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ON IMPRIMITIVITY HILBERT BIMODULES OVER COMMUTATIVE H^*-ALGEBRAS

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Abstract. In this paper, we introduce the notion of imprimitivity Hilbert H^*-bimodule and describe some properties of it. Moreover, we show that if $\mathcal{A}$ and $\mathcal{B}$ are proper and commutative H^*-algebras, ${}_{\mathcal{A}}E_{\mathcal{B}}$ is a Hilbert H^*-bimodule and e_1 is a minimal projection in $\mathcal{A}$ with ${}_{\mathcal{A}}[x|x] = e_1$ for some $x \in \mathcal{A}$, then $[x|x]_{\mathcal{B}}$ is a minimal projection in $\mathcal{B}$, too. Furthermore, the existence of orthonormal bases for such spaces is studied.

1. Introduction and Preliminaries

An H^*-algebra, introduced by Ambrose [1] is a complex Banach algebra $\mathcal{A}$ satisfying the following conditions:

(i) $\mathcal{A}$ is a Hilbert space under an inner product $\langle \cdot, \cdot \rangle$;
(ii) for each a in $\mathcal{A}$ there is an element a^* in $\mathcal{A}$, the so-called adjoint of a, such that $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ab, c\rangle = \langle a, cb^*\rangle$, for all $b, c \in \mathcal{A}$.

Recall that $\mathcal{A}_0 = \{a \in \mathcal{A} : a\mathcal{A} = \{0\}\} = \{a \in \mathcal{A} : \mathcal{A}a = \{0\}\}$ is called the annihilator ideal of $\mathcal{A}$. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. Ambrose [1] proved that an H^*-algebra is proper if and only if every element has a unique adjoint. The trace-class $\tau(\mathcal{A})$ of a proper H^*-algebra $\mathcal{A}$ is defined by the set $\tau(\mathcal{A}) = \{ab : \ a, b \in \mathcal{A}\}$. It is known that $\tau(\mathcal{A})$ is an ideal of $\mathcal{A}$, which is a Banach $*$-algebra under a suitable norm $\tau_{\mathcal{A}}(\cdot)$. The norm $\tau_{\mathcal{A}}$ is related to the given norm $\|\cdot\|$ on $\mathcal{A}$ by $\tau_{\mathcal{A}}(a^*a) = \|a\|^2$ for all $a \in \mathcal{A}$. The trace functional $\mathrm{tr}_{\mathcal{A}}$ on $\tau(\mathcal{A})$ is defined by $\mathrm{tr}_{\mathcal{A}}(ab) = \langle a, b^*\rangle = \langle b, a^*\rangle = \mathrm{tr}_{\mathcal{A}}(ba)$ for each $a, b \in \mathcal{A}$. In particular $\mathrm{tr}_{\mathcal{A}}(aa^*) = \mathrm{tr}_{\mathcal{A}}(a^*a) = \|a\|^2$ for each $a \in \mathcal{A}$. A nonzero element $e \in \mathcal{A}$ is called a projection, if it is self-adjoint and idempotent. In addition, if $e\mathcal{A}e = \mathbb{C}e$ then,

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it is called a minimal projection. Each simple H^*-algebra (that is, an H^*-algebra without nontrivial closed two-sided ideals) contains minimal projections. It is known that all minimal projections in a simple H^*-algebra have equal norms equal to α for some $\alpha \geq 1$ [2]. Two idempotents e and e' are doubly orthogonal if $\langle e, e' \rangle = 0$ and $ee' = e'e = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotents. Every proper H^*-algebra contains a maximal family of doubly orthogonal primitive self-adjoint idempotents [1]. Recall that in a commutative H^*-algebra an element is a primitive projection if and only if it is a minimal projection [7, Lemma 1.1]. There are many scholars have worked on H^*-algebras and developed the topic in several directions, see [1, 3, 8–10] and references cited therein.

Proposition 1.1. *Let $\mathcal{A}$ be a proper commutative H^*-algebra. If e and e' are distinct minimal projections in $\mathcal{A}$, then they are doubly orthogonal.*

Proof. We are going to show that $ee' = 0$. If on the contrary $ee' \neq 0$, then commutativity of $\mathcal{A}$ and minimality of the projections e and e', imply that $ee' = ee'e = \lambda_1 e = \lambda_2 e'$ for some nonzero and distinct scalars λ_1 and λ_2. On the other hand, since e, e' and ee' are idempotents, then $(\lambda_1 e)^2 = \lambda_1 e = ee' = (\lambda_2 e')^2 = \lambda_2 e'$ and so $\lambda_1 = \lambda_2 = 1$, which gives $e = e'$ a contradiction. Thus $ee' = e'e = 0$ and therefore $\langle e, e' \rangle = \mathrm{tr}_{\mathcal{A}}(ee') = 0$. ☐

An immediate consequence of the above proposition is the following result.

Corollary 1.1. *Each commutative H^*-algebra has a unique maximal family of doubly orthogonal minimal projections which contains all of its minimal projections.*

Let us recall the definition of a Hilbert H^*-module.

Definition 1.1. [2] A Hilbert H^*-module over a proper H^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E on which there is a mapping $[\cdot|\cdot] : E \times E \to \tau(\mathcal{A})$ (called $\tau(\mathcal{A})$-valued product), satisfying

(i) $[\alpha x|y] = \alpha[x|y]$;
(ii) $[x + y|z] = [x|z] + [y|z]$;
(*iii*) $[ax|y] = a[x|y]$;
(iv) $[x|y]^* = [y|x]$;
(v) for each nonzero element x in E there is a nonzero element a in $\mathcal{A}$ such that $[x|x] = a^*a$;
(vi) E is a Hilbert space under the inner product $(x, y) = \mathrm{tr}_{\mathcal{A}}([x|y])$;

for each $\alpha \in \mathbb{C}$, $x, y, z \in E$ and $a \in \mathcal{A}$.

The Hilbert H^*-module E is full [7] if the ideal $[E|E] = \mathrm{span}\{[x|y] : x, y \in E\}$, is dense in $\tau(\mathcal{A})$ under the norm $\tau_{\mathcal{A}}(\cdot)$.

Example 1.1. [2] Let H be an infinite dimensional Hilbert space and $\mathcal{HS}(H)$ be the standard H^*-algebra of Hilbert-Schmidt operators on it. Let us denote by $\Theta_{x,y}$ the rank 1 operator on H defined by $\Theta_{x,y}(z) = (z, y)x$. It is well known that H may be regarded as a Hilbert H^*-module over $\mathcal{HS}(H)$. Given $x \in H$ and $T \in \mathcal{HS}(H)$, Tx is interpreted as the action of T and $\mathcal{HS}(H)$-valued product on H is defined by $[x|y] = \Theta_{x,y}$. Since $\mathrm{tr}_{\mathcal{HS}(H)}\Theta_{x,y} = (x, y)$, then the resulting norm on H coincides with the original one.

