \\ =& -T(\mathcal{D}(X, Y)X_1, X_2, \dots, X_k) - \cdots - T(X_1, X_2, \dots, \mathcal{D}(X, Y)X_k) \end{aligned}$$

and for $A \in \mathcal{T}^0_2(M)$ one can also define a tensor $Q(A, T) \in \mathcal{T}^0_{k+2}(M)$ as follows ([26, 30, 34, 56, 68])

$$\begin{aligned} & Q(A, T)(X_1, X_2, \dots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, X_2, \dots, X_k) \\ =& -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \cdots - T(X_1, X_2, \dots, (X \wedge_A Y)X_k). \end{aligned}$$

From the above discussion, we can state the following.

Lemma 2.1. *Let $\mathcal{H}$ be an endomorphism over the Lie algebra $\chi(M)$ of a semi-Riemannian mannifold M, $n \geq 3$, and A, E be two $(0,2)$-tensors on M. Then*

$$\mathcal{H}(A \wedge E) = (A \wedge \mathcal{H}E) + (E \wedge \mathcal{H}A).$$

Lemma 2.2 (see, e.g., [32]). *On a semi-Riemannian manifold M, $n \geq 3$, the following conditions hold:*

$$g \wedge Q(g, S) = Q(g, g \wedge S) = -\frac{1}{2}Q(S, g \wedge g) = -Q(S, G).$$

Definition 2.1. [9, 56, 67] For $T \in \mathcal{T}^0_k(M)$ and $D \in \mathcal{T}^0_4(M)$, a semi-Riemannian manifold M, $n \geq 3$, is said to be T-semisymmetric type if $D \cdot T = 0$.

In particular, for $D = R$ and $T = R$ (resp., S, P, C, W, K), the manifold is called semisymmetric (resp., Ricci, projectively, conformally, concircularly, conharmonically semisymmetric).

Definition 2.2. [2, 20, 22, 56] For $T \in \mathcal{T}^0_k(M)$ and $D_1, D_2, \ldots, D_r \in \mathcal{T}^0_4(M)$, a semi-Riemannian manifold M, $n \geq 3$, is said to be T-pseudosymmetric type if the tensors $D_1 \cdot T$, $D_2 \cdot T$, …, $D_r \cdot T$ are linearly dependent.

In particular, if $r = 2$, $D_1 = R$, $D_2 = G$ and $T = R$ (resp., S, P, C, W, K), then the manifold is called Deszcz pseudosymmetric (resp., Ricci, projectively, conformally, concircularly, conharmonically pseudosymmetric). Especially, if $r = 2$, $D_1 = C$, $D_2 = G$ and $T = C$, then M is called a manifold of pseudosymmetric Weyl tensor. Again if $r = 2$, $D_1 = R$, $\mathcal{D}_2(X, Y) = X \wedge_S Y$ and $T = R$, then the manifold is called Ricci generalized pseudosymmetric. Thus a manifold is Deszcz pseudosymmetric, Ricci-pseudosymmetric [5, 22], manifold of pseudosymmetric Weyl tensor [5, 22] and Ricci generalized pseudosymmetric [15, 16, 22] respectively if and only if

$$\begin{aligned}
R \cdot R =& L_R Q(g, R) \quad \text{holds on } U_R = \left\{x \in M : R - \frac{\kappa}{n(n-1)}G \neq 0 \text{ at } x\right\},\\
R \cdot S =& L_R Q(g, S) \quad \text{holds on } U_S = \left\{x \in M : S - \frac{\kappa}{n}g \neq 0 \text{ at } x\right\},\\
C \cdot C =& L_C Q(g, C) \quad \text{holds on } U_C = \{x \in M : C \neq 0 \text{ at } x\} \text{ and}\\
R \cdot R =& L Q(S, R) \quad \text{holds on } U_1 = \{x \in M : Q(S, R) \neq 0 \text{ at } x\}.
\end{aligned}$$

We note that $U_S \cup U_C = U_R$. We mention that Deszcz pseudosymmetric manifolds are also named as pseudosymmetric in the sense of Ryszard Deszcz [71, 73] or Deszcz symmetric spaces (see, e.g., [12, 14, 72, 73]).

Definition 2.3 ([23–25], see also [7, 42, 43, 51]). A semi-Riemannian manifold M, $n \geq 4$, is said to be a Roter type manifold if its curvature tensor can be expressed as

$$R = N_1 g \wedge g + N_2 g \wedge S + N_3 S \wedge S, \tag{2.1}$$

where $N_1, N_2, N_3 \in C^\infty(M)$ and such a manifold is denoted by RT_n.

For details about RT_n we refer the reader to see [24, 28, 38–40, 42, 44, 45] and also references therein. It seems that [47] is the first paper containing results on semi-Riemannian manifolds satisfying (2.1). For any $(0,4)$ tensor T and any two $(0,2)$ tensors A, E on the manifold M, the tensor $W(T,A,E)$, given by

$$W(T,A,E) = T - N_1 A \wedge A - N_2 A \wedge E - N_3 E \wedge E$$

is said to be a Roter type tensor. If $W(T,A,E)$ vanishes identically on M, then the manifold is said to satisfy Roter type condition. Especially, if $W(R,g,S) = 0$ on M, then M is called Roter type manifold.

Some comments on pseudosymmetric manifolds (Deszcz symmetric spaces) and Roter type manifolds (named also Roter spaces) are given in Section 1 of [14]: "From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms." and "From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms."

Definition 2.4. [51] A semi-Riemannian manifold M, $n \geq 4$, is said to be a generalized Roter type if its curvature tensor can be expressed as

$$R = L_1 g \wedge g + L_2 g \wedge S + L_3 S \wedge S + L_4 g \wedge S^2 + L_5 S \wedge S^2 + L_6 S^2 \wedge S^2, \tag{2.2}$$

where $L_i \in C^\infty(M)$, $i = 1, 2, \ldots, 6$ and such a manifold is denoted by GRT_n.

For any $(0,4)$ tensor T and any three $(0,2)$ tensors A, E, F on M, the tensor $GW(T,A,E,F)$, given by

$$GW(T,A,E,F) = T - L_1 A \wedge A - L_2 A \wedge E - L_3 E \wedge E - L_4 A \wedge F - L_5 E \wedge F - L_6 F \wedge F$$

is said to be a generalized Roter type tensor. If $GW(T,A,E,F)$ vanishes identically on M, then the manifold is said to satisfy generalized Roter type condition. Especially, if $GW(R,g,S,S^2) = 0$ on M, then M is called generalized Roter type manifold.

