

## WEAKLY TRIPOTENT RINGS

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ABSTRACT. We completely characterize those rings  $R$ , calling them *weakly tripotent*, whose elements satisfy the equations  $x^3 = x$  or  $x^3 = -x$ . This enlarges a result due to Hirano-Tominaga in Bull. Austral. Math. Soc. (1988) concerning tripotent rings.

### 1. INTRODUCTION AND BACKGROUND

Everywhere in the text of the present paper, all our rings  $R$  are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with [6], and the specific ones will be listed explicitly in the sequel. For instance,  $U(R)$  denotes the unit group of such a ring  $R$ ,  $\text{Inv}(R)$  is its subset consisting of all involutions (i.e., torsion units of order not exceeding 2) which is actually a subgroup when the former ring is commutative,  $\text{Id}(R)$  stands for the set of all idempotents in  $R$ , and  $J(R)$  designates the Jacobson radical of  $R$ . Recall that a ring  $R$  is *semiprimitive* (or, in other terms, Jacobson semi-simple), provided  $J(R) = \{0\}$ .

Imitating [2], we shall say that a ring is *invo-clean* if each its element is the sum of an involution and an idempotent, and mimicking [3] a ring is *weakly invo-clean* if each its element is the sum or the difference of an involution and an idempotent.

It is well known that a ring is said to be *tripotent* if any its element satisfies the equation  $x^3 = x$ . Such an element  $x$  is also called *tripotent*.

This motivates us to state the next more general concept.

**Definition 1.1.** A ring is called *weakly tripotent* if any its element satisfies the equations  $x^3 = x$  or  $x^3 = -x$ .

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Obvious examples of weakly tripotent rings are  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$  and  $\mathbb{Z}_5$ , whereas  $\mathbb{Z}_4$  and  $\mathbb{Z}_5 \times \mathbb{Z}_5$  need *not* be so.

The brief historical retrospection of the development of this and some other similar notions is as follows: A classical type of rings is the class of *boolean* rings that are rings whose elements are idempotents, i.e., having all elements satisfying the equation  $x^2 = x$ . Equivalently, any boolean ring is a subdirect product of isomorphic copies of the field  $\mathbb{Z}_2$ . Furthermore, rings with elements satisfying the equations  $x^2 = x$  or  $x^2 = -x$ , called there *weakly nil-clean*, were explored in [4] proving that they are isomorphic to either a boolean ring, or to  $\mathbb{Z}_3$ , or to a direct product of two such rings.

On the other hand, concerning cubic equations, in [5] were studied those rings whose elements are solutions of the equation  $x^3 = x$ . It was shown there that these rings are necessarily commutative being a subdirect product of family of copies of the fields  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$ . Even something more, each their element is a sum of two (commuting) idempotents.

However, for the cubic equation  $x^3 = x$ , which is implied by both the equations  $x^2 = x$  and  $x^2 = -x$  considered above, can be made the following reduction:  $x^3 = x$  is equivalent to  $x^2 = vx$ , where  $v = x^2 + x - 1$  is an involution, that is,  $v^2 = 1$  (see, e.g., [1]).

Returning to our weakly tripotent rings, the equations  $x^3 = x$  and  $x^3 = -x$  can be written in a more general form as  $x^3 = wx$  for some involution  $w$  with  $w^2 = 1$  which certainly amounts to the equation  $x^5 = x$ . Etc., this could be successfully adapted for any natural number  $n$ -compare with the discussion at the end of the paper.

Our objective here is to extend significantly the aforementioned articles [5] and [4], by classifying up to an isomorphism weakly tripotent rings in a different aspect, doing that in Theorem 2.1 quoted below.

## 2. MAIN RESULTS

We begin with the following very simple but useful technicality.

**Lemma 2.1.** *In a weakly tripotent ring  $R$  the equalities  $6 = 0$  or  $10 = 0$  are fulfilled, and hence  $R \cong R_2 \times R_3$  or  $R \cong R_2 \times R_5$ , where  $R_2 = \{0\}$  or  $R_2$  is a weakly tripotent ring of characteristic 2,  $R_3 = \{0\}$  or  $R_3$  is a weakly tripotent ring of characteristic 3 and  $R_5 = \{0\}$  or  $R_5$  is a weakly tripotent ring of characteristic 5.*

*Proof.* Write  $2^3 = 2$  or  $2^3 = -2$ , so that  $6 = 0$  or  $10 = 0$  are valid. Since  $(2, 3) = 1$  and  $(2, 5) = 1$ , the second part-half is now an immediate consequence of the Chinese Remainder Theorem.  $\square$

We will be now focussed on weakly tripotent rings of characteristics 2, 3 and 5, respectively.