For a Hilbert H^*-module E over a proper H^*-algebra $\mathcal{A}$ the following relations between the two norms $\|.\|$ and $\tau_{\mathcal{A}}$ hold (see [2]):

$$\|x\|^2 = \mathrm{tr}_{\mathcal{A}}([x|x]) = \tau_{\mathcal{A}}([x|x]), \quad \text{for all } x \in E,$$

$$\|[x|y]\| \leq \tau_{\mathcal{A}}([x|y]) \leq \|x\|\|y\|, \quad \text{for all } x, y \in E,$$

$$\|ax\| \leq \|a\|\|x\|, \quad \text{for all } a \in \mathcal{A}, x \in E.$$

Definition 1.2. [2] An element $u \in E$ is a basic element if there exists a minimal projection $e \in \mathcal{A}$ (called the supporting projection) such that $[u|u] = e$. An orthonormal system in E is a family of basic elements $\{u_\lambda\}$, $\lambda \in \Lambda$ satisfying $[u_\lambda|u_\mu] = 0$, for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. An orthonormal basis in E is an orthonormal system generating a dense submodule of E.

Note that if $\{u_\lambda\}$ is an orthonormal basis for E, then for each $x \in E$, $x = \sum_\lambda [x|u_\lambda]u_\lambda$ (Fourier expansion) (see [2]). We recall that each Hilbert H^*-module E contains basic elements, orthonormal systems and orthonormal bases and moreover, all orthonormal bases for E have the same cardinal number called the hilbertian dimension of E.

Lemma 1.1. [2] *Let E be a Hilbert module over an arbitrary H^*-algebra $\mathcal{A}$, $e \in \mathcal{A}$ be a projection* (*not necessarily minimal*) *and let $x \in E$ be such that $[x|x] = e$. Then $ex = x$.*

In the above lemma one observes that if $[x|x] = \lambda e$ for some scalar λ and some projection e in $\mathcal{A}$, then

$$[ex - x|ex - x] = [ex|ex] - [ex|x] - [x|ex] + [x|x] = \lambda e^3 - \lambda e^2 - \lambda e^2 + \lambda e = 0,$$

which implies that $ex = x$. Let E be a Hilbert H^*-module over an H^*-algebra $\mathcal{A}$ and let $e \in \mathcal{A}$ be a minimal projection. Then $E_e = \{x \in E : [x|x] = \lambda e, \lambda \geq 0\}$ is a closed subspace of the Hilbert space E. If $\mathcal{A}$ is a simple H^*-algebra, then the subspace E_e generates a dense submodule in E (see [2]). For emphasizing its H^*-algebra, we denote E_e by $({}_{\mathcal{A}}E)_e$ (or $(E_{\mathcal{A}})_e$ in right module case). For more details on this issue see [2]. Also, for general facts about Hilbert H^*-modules we refer the interested reader to [2, 4, 7, 10, 11].

We introduce the notion of imprimitivity Hilbert H^*-bimodule and describe some properties of it. In this paper, we show that if $\mathcal{A}$ and $\mathcal{B}$ are proper and commutative H^*-algebras, ${}_{\mathcal{A}}E_{\mathcal{B}}$ is an imprimitivity Hilbert H^*-bimodule and e_1 is a minimal projection in $\mathcal{A}$ with ${}_{\mathcal{A}}[x|x] = e_1$ for some $x \in \mathcal{A}$, then $[x|x]_{\mathcal{B}}$ is a minimal projection in $\mathcal{B}$, too. Furthermore, the existence of orthonormal bases for such spaces is studied.

2. Main Results

In this section, we state the notions of Hilbert H^*-bimodule and imprimitivity Hilbert H^*-bimodule. We then investigate the existence of orthonormal bases for imprimitivity Hilbert bimodules over the commutative H^*-algebras. Before giving our results, we state two interesting facts related to Hilbert modules over the commutative H^*-algebras which will be used in the sequel.

Proposition 2.1. *Let E be a Hilbert module over a commutative H^*-algebra $\mathcal{A}$. If $\{u_\lambda\}$, $\lambda \in \Lambda$ is an orthonormal basis for E and $x \in E$, then $x = \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda$ for some $\mu_\lambda \in \mathbb{C}$.*

Proof. Let $\{e_i\}$, $i \in I$, be the maximal family of doubly orthogonal minimal projections in $\mathcal{A}$ as Corollary 1.1. Let's also suppose that each u_λ, $\lambda \in \Lambda$, has supporting projection e_{i_λ} for some $i_\lambda \in I$. Since $x = \sum_{\lambda\in\Lambda}[x|u_\lambda]u_\lambda$, then applying [1, Theorem 4.1] and by the commutativity of $\mathcal{A}$, we get $[x|u_\lambda] = \sum_{i\in I} \mu_{\lambda,i} e_i$, for each $\lambda \in \Lambda$ and some scalars $\mu_{\lambda,i}$. Thus we have $x = \sum_{\lambda\in\Lambda}\sum_{i\in I} \mu_{\lambda,i} e_i u_\lambda$.

On the other hand, $e_i u_\lambda = 0$ for all $i \neq i_\lambda$. Indeed, by applying Proposition 1.1 we conclude that $[e_i u_\lambda|e_i u_\lambda] = e_i[u_\lambda|u_\lambda] = e_i e_{i_\lambda} = 0$ for each $i \neq i_\lambda$. Therefore, $x = \sum_{\lambda\in\Lambda} \mu_{\lambda,i_\lambda} e_{i_\lambda} u_\lambda = \sum_{\lambda\in\Lambda} \mu_{\lambda,i_\lambda} u_\lambda$ by Lemma 1.1. □

Proposition 2.2. *Let E be a full Hilbert module over a commutative H^*-algebra $\mathcal{A}$, $e_0 \in \mathcal{A}$ be a minimal projection and $\{u_\lambda\}$, $\lambda \in \Lambda$, be an orthonormal basis for E. Then $e_0 = [u_{\lambda_0}|u_{\lambda_0}]$ for some $\lambda_0 \in \Lambda$.*