A RT_n (resp., GRT_n) manifold is called proper if it is a non-conformally flat manifold (resp., a not RT_n manifold). We mention that proper GRT_n manifold was already investigated in [49] (see also [43], eq. (5.5)) and recently in [31–33, 35, 50, 51, 57] and [58]. But the name "generalized Roter type" was first used in [51]. We mention that in [57] the present authors studied the characterization of a warped product generalized Roter type manifold.

Definition 2.5. [8] A semi-Riemannian manifold M, $n \geq 3$, is said to be Einstein if its Ricci tensor S is linearly dependent with its metric tensor g.

For an Einstein manifold we have $S = \frac{\kappa}{n} g$.

Contracting (2.1) and (2.2), we get some generalizations of Einstein manifold.

Definition 2.6. [8] A semi-Riemannian manifold M, $n \geq 3$, is said to be an Einstein manifold of level 2, 3 and 4 (briefly, Ein(2), Ein(3) and Ein(4)) if

$$S^2 + a_1 S + a_2 g = 0,$$

$$S^3 + a_3 S^2 + a_4 S + a_5 g = 0$$

and

$$S^4 + a_6 S^3 + a_7 S^2 + a_8 S + a_9 g = 0,$$

holds respectively for some $a_i \in C^\infty(M)$, $i = 1, 2, \ldots, 9$.

We note that an Einstein manifold is Ein(1) and conversely. We also note that every proper GRT_n is Ein(4) and every proper RT_n is Ein(2).

Definition 2.7. A semi-Riemannian manifold M, $n \geq 3$, is said to be of quasi-constant curvature ([13], [53] and also references therein) if R can be expressed as

$$R = \gamma G + \delta g \wedge (\mu \otimes \mu),$$

where γ, $\delta \in C^\infty(M)$ and $\mu \in \mathcal{T}_1^0(M)$.

Again, a semi-Riemannian manifold M, $n \geq 3$, is said to be quasi-Einstein (see, e.g., [12, 30, 37, 54, 66] and also references therein) if its Ricci tensor is given by

$$S = \alpha g + \beta \eta \otimes \eta,$$

where α, $\beta \in C^\infty(M)$ and $\eta \in \mathcal{T}_1^0(M)$.

3. Semi-Riemannian Manifolds Satisfying Generalized Roter Type Condition

Proposition 3.1. *A semi-Riemannian* Ein(2) *manifold M is a GRT_n if and only if it is a RT_n.*

Proof. Let M be a GRT_n. Since M is an Ein(2) manifold, S^2 can be expressed as a linear combination of S and g. Thus putting the value of S^2 in terms of S and g in (2.2) we get our assertion. The converse part is obvious. □

Proposition 3.2. *A semi-Riemannian manifold M satisfying the Roter type condition* (2.1) *with $N_3 \neq 0$, also realizes the generalized Roter type condition* (2.2) *such that*

$$L_1 = -\frac{b^2 L_6 + 2b L_4 N_3 - 4N_1 N_3^2}{4N_3^2},$$

$$L_2 = -\frac{ab L_6 + a L_4 N_3 + b L_5 N_3 - 2N_2 N_3^2}{2N_3^2} \quad \textit{and}$$

$$L_3 = -\frac{a^2 L_6 + 2a L_5 N_3 - 4N_3^3}{4N_3^2},$$

where $a = (n-2)N_2 + 2N_3\kappa$ and $b = 2(n-1)N_1 + N_2\kappa$.

Proof. Contracting the equation (2.1), we get

$$a_1 g + a_2 S + a_3 S^2 = 0,$$

where $a_1 = 2N_1(n-1) + N_2\kappa$, $a_2 = N_2(n-2) + 2N_3\kappa$, $a_3 = -2N_3$. Again since $N_3 \neq 0$, $a_3 \neq 0$, and hence we can evaluate S^2 in terms of S and g ([28] and [29]).

Now putting the value of S^2 in terms of S and g in (2.2) we get another Roter type condition. Then comparing this Roter type condition with (2.1) we get the result. $\square$

Note. It is obvious that a semi-Riemannian manifold satisfying (2.1) satisfies (2.2) with $L_1 = N_1$, $L_2 = N_2$, $L_3 = N_3$ and $L_4 = L_5 = L_6 = 0$, which can be obtained from the above proposition as a special case.

Similarly as the proof of Proposition 3.2, we can obtain the following.

Proposition 3.3. *A non flat semi-Riemannian manifold M of constant curvature fulfills* (2.1) *such that*

$$\frac{N_1 n^2 + \kappa(N_2 n + N_3 \kappa)}{n} = \frac{\kappa}{2(n-1)}.$$

It also satisfies (2.2) *such that*

$$\frac{L_1 n^4 + \kappa\{L_2 n^3 + \kappa[n^2(L_3 + L_4) + L_5 n\kappa + L_6 \kappa^2]\}}{n^3} = \frac{\kappa}{2(n-1)}.$$

Proposition 3.4. *A semi-Riemannian manifold M satisfying generalized Roter type condition is a manifold of constant curvature if and only if it is Einstein.*

Proof. If M is Einstein, then $S^2 = \left(\frac{\kappa}{n}\right)^2 g$. Hence from (2.2), we get

$$R = \frac{2}{n^4}\left\{L_1 n^4 + \kappa[L_2 n^3 + \kappa((L_3 + L_4)n^2 + L_5 n\kappa + L_6 \kappa^2)]\right\} G,$$

which implies that the manifold is of constant curvature. The converse is obvious. $\square$

Proposition 3.5. *A semi-Riemannian manifold M satisfying generalized Roter type condition is a manifold of quasi-constant curvature if and only if it is quasi-Einstein.*

Proof. If M is a quasi-Einstein, then $S^2 = \alpha^2 g + \beta(2\alpha + \beta||\eta||^2)\eta \otimes \eta$. Hence from (2.2), we get

$$R = \gamma G + \delta g \wedge (\eta \otimes \eta),$$

where $\gamma = 2\left[L_1 + \alpha(L_2 + \alpha(L_3 + L_4 + \alpha(L_5 + L_6\alpha)))\right]$ and $\delta = \beta[L_2 + \alpha(2(L_3 + L_4) + 3L_5\alpha + 4L_6\alpha^2) + (L_4 + \alpha(L_5 + 2L_6\alpha))\beta||\eta||^2]$. Thus the manifold is of quasi-constant curvature. The converse part is obvious. $\square$

If an operator is linear over $\mathbb{R}$ only, then it is called 1st type, and if it linear over $C^\infty(M)$ also, then it is called 2nd type (see [56]). Again if the operator commutes with contraction then it is called commutative. For details about the classification of such curvature restriction operators, we refer the reader to see [56].