**Proposition 2.1.** *Let  $R$  be a weakly tripotent ring.*

- (i) *If  $2 = 0$ , then  $R$  is boolean and so it is a subdirect product of  $\prod_\lambda \mathbb{Z}_2$ , where  $\lambda$  is an ordinal.*

- (ii) If  $3 = 0$ , then  $R$  is tripotent and thus it is a subdirect product of  $\prod_{\mu} \mathbb{Z}_3$ , where  $\mu$  is an ordinal.
- (iii) If  $5 = 0$ , then  $R$  is isomorphic to  $\mathbb{Z}_5$ , that is,  $R \cong \mathbb{Z}_5$ .

*Proof.* (i) Each of the equations  $x^3 = x$  and  $x^3 = -x$  implies that  $x^5 = x$  and thus, as it is well-known,  $R$  must be a subdirect product of isomorphic copies of the fields  $\mathbb{Z}_2, \mathbb{Z}_3$  and/or  $\mathbb{Z}_5$ . But  $2 = 0$  yields that the only members of this subdirect product are these isomorphic to  $\mathbb{Z}_2$ , as expected.

(ii) Each of the equations  $x^3 = x$  and  $x^3 = -x$  implies that  $x^5 = x$  and so, as it is well-known,  $R$  has to be a subdirect product of isomorphic copies of the fields  $\mathbb{Z}_2, \mathbb{Z}_3$  and/or  $\mathbb{Z}_5$ . But  $3 = 0$  yields that the only members of this subdirect product are isomorphic to  $\mathbb{Z}_3$ , as expected.

(iii) As in the preceding two points,  $R$  is a subdirect product of  $\prod_{\nu} \mathbb{Z}_5$  for some ordinal  $\nu$  (see, e.g., [6]). We claim that  $\nu = 1$ , that is,  $R$  is embedded in  $\mathbb{Z}_5$  which forces their isomorphism, as promised. In order to show that, we consider all elements of the kind  $(a, b)$ , where  $a, b$  belong to the set  $\{0, 1, 2, 3, 4\}$ . However, all elements in  $R$  are  $\bar{0} = (0, 0), \bar{1} = (1, 1), \bar{2} = (2, 2), \bar{3} = (3, 3), \bar{4} = (4, 4)$ , which are exactly these  $\bar{a} = (a, a)$  for  $b = a$ . Indeed, to verify this, one must to see that the elements  $0, 1, 4$  are solutions of the equation  $x^3 = x$  while the elements  $2, 3$  are solutions of the equation  $x^3 = -x$ . Therefore, if some  $(a, b)$  lies in  $R$  for  $a \neq b$ , then  $2(a, b) = (2a, 2b)$  or  $(a, b) + (a, a) = (2a, a+b)$  have again to lie in  $R$  whence by a direct check, which we leave to the reader, we will obtain that  $2a, 2b$  and/or  $a + b$  are solutions of either of the different equations  $x^3 = x$  or  $x^3 = -x$ , which is a contradiction. That is why,  $R$  cannot be properly embedded in  $\mathbb{Z}_5 \times \mathbb{Z}_5$ , and so the claim is sustained after all.

We also have the following parallel confirmation of the validity of point (iii) like this: Let  $P$  be the subring of  $R$  generated by 1, and thus note that  $P \cong \mathbb{Z}_5$ . We claim that  $P = R$ , so we assume in a way of contradiction that there exists  $b \in R \setminus P$ . With no loss of generality, we shall also assume that  $b^3 = b$  since  $b^3 = -b$  obviously implies that  $(2b)^3 = 2b$  as  $5 = 0$  and  $b \notin P \iff 2b \notin P$ .

Let us now  $(1+b)^3 = -(1+b)$ . Hence  $b = b^3$  along with  $5 = 0$  enable us that  $b^2 = 1$ . This allows us to conclude that  $(1+2b)^3 \neq \pm(1+2b)$ , however. In fact, if  $(1+2b)^3 = 1+2b$ , then one deduces that  $2b = 3$  and, by multiplying with 3, that  $b = -1 \in P$  which is manifestly untrue. If now  $(1+2b)^3 = -1-2b$ , then one infers that  $2b = 2 \in P$  which is false. That is why,  $(1+b)^3 = 1+b$ . This, in turn, guarantees that  $b^2 = -b$ . Moreover,  $b^3 = b$  is equivalent to  $(-b)^3 = -b$  as well as  $b^3 = -b$  to  $(-b)^3 = -(-b)$  and thus, by what we have proved so far applied to  $-b \notin P$ , it follows that  $-b = b^2 = (-b)^2 = -(-b) = b$ . Consequently,  $2b = 0 = 6b = b \in P$  because  $5 = 0$ , which is the wanted contradiction. We thus conclude that  $P = R$ , as expected. □

We may discuss the last statement in the following way.

*Remark 2.1.* Actually,  $x^3 = x$  for all  $x$  with  $2 = 0$  ensures that  $x^2 = x$ , while  $x^3 = x$  or  $x^3 = -x$  for all  $x$  with  $3 = 0$  assures that  $x^3 = x$ .