Proof. On the contrary, we suppose that

$$e_0 \neq [u_\lambda|u_\lambda], \tag{2.1}$$

for all $\lambda \in \Lambda$. By the fullness of E we get $e_0 = \sum_{t\in J}[x_t|y_t]$, for some index set J and x_t and y_t in E. Regarding to Proposition 2.1 it follows that $x_t = \sum_{\lambda\in\Lambda} \mu_{t,\lambda} u_\lambda$ and $y_t = \sum_{\lambda\in\Lambda} \mu'_{t,\lambda} u_\lambda$, for each $t \in J$ and some scalars $\mu_{t,\lambda}$ and $\mu'_{t,\lambda}$. Therefore, we can write

$$e_0 = \sum_{t\in J}[x_t|y_t] = \sum_{t,\lambda} \mu_{t,\lambda}\overline{\mu'_{t,\lambda}}[u_\lambda|u_\lambda]. \tag{2.2}$$

Accordingly, by (2.1) and (2.2) and applying Proposition 1.1 we observe that,

$$e_0 = e_0^2 = \sum_{t,\lambda} \mu_{t,\lambda}\overline{\mu'_{t,\lambda}}[u_\lambda|u_\lambda]e_0 = 0,$$

which gives a contradiction to the fact $e_0 \neq 0$. □

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two proper H^*-algebras. By a Hilbert bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ we mean a left Hilbert $\mathcal{A}$-module with the $\tau(\mathcal{A})$-valued product ${}_{\mathcal{A}}[\cdot|\cdot] : E \times E \to \tau(\mathcal{A})$ and a right Hilbert $\mathcal{B}$-module with the $\tau(\mathcal{B})$-valued product $[\cdot|\cdot]_{\mathcal{B}} : E \times E \to \tau(\mathcal{B})$ such that

(i) $(ax)b = a(xb)$;

(ii) ${}_{\mathcal{A}}[xb|y] ={}_{\mathcal{A}} [x|yb^*]$;
(iii) $[x|ay]_{\mathcal{B}} = [a^*x|y]_{\mathcal{B}}$;
for all $x, y \in_{\mathcal{A}} E_{\mathcal{B}}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Further, Hilbert H^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ is called full, if it is full both as a left and as a right Hilbert module over $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition 2.2. A Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule E is called an imprimitivity bimodule if

$$ {}_{\mathcal{A}}[x|y]z = x[y|z]_{\mathcal{B}}, $$

where $x, y, z \in_{\mathcal{A}} E_{\mathcal{B}}$.

Example 2.1. Suppose $\mathcal{A}$ is a proper H^*-algebra. It is easy to verify that $\mathcal{A}$ is a full Hilbert H^*-bimodule over $\mathcal{A}$ with the maps ${}_{\mathcal{A}}[a_1|a_2] = a_1 a_2^*$ and $[a_1|a_2]_{\mathcal{A}} = a_1^* a_2$, $a_1, a_2 \in \mathcal{A}$.

We point out that each Hilbert H^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ is a Hilbert space under both inner products ${}_{\mathcal{A}}(x, y) = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|y])$, $(x, y)_{\mathcal{B}} = \mathrm{tr}_{\mathcal{B}}([x|y]_{\mathcal{B}})$ and therefore it has two norms, usually different, as follows

$$ {}_{\mathcal{A}}\|x\| = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x])^{\frac{1}{2}}, \quad \|x\|_{\mathcal{B}} = \mathrm{tr}_{\mathcal{B}}([x|x]_{\mathcal{B}})^{\frac{1}{2}}, \quad x \in E. $$

We however have the following result in the particular case $\mathcal{A} = \mathcal{B}$.

Theorem 2.1. *Let E be a Hilbert H^*-bimodule over an H^*-algebra $\mathcal{A}$, then*

$$ {}_{\mathcal{A}}\|xa\| \le \|a\|_{\mathcal{A}}\|x\|, \quad \|ax\|_{\mathcal{A}} \le \|a\|\|x\|_{\mathcal{A}}, $$

for each $a \in \mathcal{A}$ and $x \in E$.

Proof. We are going to show that ${}_{\mathcal{A}}\|xa\| \le \|a\|_{\mathcal{A}}\|x\|$. Without loss of generality, we can assume that $\|a\| \le 1$. Take $b = aa^*$. Then $b - b^2 = h^2$ for some positive element $h \in \tau(\mathcal{A})$ (see [5, p. 34]). Therefore we can write $\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x(b - b^2)|x]) = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xh|xh]) ={}_{\mathcal{A}} \|xh\|^2 \ge 0$ and thus we have

$$ \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xbb^*]) = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb^2|x]) \le \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|x]). \tag{2.3} $$

On using (2.3), we get

$$ \begin{aligned} 0 \le&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x - xb|x - xb]) \\ =&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xb]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|x]) + \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|xb]) \\ =&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xb]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|x]) + \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xbb^*]) \\ \le&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xb]) - \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|x]) + \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xb|x]). \end{aligned} $$

It enforces that $\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xb]) \le \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x])$. Hence

$$ \begin{aligned} {}_{\mathcal{A}}\|xa\|^2 =&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[xa|xa]) = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xaa^*]) = \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|xb]) \\ \le&\mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x]) ={}_{\mathcal{A}} \|x\|^2, \end{aligned} $$

as desired. The proof of the other part is similar and therefore, to avoid repeation we remove it. ☐

Now we are in a position to state and prove our main result.

Theorem 2.2. *Suppose that E is an imprimitivity Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule over the commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and $x \in E$. Then ${}_{\mathcal{A}}[x|x]$ is a minimal projection in $\mathcal{A}$ if and only if $[x|x]_{\mathcal{B}}$ is a minimal projection in $\mathcal{B}$. Furthermore, x is in the Hilbert space $({}_{\mathcal{A}}E)_e$ for some minimal projection $e \in \mathcal{A}$ if and only if x is in the Hilbert space $(E_{\mathcal{B}})_{e'}$ for some minimal projection $e' \in \mathcal{B}$.*

Proof. Consider ${}_{\mathcal{A}}[x|x] = e_1$ for some minimal projection e_1 in $\mathcal{A}$. Then $e_1 x = x$ by Lemma 1.1 and therefore it establishes

$$[x|x]_{\mathcal{B}} = [x|e_1 x]_{\mathcal{B}} = \Big[x\Big|_{\mathcal{A}}[x|x]x\Big]_{\mathcal{B}} = \Big[x\Big|x[x|x]_{\mathcal{B}}\Big]_{\mathcal{B}} = [x|x]^2_{\mathcal{B}}.$$