Theorem 3.1. *Let M be a semi-Riemannian manifold satisfying generalized Roter type condition. Then the structures defined by a commutative 2nd type restriction imposed on R and S are equivalent.*

Proof. Consider a commutative 2nd type restriction operator $\mathcal{L}$. Then the corresponding geometric structure due to a tensor T is given by $\mathcal{L}(T) = 0$. Now as $\mathcal{L}$ is commutative so by contracting $\mathcal{L}(R) = 0$, we get $\mathcal{L}(S) = 0$. Hence to prove the theorem it is sufficient to show $\mathcal{L}(S) = 0 \Rightarrow \mathcal{L}(R) = 0$.

Let us now consider $\mathcal{L}(S) = 0$. Then from Lemma 2.1,

$$\mathcal{L}(S \wedge S) = 2S \wedge \mathcal{L}S = 0.$$

Since $\mathcal{L}$ is commutative, contracting the above, we get

$$\mathcal{L}(S^2 - (n-2)S) = 0 \Rightarrow \mathcal{L}(S^2) = 0.$$

Again since $\mathcal{L}$ is of 2nd type and commutative, applying $\mathcal{L}$ on (2.2) and using Lemma 2.1, we have

$$\begin{aligned}\mathcal{L}(R) =& 0 + L_2 g \wedge \mathcal{L}(S) + 2L_3 S \wedge \mathcal{L}(S) + L_4 g \wedge \mathcal{L}(S^2)\\ &+ L_5 S^2 \wedge \mathcal{L}(S) + L_5 S \wedge \mathcal{L}(S^2) + 2L_6 S^2 \wedge \mathcal{L}(S^2) = 0.\end{aligned}$$

This proves the theorem. □

Since the curvature restriction operators of semisymmetric and pseudosymmetric structures are commutative and of 2nd type, we can state the following.

Corollary 3.1. *On a GRT_n, semisymmetry and Ricci-semisymmetry are equivalent.*

Corollary 3.2. *On a GRT_n, pseudosymmetry and Ricci-pseudosymmetry are equivalent.*

Corollary 3.3 (see [19] and references therein)**.** *On a RT_n or a conformally flat manifold, the semisymmetry* (*resp., pseudosymmetry*) *and Ricci-semisymmetry* (*resp., Ricci-pseudosymmetry*) *are equivalent.*

Corollary 3.4. *If D_1 and D_2 are two generalized curvature tensors on a GRT_n, then*

(i) $D_1 \cdot R = 0 \Leftrightarrow D_1 \cdot S = 0$ *and*

(ii) $D_1 \cdot R = LD_2 \cdot R \Leftrightarrow D_1 \cdot S = LD_2 \cdot S$,

where L is a smooth function on $\{x \in M : D_2 \cdot S \neq 0$ at $x\}$.

Remark 3.1. We note that on a GRT_n to study any 2nd type commutative semisymmetric type or pseudosymmetric type condition imposed on R, it is sufficient to study the condition on S.

Corollary 3.5. *If D is a generalized curvature tensor on a GRT_n, then*

(i) $D \cdot S = 0 \Rightarrow D \cdot R = 0$ *and hence* $D \cdot C = 0$;

(ii) $D \cdot S = L_S Q(g, S) \Rightarrow D \cdot R = L_S Q(g, R)$ *and hence* $D \cdot C = L_S Q(g, C)$.

Especially, if D is R, C, W and K, then we can obtain the consequent results.

Proposition 3.6. [16] *Let M, $n \geq 3$, be a semi-Riemannian manifold. Let A be a non-zero symmetric* $(0, 2)$*-tensor and B be a generalized curvature tensor. If at $x \in M$, $Q(A, B) = 0$, then*

(i) B *and* $A \wedge A$ *are linearly dependent if* $A(X,Y) \neq \frac{1}{A(V,V)} A(V,X)A(V,Y)$*;*

(ii) $\sum_{X,Y,Z} a(X)\mathcal{B}(X,Y) = 0$ *if* $A(X,Y) = \frac{1}{A(V,V)} A(V,X)A(V,Y)$*;*

where V *is a vector at* x *such that* $A(V,V) \neq 0$ *and* $\sum\limits_{X,Y,Z}$ *denotes the cyclic sum.*

Moreover we have the following.

Proposition 3.7 ([43], Theorems 2.8 and 2.9)**.** *Let* A *be a symmetric* $(0,2)$*-tensor and* B *be a generalized curvature tensor at a point* x *of a semi-Riemannian manifold* M*,* $n \geq 4$*. Moreover, let* $\operatorname{rank}(A - \rho g) > 1$ *for any* $\rho \in \mathbb{R}$*. Then the Tachibana tensors* $Q(g,B)$*,* $Q(g, g \wedge A)$*,* $Q(g, A \wedge A)$ *and* $Q(A,B)$ *are linearly dependent at* x *if and only if the tensor* B *has at this point a decomposition of the form*

$$B \;=\; \frac{\alpha}{2} A \wedge A + \beta\, g \wedge A + \gamma\, G,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$*.*

Proposition 3.8. [36] *Let* M*,* $n \geq 3$*, be a semi-Riemannian manifold. If at a point* x *in* M*,*

$$S = \mu g + \rho a \otimes a \quad \text{and} \quad \sum_{X,Y,Z} a(X)\mathcal{B}(X,Y) = 0,$$

for some non-zero vector a*, where* $\mathcal{B}(X,Y)$ *is the corresponding endomorphism of* $B = R - \gamma G$ *and* $\mu, \rho, \gamma \in \mathbb{R}$*. Then at* x*, we have*

$$R \cdot R = \frac{\kappa}{n(n-1)} Q(g,R) \quad \text{and} \quad R \cdot R = Q(S,R) - \frac{(n-2)\kappa}{n(n-1)} Q(g,C).$$

Proof of Theorem II. It is given that

$$\begin{aligned} R \cdot S &= L_S Q(g,S), \\ R \cdot R - Q(S,R) &= L_1 Q(g,C) \ \text{ and} \\ R \cdot C &= L_2 Q(S,C). \end{aligned}$$