We now come to our chief tool here, which is the following one.

**Theorem 2.1.** *Suppose  $R$  is a ring. Then the following five items are tantamount:*

- (0)  $R$  is weakly tripotent;
- (1) all elements of  $R$  satisfy the equations  $x^3 = x$  or  $x^3 = -x$ ;
- (2)  $R$  is commutative such that every element is a sum of two idempotents with  $6 = 0$ , or  $R$  is commutative such that every element is a sum or a difference of an involution and an idempotent with  $10 = 0$ ;
- (3)  $R$  is a subdirect product of isomorphic copies of  $\mathbb{Z}_2$  and/or  $\mathbb{Z}_3$ , or of  $\mathbb{Z}_2$  and/or a single isomorphic copy of  $\mathbb{Z}_5$ ;
- (4)  $R$  is commutative semiprimitive weakly invo-clean with  $6 = 0$  or  $10 = 0$ .

*Proof.* First of all, notice that the equivalence (0) $\Leftrightarrow$ (1) is just Definition 1.1 alluded to above.

(1) $\Leftrightarrow$ (3) If (1) is valid, then we can combine Lemma 2.1 together with Proposition 2.1.

Conversely, if (3) holds, then all elements of  $R$  are solutions of the equations  $x^3 = x$  or  $x^3 = -x$ , because the elements from  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  satisfy  $x^3 = x$ , whereas the elements of  $\mathbb{Z}_2$  satisfy  $x^2 = x = -x$  and hence  $x^3 = -x$  as well as the elements of  $\mathbb{Z}_5$  satisfy the same equation  $x^3 = -x$  along with the equation  $x^3 = x$ , as needed.

(3) $\Rightarrow$ (2) If  $R$  is in the first subdirect product, as noted before, it follows from [5] that any its element is the sum of two idempotents, so we may assume that  $R$  is embedded in the second subdirect product. Knowing that each element  $c$  of the subdirect product of copies of  $\mathbb{Z}_2$  is an idempotent  $e$  and that each element  $d$  of  $\mathbb{Z}_5$  is a sum or a difference of an involution and an idempotent, say  $v + f$  or  $v - f$ , we detect that  $(e, v + f) = (1, v) + (1 - e, f)$  which is again the sum of an involution and an idempotent as  $2 = 0$  in  $\mathbb{Z}_2$  and that  $(e, v - f) = (1, v) - (1 - e, f)$  which is also the difference of an involution and an idempotent. With this at hand, point (2) is true.

(2) $\Rightarrow$ (4) If  $6 = 0$ , what suffices to prove is that the sum  $e + f$  for some  $e, f \in Id(R)$  is a sum or a difference of an involution and an idempotent. In fact, when  $2 = 0$ ,  $e + f$  must be an idempotent, say  $h$ , and hence  $h = 1 - (1 - h)$  satisfies our requirement. If now  $3 = 0$ , then  $e + f = (1 + e) - (1 - f)$  where  $(1 + e)^2 = 1 + 3e = 1$  and  $(1 - f)^2 = 1 - f$ , as required. Since with the aid of the Chinese Remainder Theorem  $R$  can be decomposed as the direct product of two such rings having characteristics 2 and 3, respectively, we are set.

Likewise, one observes that the sum  $e + f$  satisfies  $(e + f)^3 = e + f$  which is definitely right because  $(e + f)^3 = e + 6ef + f = e + f$  taking into account that  $6 = 0$ .

(3) $\Leftrightarrow$ (4) This equivalence follows by an application of [3] (cf. [2] too). □

Consulting with the proof of this theorem, one deduces that  $\mathbb{Z}_2 \times \mathbb{Z}_5$  is weakly tripotent, but  $\mathbb{Z}_3 \times \mathbb{Z}_5$  is not so. We close the work with the following challenging question.

*Problem 2.1.* What is the isomorphic structure of the so-called *weakly  $n$ -potent* rings, where  $n \in \mathbb{N}$ , i.e., rings whose elements satisfy the equations  $x^n = x$  or  $x^n = -x$ ?

What can be currently said is that  $x^n = vx$ , for  $v^2 = 1$ , is amounting to  $x^{2n-1} = x$ , where  $v = x^{2n-2} + x^{n-1} - 1$ ,  $x^{2n-2}$  is an idempotent and  $x^{n-1}$  is a tripotent.

In particular, for  $n = 3$ , one may check that the equation  $x^3 = vx$ , that is,  $x = vx^3$  is equivalent to  $x^5 = x$ , where  $v = x^4 + x^2 - 1$  is an involution, i.e.,  $v^2 = 1$ . Certainly, weakly  $n$ -potent rings must be commutative reduced and, therefore, they have to be a subdirect product of domains, each of which satisfies these two conditions.

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