Since $[x|x]_{\mathcal{B}} = b^*b$ for some nonzero $b \in \mathcal{B}$, then $[x|x]_{\mathcal{B}}$ is a projection. It remains to prove that it is a minimal projection. For this purpose, let $\{e'_j\}$, $j \in J$ be the maximal family of minimal projections in $\mathcal{B}$. In view of [1, Lemma 4.1] and [7, Lemma 1.1], $[x|x]_{\mathcal{B}} = \sum_{j\in J} t'_j e'_j$ for some nonnegative numbers t'_j, $j \in J$. Put $[x|x]_{\mathcal{B}} = \sum_{j\in J_0} t'_j e'_j$, where $J_0 = \{j \in J : t'_j \neq 0\}$. Now, since $[x|x]_{\mathcal{B}}$ is idempotent, so we get $[x|x]_{\mathcal{B}} = \sum_{j\in J_0} e'_j$. We claim that $[x|x]_{\mathcal{B}} = e'_j$ for some $j \in J_0$. First, on the contrary suppose that $[x|x]_{\mathcal{B}} = e'_{j_1} + e'_{j_2}$, for distinct elements $j_1, j_2 \in J_0$. Applying again Lemma 1.1, for each $a \in \mathcal{A}$, we have

$$\begin{aligned} e_1 a &= {}_{\mathcal{A}}[x|x]a \\ &= {}_{\mathcal{A}}\Big[{}_{\mathcal{A}}[x|x]x\Big|{}_{\mathcal{A}}[x|x]x\Big]a = {}_{\mathcal{A}}\Big[x[x|x]_{\mathcal{B}}\Big|x[x|x]_{\mathcal{B}}\Big]a \\ &= {}_{\mathcal{A}}[x(e'_{j_1} + e'_{j_2})|x(e'_{j_1} + e'_{j_2})]a = {}_{\mathcal{A}}[xe'_{j_1} + xe'_{j_2}|xe'_{j_1} + xe'_{j_2}]a \\ &= {}_{\mathcal{A}}[xe'_{j_1}|xe'_{j_1}]a + {}_{\mathcal{A}}[xe'_{j_1}|xe'_{j_2}]a + {}_{\mathcal{A}}[xe'_{j_2}|xe'_{j_1}]a + {}_{\mathcal{A}}[xe'_{j_2}|xe'_{j_2}]a. \end{aligned}$$

The double orthogonality of e_j s' ensures that

$$e_1 a = {}_{\mathcal{A}}[xe'_{j_1}|xe'_{j_1}]a + {}_{\mathcal{A}}[xe'_{j_2}|xe'_{j_2}]a. \tag{2.4}$$

Assume that $\{u_\lambda\}$, $\lambda \in \Lambda$, is an orthonormal basis for the right Hilbert $\mathcal{B}$-module E. According to Proposition 2.1 we observe that $x = \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda$ for some scalars μ_λ, $\lambda \in \Lambda$, and therefore, $e'_{j_1} + e'_{j_2} = [x|x]_{\mathcal{B}} = [\sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda|\sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda]_{\mathcal{B}} = \sum_{\lambda\in\Lambda} |\mu_\lambda|^2 [u_\lambda|u_\lambda]_{\mathcal{B}}$. So, there exists λ_1 and λ_2 in Λ such that $[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}} = e'_{j_1}$, $[u_{\lambda_2}|u_{\lambda_2}]_{\mathcal{B}} = e'_{j_2}$. Regarding to (2.4) we derive that

$$\begin{aligned} e_1 a &= {}_{\mathcal{A}}\Big[x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}}\Big|x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}}\Big]a + {}_{\mathcal{A}}\Big[x[u_{\lambda_2}|u_{\lambda_2}]_{\mathcal{B}}\Big|x[u_{\lambda_2}|u_{\lambda_2}]_{\mathcal{B}}\Big]a \\ &= {}_{\mathcal{A}}\Big[{}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}\Big|{}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}\Big]a + {}_{\mathcal{A}}\Big[{}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}\Big|{}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}\Big]a \\ &= {}_{\mathcal{A}}[x|u_{\lambda_1}]_{\mathcal{A}}[x|u_{\lambda_1}]^*_{\mathcal{A}}[u_{\lambda_1}|u_{\lambda_1}]a + {}_{\mathcal{A}}[x|u_{\lambda_2}]_{\mathcal{A}}[x|u_{\lambda_2}]^*_{\mathcal{A}}[u_{\lambda_2}|u_{\lambda_2}]a. \end{aligned}$$

Both of statements in the right hand side of the above relation are nonzero. Indeed, we have $[{}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}|{}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}]_{\mathcal{B}} = [x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}}|x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}}]_{\mathcal{B}} = [x|x]_{\mathcal{B}}[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}} = (e'_{j_1} + e'_{j_2})e'_{j_1} = e'_{j_1}$ and similarly $[{}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}|{}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}]_{\mathcal{B}} = e'_{j_2}$. Whence ${}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}$ and ${}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}$ and consequently ${}_{\mathcal{A}}[{}_{\mathcal{A}}[x|u_{\lambda_i}]u_{\lambda_i}|{}_{\mathcal{A}}[x|u_{\lambda_i}]u_{\lambda_i}]$, $i = 1, 2$ are nonzero.

Next, put ${}_{\mathcal{A}}[x|u_{\lambda_1}] = g$, ${}_{\mathcal{A}}[u_{\lambda_1}|u_{\lambda_1}] = h^*h$, ${}_{\mathcal{A}}[x|u_{\lambda_2}] = g'$ and ${}_{\mathcal{A}}[u_{\lambda_2}|u_{\lambda_2}] = h'^*h'$, for some g, g', h, h' in $\mathcal{A}$. Thus, we derive that

$$e_1 a = gg^*h^*ha + g'g'^*h'^*h'a = (gh)(gh)^*a + (g'h')(g'h')^*a.$$

Take $gh = k$ and $g'h' = k'$, so

$$e_1 a = (kk^* + k'k'^*)a. \tag{2.5}$$

Let $\{e_i\}$, $i \in I$,be the maximal family of minimal projections in $\mathcal{A}$ containing e_1. Without loss of generality, we may assume that $e_{i_1} = e_1$, where $i_1 \in I$. If we put $k = \sum_{i\in I} t_i e_i$ and $k' = \sum_{i\in I} s_i e_i$, then $e_1 = kk^* + k'k'^* = \sum_{i\in I}|t_i|^2 e_i + \sum_{i\in I}|s_i|^2 e_i$. On the other hand,