Now comparing last two results we get

$$\begin{aligned}
& Q(S,R) + L_1 Q(g,C) - \frac{1}{n-2} R \cdot g \wedge S = L_2 Q(S,C) \\
\Rightarrow\ & Q(S,R) + L_1 \left[Q(g,R) - \frac{1}{n-2} Q(g, g \wedge S) \right] - \frac{1}{n-2} g \wedge (R \cdot S) \\
& = L_2 \left[Q(S,R) - \frac{1}{n-2} Q(S, g \wedge S) + \frac{\kappa}{2(n-1)(n-2)} Q(S, g \wedge g) \right] \\
\Rightarrow\ & Q(S,R) + L_1 \left[Q(g,R) + \frac{1}{2(n-2)} Q(S, g \wedge g) \right] - \frac{L_S}{n-2} Q(g, g \wedge S) \\
& = L_2 \left[Q(S,R) + \frac{1}{2(n-2)} Q(g, S \wedge S) + \frac{\kappa}{2(n-1)(n-2)} Q(S, g \wedge g) \right]
\end{aligned}$$

(by using the Lemma 2.2)

$$\Rightarrow (1-L_2)Q(S,R)+L_1Q(g,R)+\frac{1}{2(n-2)}\left[L_1+L_S-\frac{L_2\kappa}{n-1}\right]Q(S,g\wedge g)$$
$$-\frac{L_2}{2(n-2)}Q(g,S\wedge S)=0.$$

Then we can write

$$\alpha_1Q(S,R)+\alpha_2Q(g,R)+\alpha_3Q(S,g\wedge g)+\alpha_4Q(g,S\wedge S)=0,$$

where $\alpha_1=1-L_2$, $\alpha_2=L_1$, $\alpha_3=\frac{1}{2(n-2)}\left[L_1+L_S-\frac{L_2\kappa}{n-1}\right]$ and $\alpha_4=-\frac{L_2}{2(n-2)}$.

Again as $L_S=\frac{\kappa}{n}L_2$ at x, $L_S\neq 0$, under given hypothesis of the theorem, from Remark 3.1 of [36] we have $L_S=\frac{\kappa}{n(n-1)}$ and $L_2=\frac{1}{n-1}\neq 1$ and thus $\alpha_1\neq 0$. We note that $L_S\neq 0$ implies $\kappa\neq 0$. Therefore we restrict our consideration to the set $U=U_S\cap U_1$. Hence the above equation can be written as

$$Q\left(\alpha_1S+\alpha_2g,R+\frac{\alpha_3}{\alpha_1}g\wedge g-\frac{\alpha_4}{\alpha_1}g\wedge S\right)=0.$$

Let us now consider the above situation as $Q(A,B)=0$, where $A=\alpha_1S+\alpha_2g$ and

$$B=R+\frac{\alpha_3}{\alpha_1}g\wedge g-\frac{\alpha_4}{\alpha_1}g\wedge S.$$

Then two cases arise.

Case 1. Let $x\in M$, $\operatorname{rank}(A)>1$. Then from Proposition 3.6 (also from Proposition 3.7) we have

$$B=\lambda_1A\wedge A,$$

i.e., M satisfies Roter type condition at x. Then from Theorem 3.1 and also by Corollary 3.2, we can conclude that the manifold satisfies the pseudosymmetry condition at x, since M is a Ricci-pseudosymmetric manifold.

Case 2. Let $x\in M$, $\operatorname{rank}(A)=1$ and $A\neq 0$. Then there exists some $V\in T_x(M)$ such that $A(V,V)\neq 0$ and by Proposition 3.6, we have

$$\alpha_1S+\alpha_2g=\frac{1}{\rho}a\otimes a$$

and

$$a(X)\mathcal{B}(Y,Z)+a(Y)\mathcal{B}(Z,X)+a(Z)\mathcal{B}(X,Y)=0,$$

where $a(X)=A(X,V)$ and $\rho=A(V,V)$. Thus B is some linear combination of R, G and $g\wedge(a\otimes a)$, say,

$$B=\beta_1R+\beta_2G+\beta_3g\wedge(a\otimes a).$$

Since $\tilde{B}=g\wedge(a\otimes a)$ always satisfies the condition

$$a(X)\tilde{\mathcal{B}}(Y,Z)+a(Y)\tilde{\mathcal{B}}(Z,X)+a(Z)\tilde{\mathcal{B}}(X,Y)=0,$$

$\bar{B}=\beta_1R+\beta_2G$ satisfies

$$a(X)\bar{\mathcal{B}}(Y,Z)+a(Y)\bar{\mathcal{B}}(Z,X)+a(Z)\bar{\mathcal{B}}(X,Y)=0.$$

Thus by Proposition 3.8, we have $R \cdot R = \frac{\kappa}{n(n-1)} Q(g, R)$ on U. This completes the proof.

Proof of Theorem I. If $\kappa = 0$, then the result easily follows from Theorem 4.2 of [5]. Next consider $\kappa \neq 0$. Since M is Ricci-semisymmetric and $R \cdot R = Q(S, R)$, from Proposition 5.1 of [5] we have

$$C \cdot C = \frac{n-3}{n-2} R \cdot R + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \tau \right) Q(g, C),$$

where $\tau = \frac{tr(S^2)}{\kappa}$. Again since $C \cdot C = LQ(g, C)$ for some scalar L (say), we obtain

$$\begin{aligned}
&\frac{n-3}{n-2} Q(S, R) + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \tau \right) Q(g, C) = LQ(g, C) \\
\Rightarrow &\frac{n-3}{n-2} Q(S, R) + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \tau - L(n-2) \right) \left[Q(g, R) - \frac{1}{n-2} Q(g, g \wedge S) \right] = 0 \\
\Rightarrow &\frac{n-3}{n-2} Q(S, R) + \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \tau - L(n-2) \right) \left[Q(g, R) + \frac{1}{n-2} Q(S, G) \right] = 0 \\
&\quad \text{(by using the Lemma 2.2)} \\
\Rightarrow &Q(\alpha_1 S + \alpha_2 g, R + \alpha_3 G) = 0,
\end{aligned}$$

where $\alpha_1 = \frac{n-3}{n-2}$, $\alpha_2 = \frac{1}{n-2} \left[\frac{\kappa}{n-1} - \tau - L(n-2) \right]$ and $\alpha_3 = \frac{\alpha_2}{(n-3)}$. Since $n > 3$, $\alpha_1 \neq 0$ and hence similar to the proof of the Theorem II, we get our assertion.