$$\begin{aligned}(gh^*h)(g'h'^*h') &= {}_{\mathcal{A}}[x|u_{\lambda_1}]_{\mathcal{A}}[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{A}}[x|u_{\lambda_2}]_{\mathcal{A}}[u_{\lambda_2}|u_{\lambda_2}]\\
&= {}_{\mathcal{A}}\Big[{}_{\mathcal{A}}[x|u_{\lambda_1}]u_{\lambda_1}\Big|u_{\lambda_1}\Big]_{\mathcal{A}}\Big[{}_{\mathcal{A}}[x|u_{\lambda_2}]u_{\lambda_2}\Big|u_{\lambda_2}\Big]\\
&= {}_{\mathcal{A}}\Big[x[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{B}}\Big|u_{\lambda_1}\Big]_{\mathcal{A}}\Big[x[u_{\lambda_2}|u_{\lambda_2}]_{\mathcal{B}}\Big|u_{\lambda_2}\Big]\\
&= {}_{\mathcal{A}}[xe'_{j_1}|u_{\lambda_1}]_{\mathcal{A}}[xe'_{j_2}|u_{\lambda_2}] = {}_{\mathcal{A}}\Big[{}_{\mathcal{A}}[xe'_{j_1}|u_{\lambda_1}]xe'_{j_2}\Big|u_{\lambda_2}\Big]\\
&= {}_{\mathcal{A}}\Big[xe'_{j_1}[u_{\lambda_1}|xe'_{j_2}]_{\mathcal{B}}\Big|u_{\lambda_2}\Big] = {}_{\mathcal{A}}\Big[xe'_{j_1}e'_{j_2}[u_{\lambda_1}|x]_{\mathcal{B}}\Big|u_{\lambda_2}\Big]\\
&= {}_{\mathcal{A}}[0|u_{\lambda_2}] = 0,\end{aligned}$$

which in turn implies that

$$kk^*k'k'^* = {}_{\mathcal{A}}[x|u_{\lambda_1}]_{\mathcal{A}}[x|u_{\lambda_1}]^*_{\mathcal{A}}[u_{\lambda_1}|u_{\lambda_1}]_{\mathcal{A}}[x|u_{\lambda_2}]_{\mathcal{A}}[x|u_{\lambda_2}]^*_{\mathcal{A}}[u_{\lambda_2}|u_{\lambda_2}] = 0. \tag{2.6}$$

Clearly, $kk^* + k'k'^* = \sum_{i\in I}|t_i|^2 e_i + \sum\limits_{i\in I}|s_i|^2 e_i$ have a nonzero scalar t_{i_2} or s_{i_2} for some $i_2 \neq i_1$. Otherwise, $kk^* + k'k'^* = |t_{i_1}|^2 e_{i_1} + |s_{i_1}|^2 e_{i_1}$ and so $kk^*k'k'^* = |t_{i_1}s_{i_1}|^2 e_{i_1}^2 = |t_{i_1}s_{i_1}|^2 e_{i_1} \neq 0$ which is in contradiction with (2.6).

On the other hand, if such t_{i_2} or s_{i_2} occurs in $kk^* + k'k'^*$, then substituting a with e_{i_2} in (2.5), we get $e_1 e_{i_2} = (kk^* + k'k'^*)e_{i_2}$. It leads to a contradiction, since the right hand side of this equality is greater than $|t_{i_2}|^2 e_{i_2}$ or $|s_{i_2}|^2 e_{i_2}$ or sum of them but the left hand side is equal to zero. Therefore $[x|x]_{\mathcal{B}}$ cannot be of the form $e'_{j_1} + e'_{j_2}$. Repeating the above procedure, we realize that $[x|x]_{\mathcal{B}}$ cannot be appear as the form $e'_{j_1} + \cdots + e'_{j_n}$ where $n > 2$. Hence $[x|x]_{\mathcal{B}} = e'_j$ for some $j \in J_0$ and so the claim holds.

Finally, if $x \in ({}_{\mathcal{A}}E)_e$ for some minimal projection e in $\mathcal{A}$, then ${}_{\mathcal{A}}[x|x] = \lambda e$ for some $\lambda > 0$. Therefore ${}_{\mathcal{A}}[(\sqrt{\lambda})^{-1}x|(\sqrt{\lambda})^{-1}x] = e$ and so using the first part $[\sqrt{\lambda}x|\sqrt{\lambda}x]_{\mathcal{B}}$ is a minimal projection in $\mathcal{B}$, too. This completes the proof. □

Theorem 2.3. *Let E be an imprimitivity Hilbert H^*-bimodule over commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$. If x and y are two nonzero elements in E such that ${}_{\mathcal{A}}[x|x]$ and ${}_{\mathcal{A}}[y|y]$ are scalar multiplication of some minimal projections in $\mathcal{A}$, then the following four statements are equivalent:*

(i) *x, y are in Hilbert space $({}_{\mathcal{A}}E)_e$ for some minimal projection e in $\mathcal{A}$;*
(ii) $[x|y]_{\mathcal{B}} \neq 0$*;*
(iii) *x, y are in Hilbert space $(E_{\mathcal{B}})_{e'}$ for some minimal projection e' in $\mathcal{B}$;*
(iv) ${}_{\mathcal{A}}[x|y] \neq 0$.

Proof. (i)⇒(ii) Let us assume that ${}_{\mathcal{A}}[x|x] = \lambda e$ and ${}_{\mathcal{A}}[y|y] = \mu e$ for some positive scalars λ and μ. According to Lemma 1.1 and imprimitivity of E we conclude that

$$[x|x]_{\mathcal{B}} = \Big[\frac{1}{\mu}{}_{\mathcal{A}}[y|y]x\Big|x\Big]_{\mathcal{B}} = \frac{1}{\mu}\Big[x\Big|{}_{\mathcal{A}}[y|y]x\Big]_{\mathcal{B}} = \frac{1}{\mu}\Big[x\Big|y[y|x]_{\mathcal{B}}\Big]_{\mathcal{B}} = \frac{1}{\mu}[x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}},$$

which implies that $[x|y]_{\mathcal{B}} \neq 0$.
(ii)⇒(i) Suppose, on the contrary that, ${}_{\mathcal{A}}[x|x] = \lambda_1 e_1$ and ${}_{\mathcal{A}}[y|y] = \lambda_2 e_2$, for distinct minimal projections e_1 and e_2 in $\mathcal{A}$. These conditions assure us $e_1 x = x$ and $e_2 y = y$. Thus we get $[x|y]_{\mathcal{B}} = [e_1 x|e_2 y]_{\mathcal{B}} = [e_1 e_2 x|y]_{\mathcal{B}} = [0|y]_{\mathcal{B}} = 0$ which contradicts assertion (ii).
(i)⇒(iii) Put ${}_{\mathcal{A}}[x|x] = \lambda e$ and ${}_{\mathcal{A}}[y|y] = \mu e$ for some positive scalars λ and μ. Applying a similar argument as before we observe that

$$[y|y]_{\mathcal{B}} = \Big[\frac{1}{\lambda}{}_{\mathcal{A}}[x|x]y\Big|y\Big]_{\mathcal{B}} = \frac{1}{\lambda}\Big[x[x|y]_{\mathcal{B}}\Big|y\Big]_{\mathcal{B}} = \frac{1}{\lambda}[x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}},$$

which let us conclude that $[x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}} \neq 0$. According to [1, Lemma 2.3], $([x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}})^2 \neq 0$ and so $[x|x]_{\mathcal{B}}[y|y]_{\mathcal{B}} = \frac{1}{\lambda\mu}([x|y]_{\mathcal{B}}[y|x]_{\mathcal{B}})^2 \neq 0$. It enforces that $x, y \in (E_{\mathcal{B}})_{e'}$ for some minimal projection e' in $\mathcal{B}$.

Implications (iii)⇒(i) and (iii)⇔(iv) are proved in similar ways and so we omit them. □

Corollary 2.1. *Suppose that E is an imprimitivity Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule over the commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and also assume that $\{u_\lambda\}$, $\lambda \in \Lambda$ is an orthonormal system for Hilbert H^*-module ${}_{\mathcal{A}}E$. Then $\{u_\lambda\}$ is an orthonormal system for right Hilbert H^*-module $E_{\mathcal{B}}$ if and only if each u_λ, $\lambda \in \Lambda$, has its exclusive supporting projection in $\mathcal{A}$, it means that if λ_1, λ_2 are distinct elements in Λ with ${}_{\mathcal{A}}[u_{\lambda_1}|u_{\lambda_1}] = e_{\lambda_1}$ and ${}_{\mathcal{A}}[u_{\lambda_2}|u_{\lambda_2}] = e_{\lambda_2}$ for some minimal projections e_{λ_1} and e_{λ_2} in $\mathcal{A}$, then $e_{\lambda_1} \neq e_{\lambda_2}$.*

Proof. Suppose that $\{u_\lambda\}$ is an orthonormal system for Hilbert module $E_{\mathcal{B}}$. We assert that each u_λ, $\lambda \in \Lambda$, has its exclusive supporting projection in $\mathcal{A}$. If not, then there are distinct elements u_μ and u_ν in $\{u_\lambda\}$ with the same supporting projection e in $\mathcal{A}$. Whence $u_\mu, u_\nu \in E_e$ and by Theorem 2.3 we have that $[u_\mu|u_\nu]_{\mathcal{B}} \neq 0$, which leads to a contradiction. So each u_λ, $\lambda \in \Lambda$, has its exclusive supporting projection in $\mathcal{A}$. The reverse direction is a straightforward consequence of Theorems 2.2 and 2.3. □

Up to now we discussed the existence of basic elements and orthonormal systems for a particular class of Hilbert H^*-bimodules. We are interested to prove the existence of orthonormal bases in these space. We focus on this subject below.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative H^*-algebras and ${}_{\mathcal{A}}E_{\mathcal{B}}$ be an imprimitivity Hilbert H^*-bimodule. Let $\{u_\lambda\}$, $\lambda \in \Lambda$ and $\{v_\gamma\}$, $\gamma \in \Gamma$ be orthonormal bases in ${}_{\mathcal{A}}E$ and $E_{\mathcal{B}}$, respectively. Then the following conditions hold:*

(i) *for each $\lambda_0 \in \Lambda$ there is a unique $v_{\gamma_0} \in \{v_\gamma\}$ and a scalar t_{γ_0}, $\gamma_0 \in \Gamma$, with $|t_{\gamma_0}| = 1$ in which $u_{\lambda_0} = t_{\gamma_0} v_{\gamma_0}$;*
(ii) *u_{λ_0} and v_{γ_0} have the same supporting projections in $\mathcal{A}$ and also in $\mathcal{B}$;*
(iii) *there is a bijection between Λ and Γ.*

Proof. Suppose that λ_0 is any arbitrary fixed element in Λ. Regarding Proposition 2.1, $u_{\lambda_0} = \sum_{\gamma\in\Gamma'} t_\gamma v_\gamma$, where $\Gamma' = \{\gamma \in \Gamma : \ t_\gamma \neq 0\}$. We claim that there is a unique v_{γ_0} in $\{v_\gamma\}$ such that $[u_{\lambda_0}|v_{\gamma_0}] \neq 0$. First, note that for each $\gamma' \in \Gamma'$, we get

$$[u_{\lambda_0}|v_{\gamma'}]_{\mathcal{B}} = \left[\sum_{\gamma\in\Gamma'} t_\gamma v_\gamma | v_{\gamma'}\right]_{\mathcal{B}} = t_{\gamma'}[v_{\gamma'}|v_{\gamma'}]_{\mathcal{B}} \neq 0. \tag{2.7}$$

Take γ' an arbitrary fixed element in Γ' and set ${}_{\mathcal{A}}[u_{\lambda_0}|u_{\lambda_0}] = e$, ${}_{\mathcal{A}}[v_{\gamma'}|v_{\gamma'}] = e_1$ for some minimal projections e and e_1 in $\mathcal{A}$. Notice that using Theorem 2.2, ${}_{\mathcal{A}}[v_{\gamma'}|v_{\gamma'}]$ is a minimal projection in $\mathcal{A}$. From (2.7) and applying Theorem 2.3, it follows that $e = e_1$. Hence u_{λ_0} and $v_{\gamma'}$ have the same supporting projection in $\mathcal{A}$ and also in $\mathcal{B}$. Taking into account Corollary 2.1 and since $\gamma' \in \Gamma'$ was arbitrary, we deduce that there is a unique $\gamma_0 \in \Gamma.$ with $t_{\gamma_0} \neq 0$ and $t_\gamma = 0$, for all $\gamma \in \Gamma\backslash\{\gamma_0\}$. In fact, suppose that there are two distinct elements γ_1 and γ_2 in Γ' in which both of t_{γ_1} and t_{γ_2} are nonzero, then using the similar argument as above we conclude that v_{γ_1} and v_{γ_2} have the same supporting projections in $\mathcal{A}$ and also in $\mathcal{B}$. It enforces that $[v_{\gamma_1}|v_{\gamma_2}]_{\mathcal{B}} \neq 0$, which is a contradiction. Therefore $u_{\lambda_0} = t_{\gamma_0} v_{\gamma_0}$ and the claim holds.

On the other hand, if $[u_{\lambda_0}|u_{\lambda_0}]_{\mathcal{B}} = e'$ for some minimal projection e' in $\mathcal{B}$, then we have

$$e' = [u_{\lambda_0}|u_{\lambda_0}]_{\mathcal{B}} = [t_{\gamma_0} v_{\gamma_0}|t_{\gamma_0} v_{\gamma_0}]_{\mathcal{B}} = |t_{\gamma_0}|^2 [v_{\gamma_0}|v_{\gamma_0}]_{\mathcal{B}} = |t_{\gamma_0}|^2 e'.$$

It follows that $|t_{\gamma_0}| = 1$. It proves items (i) and (ii). For proving (iii) consider the mapping $\phi : \Lambda \to \Gamma$, which assigns each u_{λ_0} to v_{γ_0}, where $\lambda_0 \in \Lambda$, $\gamma_0 \in \Gamma$ and v_{γ_0} is chosen as the proof of the previous parts. It is readily verified that ϕ is an injection.