Theorem 3.2. *Let M be a semi-Riemannian manifold satisfying a generalized Roter type condition such that the associated scalars are constants or constant multiple of κ. Then the geometric structures defined by a commutative 1st type restriction imposed on R and S are equivalent.*

Proof. The proof is similar to the proof of the Theorem 3.1. □

Corollary 3.6. *If the associated scalars of a GRT_n manifold M are constants or constant multiple of κ, then*

(i) $\nabla R = 0 \Leftrightarrow \nabla S = 0$ *and*
(ii) $\nabla R = \Phi \otimes R \Leftrightarrow \nabla S = \Phi \otimes S$, *where Φ is an 1-form on M.*

Corollary 3.7. *On a conformally flat manifold M, $n \geq 4$, we have:*

(i) $\nabla R = 0 \Leftrightarrow \nabla S = 0$ *and*
(ii) $\nabla R = \Phi \otimes R \Leftrightarrow \nabla S = \Phi \otimes S$, *where Φ is an 1-form on M.*

4. Examples

Example 4.1. Let M_1 be a 5-dimensional connected semi-Riemannian manifold endowed with the semi-Riemannian metric

$$ds^2 = f \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + h(dx^5)^2 \right],$$

where f is a smooth function of x^1 and h is a smooth function of x^1 and x^2 such that f and h are non-zero at each point of M_1. The local components of the Riemann-Christoffel curvature tensor and the Ricci tensor (upto symmetry) which may not vanish identically are the following:

$$\begin{aligned}
R_{1212} =& R_{1313} = R_{1414} = \frac{(f')^2 - ff''}{2f}, \quad R_{2323} = R_{2424} = R_{3434} = -\frac{(f')^2}{4f},\\
R_{1515} =& \frac{1}{4}\left(-2hf'' - h_1 f' + \frac{2h\,(f')^2}{f} + \frac{fh_1^2}{h} - 2fh_{11}\right), \quad R_{1525} = \frac{1}{4}f\left(\frac{h_1h_2}{h} - 2h_{12}\right),\\
R_{2525} =& \frac{1}{4}f\left(-\frac{f'\,(hf' + fh_1)}{f^2} + \frac{h_2^2}{h} - 2h_{22}\right), \quad R_{3535} = R_{4545} = -\frac{f'\,(hf' + fh_1)}{4f},\\
S_{11} =& \frac{-f^2h_1^2 + fhh_1f' + 2h\left[4fhf'' + f^2h_{11} - 4h\,(f')^2\right]}{4f^2h^2}, \quad S_{12} = -\frac{h_1h_2 - 2hh_{12}}{4h^2},\\
S_{22} =& \frac{h\left[2fhf'' + 2f^2h_{22} + fh_1f' + h\,(f')^2\right] - f^2h_2^2}{4f^2h^2},\\
S_{33} =& S_{44} = \frac{2fhf'' + h\,(f')^2 + fh_1f'}{4f^2h},\\
S_{55} =& \frac{-f^2\left(h_1^2 + h_2^2\right) + 2fh\left[2h_1f' + f\,(h_{11} + h_{22})\right] + h^2\left[2ff'' + (f')^2\right]}{4f^2h},
\end{aligned}$$

where $f' = \frac{df}{dx^1}$, $f'' = \frac{d}{dx^1}\left(\frac{df}{dx^1}\right)$, $h_1 = \frac{\partial h}{\partial x^1}$, $h_2 = \frac{\partial h}{\partial x^2}$, $h_{11} = \frac{\partial}{\partial x^1}\left(\frac{\partial h}{\partial x^1}\right)$, $h_{12} = \frac{\partial}{\partial x^1}\left(\frac{\partial h}{\partial x^2}\right)$ and $h_{22} = \frac{\partial}{\partial x^2}\left(\frac{\partial h}{\partial x^2}\right)$.

Again the scalar curvature κ and the local components of the Weyl conformal curvature tensor (upto symmetry) which may not vanish identically are given by:

$$\begin{aligned}
\kappa =& -\frac{f^2\left(h_1^2 + h_2^2\right) - 2fh\left(2h_1f' + f\,(h_{11} + h_{22})\right) + 2h^2\left((f')^2 - 4ff''\right)}{2f^3h^2},\\
C_{1212} =& -C_{3434} = \frac{f\left(-h_1^2 - h_2^2 + 2hh_{11} + 2hh_{22}\right)}{24h^2},\\
C_{1313} =& C_{1414} = -C_{2323} = -C_{2424} = \frac{f\left(-h_1^2 + h_2^2 + 2hh_{11} - 2hh_{22}\right)}{24h^2},\\
C_{1323} =& C_{1424} = \frac{f\left(2hh_{12} - h_1h_2\right)}{12h^2}, \quad C_{2525} = \frac{f\left(-h_1^2 + 3h_2^2 + 2hh_{11} - 6hh_{22}\right)}{24h},\\
C_{1515} =& -\frac{f\left(-3h_1^2 + h_2^2 + 6hh_{11} - 2hh_{22}\right)}{24h}, \quad C_{1525} = -\frac{f\left(2hh_{12} - h_1h_2\right)}{6h},\\
C_{3535} =& C_{4545} = \frac{f\left(-h_1^2 - h_2^2 + 2hh_{11} + 2hh_{22}\right)}{24h}.
\end{aligned}$$

Then by straightforward calculations we can obtain the components of S^2, S^3, S^4, $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. Now we discuss the results by taking restrictions on the functions f and h.

(i) If h_1 and h_2 are non-zero functions and f is a non-constant function, then the manifold is Ein(4) but not GRT_5.

(ii) If f and h both are non-constant functions of x^1 only, then M_1 is Ein(3) and GRT_5.

(iii) If f is non-constant such that $15(f')^3 - 18ff'f'' + 4f^2f^{(3)} \neq 0$ and $h = \frac{c_1(f')^2}{f^3}$, c_1 is any arbitrary constant, then the manifold is proper RT_5 and hence also satisfies Ein(2) condition.

(iv) If f is non-constant such that $15(f')^3 - 18ff'f'' + 4f^2f^{(3)} = 0$ and $h = \frac{c_1(f')^2}{f^3}$, then the manifold is proper conformally flat, where $21(f')^3 - 22ff'f'' + 4f^2f^{(3)}$, $33(f')^3 - 46ff'f'' + 12f^2f^{(3)}$ and $-9(f')^3 - 2ff'f'' + 4f^2f^{(3)}$ are not zero simultaneously. In particular, if $f(x^1) = (x^1)^{-4}$, $x^1 > 0$ or $x^1 < 0$, then the above required conditions hold and the manifold is proper conformally flat.

(v) If f is non-constant and $h = \frac{c_1(f')^2}{f^3}$ and also $21(f')^3 - 22ff'f'' + 4f^2f^{(3)} = 33(f')^3 - 46ff'f'' + 12f^2f^{(3)} = -9(f')^3 - 2ff'f'' + 4f^2f^{(3)} = 0$, then M_1 is non-flat and is of constant curvature. In particular, if $f(x^1) = (x^1)^{-2}$, $x^1 > 0$ or $x^1 < 0$, then the above required conditions hold and the manifold is of non-zero constant curvature.