Surjectivity of ϕ follows from changing the roles of $\{u_\lambda\}$ and $\{v_\gamma\}$ in the proof of (i). □

Corollary 2.2. *Suppose that E is an imprimitivity Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule over the commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then $\{u_\lambda\}$, $\lambda \in \Lambda$, is an orthonormal basis for the Hilbert H^*-module ${}_{\mathcal{A}}E$ if and only if it is an orthonormal basis for the Hilbert H^*-module $E_{\mathcal{B}}$.*

Proof. Let $\{u_\lambda\}$, $\lambda \in \Lambda$, be an orthonormal basis in ${}_{\mathcal{A}}E$. It is an immediate consequence of Theorems 2.2 and 2.3, that $\{u_\lambda\}$, $\lambda \in \Lambda$, is an orthonormal system for $E_{\mathcal{B}}$. So it is enough to prove that $\{u_\lambda\}$ generates a dense submodule for $E_{\mathcal{B}}$. Using Theorem 2.4, we may consider $\{v_\lambda\}$, $\lambda \in \Lambda$ to be an orthonormal basis for $E_{\mathcal{B}}$ such that $u_\lambda = t_\lambda v_\lambda$ for each $\lambda \in \Lambda$ and some scalar t_λ with $|t_\lambda| = 1$. Let us denote by $\mathcal{F}$ the family of finite subsets of Λ. Now if $x \in E$, then $x = \sum_{\lambda\in\Lambda} \mu'_\lambda v_\lambda$ and thus we have

$$\left\| x - \sum_{\lambda\in\Lambda'} \mu'_\lambda v_\lambda \right\|_{\mathcal{B}} = \left\| x - \sum_{\lambda\in\Lambda'} \frac{\mu'_\lambda}{t_\lambda} u_\lambda \right\|_{\mathcal{B}}, \tag{2.8}$$

for each $\Lambda' \in \mathcal{F}$. On using (2.8) we conclude that $\{u_\lambda\}$ generates a dense submodule of $E_{\mathcal{B}}$, too. □

In the light of the previous corollary, the following definition is reasonable.

Definition 2.3. Let E be an imprimitivity Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule over the commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and $\{u_\lambda\}$, $\lambda \in \Lambda$, be an orthonormal basis for Hilbert H^*-module ${}_{\mathcal{A}}E$ (or $E_{\mathcal{B}}$). Then we say $\{u_\lambda\}$ is an orthonormal basis for Hilbert H^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$.

In the sequel, we investigate the relationship between two topologies induced by H^*-algebras $\mathcal{A}$ and $\mathcal{B}$.

In general suppose that H is a Hilbert space with both inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$ and corresponding norms $\|.\|$ and $\|.\|_1$, respectively. If $\|x\|_1 \leq \beta \|x\|$ for each $x \in H$ and some $\beta > 0$, then there is a positive operator $K \in B(H)$ (w.r.t. $\|\cdot\|$) such that K is injective and moreover $\langle x, y \rangle_1 = \langle Kx, y \rangle$, for all x, y in H. On the other hand, $\|\cdot\|$ and $\|\cdot\|_1$ give rise to the same topology if K has an inverse in $B(H)$ (see [6, Page 162]). Accordingly, if ${}_{\mathcal{A}}E_{\mathcal{B}}$ is a Hilbert H^*-bimodule, then ${}_{\mathcal{A}}\|\cdot\|$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent if and only if there is a positive invertible operator K in $B(E)$ (w.r.t. ${}_{\mathcal{A}}\|\cdot\|$) in which $\langle x, y \rangle_{\mathcal{B}} = {}_{\mathcal{A}}\langle Kx, y \rangle$, for all x, y in E. Further, some more interesting results can be found in the case that H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are commutative.

Proposition 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative H^*-algebras and ${}_{\mathcal{A}}E_{\mathcal{B}}$ be an imprimitivity Hilbert H^*-bimodule. Assume that all minimal projections in $\mathcal{A}$ and $\mathcal{B}$ have norms equal to some $\alpha \geq 1$. Then ${}_{\mathcal{A}}\|x\| = \|x\|_{\mathcal{B}}$ for each $x \in E$.*

Proof. Let $\{e_i\}$, $i \in I$, be the family of all minimal projections in $\mathcal{A}$ and $\{u_\lambda\}$, $\lambda \in \Lambda$, be an orthonormal basis for ${}_{\mathcal{A}}E_{\mathcal{B}}$ with ${}_{\mathcal{A}}[u_\lambda | u_\lambda] = e_{i_\lambda}$ for each $\lambda \in \Lambda$ and some $i_\lambda \in I$. Take $x \in E$, then $x = \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda$ for some scalars μ_λ ($\lambda \in \Lambda$) and thus we have

$$\begin{aligned}
{}_{\mathcal{A}}\|x\|^2 &= \mathrm{tr}_{\mathcal{A}}({}_{\mathcal{A}}[x|x]) = \mathrm{tr}_{\mathcal{A}}\left({}_{\mathcal{A}}\left[\sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda \Big| \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda\right]\right) \\
&= {}_{\mathcal{A}}\left(\sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda \Big| \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda\right) = \sum_{\lambda\in\Lambda} |\mu_\lambda|^2_{\mathcal{A}}(u_\lambda|u_\lambda) = \sum_{\lambda\in\Lambda} |\mu_\lambda|^2 \ {}_{\mathcal{A}}\|e_{i_\lambda}\|^2 \\
&= \sum_{\lambda\in\Lambda} |\mu_\lambda|^2 \alpha^2.
\end{aligned}$$

Since the representation of $x = \sum_{\lambda\in\Lambda} \mu_\lambda u_\lambda$ is the same with respect to both of norms ${}_{\mathcal{A}}\|\cdot\|$ and $\|\cdot\|_{\mathcal{B}}$, then similar relations proves that $\|x\|^2_{\mathcal{B}} = \sum_{\lambda\in\Lambda} |\mu_\lambda|^2 \alpha^2$. So we achieve our goal. □

Proposition 2.4. *Let ${}_{\mathcal{A}}E_{\mathcal{B}}$ be an imprimitivity Hilbert H^*-bimodule over commutative H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and let $x, y \in E$. Then ${}_{\mathcal{A}}[x|y] = 0$ if and only if $[x|y]_{\mathcal{B}} = 0$.*