(vi) Finally if both f and h are constants, then the manifold is flat.

Note 1. The inclusion relations among the various classes of manifolds of dimension ≥ 4 indicated by box are given as follows:

$$\boxed{\text{Flat}} \subset \boxed{\text{constant curvature}} \subset \boxed{\text{conformally flat}} \subset \boxed{\text{proper } RT_n} \subset \boxed{\text{proper } GRT_n}$$

$$\text{and } \boxed{\text{Ricci Flat}} \subset \boxed{\text{Einstein}} \subset \boxed{\text{Ein}(2)} \subset \boxed{\text{Ein}(3)} \subset \boxed{\text{Ein}(4)}$$

The above example ensures that the above inclusions are all proper.

Example 4.2. Let M_2 be an open connected subset of $\mathbb{R}^6$, where $x^5 > 0$, endowed with the Riemannian metric

$$ds^2 = (dx^1)^2 + e^{x^1}(dx^2)^2 + e^{x^1}(dx^3)^2 + (dx^4)^2 + e^{x^4}(dx^5)^2 + e^{x^4}(x^5+1)^2(dx^6)^2.$$

Then we can easily deduce the non-zero components of Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature (upto symmetry) as:

$$R_{1212} = R_{1313} = -\frac{e^{x^1}}{4}, \quad R_{2323} = -\frac{1}{4}e^{2x^1}, \quad R_{4545} = -\frac{e^{x^4}}{4},$$

$$e^{x^4}R_{4646} = R_{5656} = -\frac{1}{4}e^{2x^4}\left(x^5+1\right)^2,$$

$$S_{11} = S_{44} = \frac{1}{2}, \quad S_{22} = S_{33} = \frac{e^{x^1}}{2}, \quad S_{55} = \frac{e^{x^4}}{2}, \quad S_{66} = \frac{1}{2}e^{x^4}\left(x^5+1\right)^2, \quad \kappa = 3.$$

Again the non-zero components of Weyl conformal curvature tensor are given by

$$\begin{aligned}
C_{1212} =& C_{1313} = -\frac{3}{2}C_{2424} = -\frac{3}{2}C_{3434} = -\frac{3e^{x^1}}{20}, \quad C_{1414} = \frac{1}{10},\\
\frac{3}{2}C_{1515} =& -C_{4545} = \frac{3e^{x^4}}{20}, \quad C_{2525} = C_{3535} = \frac{1}{10}e^{x^1+x^4},\\
\frac{3}{2}C_{1616} =& -C_{4646} = \frac{3}{20}e^{x^4}\left(x^5+1\right)^2, \quad C_{2323} = -\frac{3e^{2x^1}}{20},\\
C_{2626} =& C_{3636} = \frac{1}{10}e^{x^1+x^4}\left(x^5+1\right)^2, \quad C_{5656} = -\frac{3}{20}e^{2x^4}\left(x^5+1\right)^2.
\end{aligned}$$

Now it is easy to check that the manifold is Einstein and also Ein(2), Ein(3) and Ein(4) as

$$\begin{aligned}
a_0\underset{2}{g} + a_1\underset{2}{S} + a_2\underset{2}{S}^2 =& 0,\\
a_3\underset{2}{g} + a_4\underset{2}{S} + a_5\underset{2}{S}^2 + a_6\underset{2}{S}^3 =& 0,\\
a_7\underset{2}{g} + a_8\underset{2}{S} + a_9\underset{2}{S}^2 + a_{10}\underset{2}{S}^3 + a_{11}\underset{2}{S}^4 =& 0,
\end{aligned}$$

where $a_2 = -2(2a_0 + a_1)$, $a_6 = -2(4a_3 + 2a_4 + a_5)$, $a_{11} = -16a_7 - 8a_8 - 4a_9 - 2a_{10}$ and a_0, a_1, a_3, a_4, a_5, a_7, a_8, a_9, a_{10} are arbitrary scalars.

We can easily check that the manifold is not a GRT_6. Although we get some dependency of $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$, given below:

$$L_0 g \wedge g + L_1 g \wedge S + L_2 S \wedge S = 0,$$

$$L_3 g \wedge g + L_4 g \wedge S + L_5 S \wedge S + L_6 g \wedge S^2 + L_7 S \wedge S^2 + L_8 S^2 \wedge S^2 = 0,$$

where $L_2 = -4(L_0+L_1)$, $L_8 = -4(4L_3+4L_4+L_5+2L_6+L_7)$ and L_i, $i = 0, 1, 3, 4, 5, 6, 7$ are arbitrary scalars. We also note that M_2 is a semisymmetric manifold.

Note 2. From Note 1 and Example 4.2, we can conclude that the following inclusion relations among the various classes of manifolds of dimension ≥ 4 are all proper

$$\begin{array}{ccccccccc}
\boxed{\text{constant curvature}} & \subset & \boxed{C=0} & \subset & \boxed{\text{proper } RT_n} & & \subset & & \boxed{\text{proper } GRT_n}\\
\cap & & & & \cap & & & & \cap\\
\boxed{\text{Einstein or Ein(1)}} & & \subset & & \boxed{\text{Ein(2)}} & \subset & \boxed{\text{Ein(3)}} & \subset & \boxed{\text{Ein(4)}}.
\end{array}$$

Example 4.3. Let M_3 be an open connected subset of $\mathbb{R}^6$ such that $x^1 < x^4$ endowed with the Riemannian metric

$$ds^2 = e^{x^1}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + e^{x^4}\left[(dx^4)^2 + (dx^5)^2 + (dx^6)^2\right].$$

Then its non-zero Riemann-Christoffel curvature tensor, Ricci tensor, scalar curvature and conformal curvature tensor components (upto symmetry) are the following:

$$R_{2323} = -\frac{e^{x^1}}{4}, \quad R_{5656} = -\frac{e^{x^4}}{4}, \quad S_{22} = S_{33} = S_{55} = S_{66} = \frac{1}{4}, \quad \kappa = 1$$

and

$$
\begin{aligned}
C_{1212} =& C_{1313} = \frac{1}{80} e^{x^1-x^4}\left(3e^{x^4}-2e^{x^1}\right), \quad C_{1515} = C_{1616} = \frac{1}{80}\left(3e^{x^1}-2e^{x^4}\right),\\
-\frac{3}{2}C_{1414} =& C_{2525} = C_{2626} = C_{3535} = C_{3636} = \frac{3}{80}\left(e^{x^1}+e^{x^4}\right),\\
C_{2323} =& -\frac{1}{40}e^{x^1-x^4}\left(e^{x^1}+6e^{x^4}\right), \quad C_{2424} = C_{3434} = -\frac{1}{80}\left(2e^{x^1}-3e^{x^4}\right),\\
C_{4545} =& C_{4646} = -\frac{1}{80}e^{x^4-x^1}\left(2e^{x^4}-3e^{x^1}\right), \quad C_{5656} = -\frac{1}{40}e^{x^4-x^1}\left(6e^{x^1}+e^{x^4}\right).
\end{aligned}
$$

Then by a straightforward calculation we get $R \cdot R = 0$ and thus the manifold is semisymmetric. Now the non-zero components (upto symmetry) of S^2 and S^3 are

$$S^2_{22} = S^2_{33} = \frac{e^{-x^1}}{16}, \quad S^2_{55} = S^2_{66} = \frac{e^{-x^4}}{16}$$

and

$$S^3_{22} = S^3_{33} = \frac{1}{64}e^{-2x^1}, \quad S^3_{55} = S^3_{66} = \frac{1}{64}e^{-2x^4}.$$

Then we can easily calculate $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$, and check that the manifold is not RT_n but special GRT_n and Ein(3) such that:

$$S^3 - \frac{e^{x^1}+e^{x^4}}{4e^{x^1+x^4}}S^2 + \frac{1}{16e^{x^1+x^4}}S = 0$$

and

$$R = \frac{2(e^{3x^1}+e^{3x^4})}{(e^{x^1}-e^{x^4})^2}S\wedge S - \frac{16e^{x^1+x^4}(e^{2x^1}+e^{2x^4})}{(e^{x^1}-e^{x^4})^2}S\wedge S^2 + \frac{32e^{2(x^1+x^4)}(e^{x^1}+e^{x^4})}{(e^{x^1}-e^{x^4})^2}S^2\wedge S^2.$$

Example 4.4. Let M_4 be an open connected semi-Riemannian manifold endowed with the semi-Riemannian metric

$$ds^2 = (f+1)(dx^1)^2 + f(dx^2)^2 + f(dx^3)^2 + \frac{(f')^2}{f(f+1)}\left[(dx^4)^2+(dx^5)^2+(dx^6)^2\right], \tag{4.1}$$

where f is a non-constant function of x^1 such that $f(f+1) \neq 0$ at any point. Then by some straightforward calculation we can get the local components of the Riemann-Christoffel curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor (upto symmetry) which may not vanish identically and the scalar curvature as follows:

$$
\begin{aligned}
R_{1212} =& R_{1313} = \frac{1}{4}\left(\frac{(2f+1)(f')^2}{f(f+1)} - 2f''\right),\\
R_{1414} =& R_{1515} = R_{1616}\\
=& \frac{f'\left[-(10f^2+9f+3)(f')^3 - 4(f^2+f)^2 f^{(3)} + 2f(f+1)(7f+3)f'f''\right]}{4f^3(f+1)^3},\\
R_{2424} =& R_{2525} = R_{2626} = R_{3434} = R_{3535} = R_{3636}
\end{aligned}
$$

$$=\frac{(f')^2\left[(2f+1)\,(f')^2-2f(f+1)f''\right]}{4f^2(f+1)^3},$$

$$R_{2323}=-\,\frac{(f')^2}{4f+4},\quad R_{4545}=R_{4646}=R_{5656}=-\frac{\left[(2f+1)\,(f')^3-2f(f+1)f'f''\right]^2}{4f^4(f+1)^5}$$

and

$$\begin{aligned}
S_{11}=&\frac{(26f^2+21f+7)\,(f')^3+12f^2(f+1)^2f^{(3)}-2f(f+1)(19f+7)f'f''}{4f^2(f+1)^2f'},\\
S_{22}=&S_{33}=\frac{8(f+1)f''+\left(-\frac{3}{f}-7\right)(f')^2}{4(f+1)^2},\\
S_{44}=&S_{55}=S_{66}=\frac{1}{4f^3(f+1)^4}\Big[8f^2(f+1)^2(f'')^2+(14f^2+11f+3)(f')^4\\
&-2f(f+1)(13f+5)(f')^2f''+4f^2(f+1)^2f^{(3)}f'\Big],\\
\kappa=&\frac{1}{2f^2(f+1)^3\,(f')^2}\left[12f^2(f+1)^2\,(f'')^2+(f(27f+17)+5)\,(f')^4\right.\\
&\left.-2f(f+1)(25f+7)\,(f')^2\,f''+12f^2(f+1)^2f^{(3)}f'\right]
\end{aligned}$$

and

$$\begin{aligned}
C_{1212}=&C_{1313}=\frac{3}{80f(f+1)^2\,(f')^2}\Big[-8f^2\,(f'')^2+27f\,(f')^4+10\,(f')^4-8f^4\,(f'')^2\\
&-16f^3\,(f'')^2+24f^{(3)}f^3f'-44f^2\,(f')^2\,f''-14f\,(f')^2\,f''\\
&+27f^2\,(f')^4+12f^{(3)}f^4f'-30f^3\,(f')^2\,f''+12f^{(3)}f^2f'\Big],\\
C_{1414}=&C_{1515}=C_{1616}=-\frac{1}{40f^3(f+1)^3}\Big[-8f^2\,(f'')^2+27f\,(f')^4+10\,(f')^4\\
&-8f^4\,(f'')^2-16f^3\,(f'')^2+24f^{(3)}f^3f'-44f^2\,(f')^2\,f''\\
&-14f\,(f')^2\,f''+27f^2\,(f')^4+12f^{(3)}f^4f'-30f^3\,(f')^2\,f''+12f^{(3)}f^2f'\Big],\\
C_{2323}=&C_{2424}=C_{2525}=C_{2626}=C_{3434}=C_{3535}=C_{3636}=\frac{-1}{80f^2(f+1)^3}\\
&\times\Big[-16f^2\,(f'')^2-21f\,(f')^4-10\,(f')^4-16f^3\,(f'')^2\\
&+4f^3f^{(3)}f'+30f^2\,(f')^2\,f''+22f\,(f')^2\,f''+4f^2f^{(3)}f'\Big],\\
C_{4545}=&C_{4646}=C_{5656}=\frac{(f')^2}{40f^3(f+1)^5}\Big[-24f^2\,(f'')^2-12f\,(f'')^2+3f\,(f')^4
\end{aligned}$$

$$- 2\left(f'\right)^4 - 12f^3\left(f''\right)^2 + 8f^3 f^{(3)} f' + 8f f^{(3)} f'$$
$$\left. + 4f\left(f'\right)^2 f'' + 4\left(f'\right)^2 f'' + 16f^2 f^{(3)} f'\right].$$