Proof. In the forward direction, suppose that ${}_{\mathcal{A}}[x|y] = 0$. Let $\{u_\lambda\}$, $\lambda \in \Lambda$ be an orthonormal basis for Hilbert H^*-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$, then for some suitable scalars t_λ and s_μ $(\lambda, \mu \in \Lambda)$, $x = \sum_{\lambda\in\Lambda'} t_\lambda u_\lambda$ and $y = \sum_{\mu\in\Lambda''} s_\mu u_\mu$, where $\Lambda' = \{\lambda \in \Lambda : t_\lambda \neq 0\}$ and $\Lambda'' = \{\mu \in \Lambda : s_\mu \neq 0\}$. These allow us to write ${}_{\mathcal{A}}[x|y] = \sum_{\lambda\in\Lambda'}\sum_{\mu\in\Lambda''} t_\lambda \overline{s_\mu}{}_{\mathcal{A}}[u_\lambda|u_\mu] = 0$, which in turn implies that $\Lambda' \cap \Lambda'' = \emptyset$. It follows from this reasoning and by applying Corollary 2.2, that $[x|y]_{\mathcal{B}} = \sum_{\lambda\in\Lambda'}\sum_{\mu\in\Lambda''} t_\lambda \overline{s_\mu}[u_\lambda|u_\mu]_{\mathcal{B}} = 0$. The inverse implication is shown similarly. □

In the sequel, we give an example to verify usefulness of our results.

Example 2.2. Let $\mathcal{A}$ be the commutative real H^*-algebra $\left\{\begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b \in \mathbb{R}\right\}$ together with the usual operations of addition and scalar multiplication and endowed with componentwise multiplication. Adjoint and inner product are defined by

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix}^* = \begin{pmatrix} a & a \\ b & b \end{pmatrix},$$

and

$${}_{\mathcal{A}}\left\langle \begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} c & c \\ d & d \end{pmatrix} \right\rangle = k(ac + bd),$$

where k is a positive number greater or equal to 1. Obviously, $\tau(\mathcal{A}) = \mathcal{A}$ and linear functional $\mathrm{tr}_{\mathcal{A}} : \tau(\mathcal{A}) \to \mathbb{R}$ defined by $\mathrm{tr}_{\mathcal{A}}\left(\begin{pmatrix} a & a \\ b & b \end{pmatrix}\right) = k(a+b)$ is positive. Similarly, consider the commutative real H^*-algebra $\mathcal{B} = \left\{\begin{pmatrix} a & b \\ a & b \end{pmatrix} : a, b \in \mathbb{R}\right\}$ together with the operations of addition, scalar multiplication, componentwise multiplication and adjoint which are defined as the similar way as $\mathcal{A}$ and inner product is defined by

$$\left\langle \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} c & d \\ c & d \end{pmatrix} \right\rangle_{\mathcal{B}} = p(ac + bd),$$

for some positive number $p \geq 1$. Evidently, $\tau(\mathcal{B}) = \mathcal{B}$ and linear functional $\mathrm{tr}_{\mathcal{B}} : \tau(\mathcal{B}) \to \mathbb{R}$ is defined by $\mathrm{tr}_{\mathcal{B}}\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = p(a + b)$ is positive. It is routine to verify that $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ are the sets of all minimal projections in $\mathcal{A}$ and $\mathcal{B}$, respectively. Now, take E the space of all 2×2 matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{R}$, and define left module multiplication $. : \mathcal{A} \times E \to E$ and right module multiplication $. : E \times \mathcal{B} \to E$ by

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

and

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & b \\ a & b \end{pmatrix}=\begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix},$$

respectively. Also, define $\tau(\mathcal{A})$- and $\tau(\mathcal{B})$-valued inner products ${}_{\mathcal{A}}[\cdot|\cdot] : E \times E \to \tau(\mathcal{A})$ and $[\cdot|\cdot]_{\mathcal{B}} : E \times E \to \tau(\mathcal{B})$ by

$${}_{\mathcal{A}}\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\middle|\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right]=\begin{pmatrix} ac & ac \\ bd & bd \end{pmatrix}$$

and

$$\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\middle|\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right]_{\mathcal{B}}=\begin{pmatrix} ac & bd \\ ac & bd \end{pmatrix},$$

respectively. It is not hard to see that E is an imprimitivity $\mathcal{A}$-$\mathcal{B}$ Hilbert bimodule.

Next, we point that the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is an orthonormal basis for ${}_{\mathcal{A}}E_{\mathcal{B}}$. This holds since

$${}_{\mathcal{A}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\middle|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$${}_{\mathcal{A}}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\middle|\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]=\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

and with the help of Corollaries 2.1 and 2.2 we get the desired result. Furthermore, assume that $p = k$, then all minimal projections in $\mathcal{A}$ and $\mathcal{B}$ have the same norm $\sqrt{k}$. Therefore, ${}_{\mathcal{A}}\|\cdot\| = \|\cdot\|_{\mathcal{B}}$ by Proposition 2.3.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative H^*-algebras and ${}_{\mathcal{A}}E_{\mathcal{B}}$ be a full Hilbert H^*-bimodule. Then H^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. Consider $\{e_i\}$, $i \in I$, is the family of all minimal projections in $\mathcal{A}$, $\{u_\lambda\}$, $\lambda \in \Lambda$ is an orthonormal basis for ${}_{\mathcal{A}}E_{\mathcal{B}}$ and a is an arbitrary element in $\mathcal{A}$. By the commutativity of $\mathcal{A}$, $a = \sum_{i\in I}\mu_i e_i$. According to Proposition 2.2, for each $i \in I$ there exists $\lambda_i \in \Lambda$ such that $e_i ={}_{\mathcal{A}} [u_{\lambda_i}|u_{\lambda_i}]$. Hence, $a = \sum_{i\in I}\mu_i \ {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}]$. Define a mapping $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(a) := \sum_{i\in I}\mu_i[u_{\lambda_i}|u_{\lambda_i}]_{\mathcal{B}}$, where $a = \sum_{i\in I}\mu_i \ {}_{\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}]$. In view of Theorem 2.3, we observe that $a = \sum_{i\in I}\mu_{i\mathcal{A}}[u_{\lambda_i}|u_{\lambda_i}] = 0$ if and only if $\varphi(a) = \sum_{i\in I}\mu_i[u_{\lambda_i}|u_{\lambda_i}]_{\mathcal{B}} = 0$. This shows that φ is well defined and injective. It is easy to verify that φ is a morphism, i.e., $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$, $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ and $\varphi(a^*) = \varphi(a)^*$, for all a_1, a_2, a in $\mathcal{A}$. The surjectivity of φ is evident. This is somewhat similar to the situation discussed for constructing φ. Therefore, φ is an isomorphism. ☐

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