Then in view of the values of the components of R and the metric tensor g we can easily calculate $R \cdot R$ and $Q(g, R)$ and we see that these two tensors are linearly dependent if

$$(3f^2 + 3f + 1)(f')^3 + f^2(f + 1)^2 f^{(3)} - 2f(f + 1)(2f + 1)f' f'' = 0. \tag{4.2}$$

Again from g and S, we can easily calculate S^2, S^3, S^4, $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. Then we can check that the manifold is an Ein(3)-manifold and also a GRT_6. We note that in [24] it was shown that every RT_n is pseudosymmetric. Now this manifold is RT_n if it is Ein(2), i.e., f satisfies one of the following:

$$(33f^2 + 31f + 10)(f')^3 + 12f^2(f + 1)^2 f^{(3)} - 2f(f + 1)(23f + 11)f' f'' = 0, \tag{4.3}$$

$$\begin{aligned} &- 4f^2(f + 1)^2(f'')^2 + (6f^2 + 5f + 2)(f')^4 - 2f(f + 1)(3f + 1)(f')^2 f'' \\ &+ 4f^2(f + 1)^2 f^{(3)} f' = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &8f^2(f + 1)^2\left(f''\right)^2 + 3(7f^2 + 7f + 2)(f')^4 - 2f(f + 1)(17f + 9)(f')^2 f'' \\ &+ 4f^2(f + 1)^2 f^{(3)} f' = 0. \end{aligned} \tag{4.5}$$

Thus if f satisfies (4.2) but not satisfies any one of (4.3), (4.4) and (4.5), then the metric given in (4.1) is pseudosymmetric, non-Roter type but of generalized Roter type. We note that if $f(x^1) = \tan^2(x^1)$, $x^1 \in (0, \frac{\pi}{2})$, then f satisfies the above conditions. In this case the non-zero components of the metric tensor, Riemann-Christoffel curvature tensor, Ricci tensor, scalar curvature and conformal curvature tensor (upto symmetry) as follows:

$$\begin{aligned} 4g_{11} =& g_{44} = g_{55} = g_{66} = 4\sec^2\left(x^1\right), \quad g_{22} = g_{33} = \tan^2\left(x^1\right), \\ 4R_{1212} =& 4R_{1313} = 4R_{2323} = R_{2424} = R_{2525} = R_{2626} = R_{3434} \\ =& R_{3535} = R_{3636} = \frac{1}{4}R_{4545} = \frac{1}{4}R_{4646} = \frac{1}{4}R_{5656} = -4\tan^2\left(x^1\right)\sec^2\left(x^1\right), \\ R_{1414} =& R_{1515} = R_{1616} = -4\sec^4\left(x^1\right), \\ S_{11} =& 5\sec^2\left(x^1\right), \quad S_{22} = S_{33} = 5\sec^2\left(x^1\right) - 4, \\ S_{44} =& S_{55} = S_{66} = 4\left(5\sec^2\left(x^1\right) - 2\right), \quad \kappa = 25 - 3\cos(2x^1) + 2\csc(x^1)^2 \end{aligned}$$

and

$$\begin{aligned} C_{1212} =& C_{1313} = \frac{3}{20}\left(3\sec^2\left(x^1\right) - 2\right), \\ C_{1414} =& C_{1515} = C_{1616} = \frac{8}{5}\left(\cos\left(2x^1\right) - 2\right)\csc^2\left(2x^1\right), \\ C_{2323} =& -\frac{3}{20}\left(\cos\left(2x^1\right) + 3\right)\tan^2\left(x^1\right), \end{aligned}$$

$$C_{2424} = C_{2525} = C_{2626} = C_{3434} = C_{3535} = C_{3636} = -\frac{1}{5}\left(\sec\left(x^1\right) - 2\right)\left(\sec\left(x^1\right) + 2\right),$$
$$C_{4545} = C_{4646} = C_{5656} = -\frac{8}{5}\left(\csc^2\left(x^1\right) - 2\sec^2\left(x^1\right)\right).$$

Then M_4 satisfies $R \cdot R = Q(g, R)$, i.e., M_4 is non-semisymmetric but pseudosymmetric and a generalized Roter type manifold.

Example 4.5 (Example 5.5 of [51]). Let M_5 be an open connected subset of $\mathbb{R}^5$ endowed with the semi-Riemannian metric

$$ds^2 = dx^2 + dy^2 + du^2 + dv^2 + \rho^2(xdu - ydv + dz)^2,$$

where ρ is a non-zero constant. In [51] it is shown that this manifold is neither a GRT_5 nor a RT_5 but it is a pseudosymmetric manifold satisfying $R \cdot R = -\frac{\rho^2}{4} Q(g, R)$.

Note 3. It is well-known that every RT_n is GRT_n as well as pseudosymmetric. From Example 4.4 and Example 4.5, we can conclude that these two generalizations of RT_n are proper and independent.

5. Conclusion

The present paper is based on the study of generalized Roter type manifolds and obtained the sufficient conditions for which a GRT_n turns into a RT_n, conformally flat manifold, manifold of constant-curvature and manifold of quasi-constant curvature. It is shown that generalized Roter type condition is sufficient for equivalency of various curvature restricted geometric structures, such as (i) semisymmetry and Ricci-semisymmetry, (ii) pseudosymmetry and Ricci-pseudosymmetry. More generally, on a GRT_n the two geometric structures formed by a commutative 2nd type restriction (i.e., the defining restriction operator commute with contraction and linear over $C^\infty(M)$) imposed on R and S are equivalent. Alternative proofs of some theorems given in [5] and [36] for the equivalency of such structures are presented here. It is also shown that special type of generalized Roter type condition is sufficient for equivalency of (i) local symmetry and Ricci-symmetry, (ii) recurrent and Ricci recurrent. We can conclude that if we extend the generalized Roter type condition to higher Ricci level, then the equivalency of above structures are also hold. The existence of non-conformally flat Roter type manifolds, non-Roter type pseudosymmetric manifolds, non-Roter type but generalized Roter type manifolds, non-Ricci flat Einstein manifolds, non-Einstein but Ein(2) manifolds, non-Ein(2) but Ein(3) manifolds and non-Ein(3) but Ein(4) manifolds are ensured by suitable examples. Also the independent existence of non-GRT_n pseudosymmetric and non-pseudosymmetric GRT_n manifolds are ensured by some suitable examples.

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