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A NEW COMPANION OF OSTROWSKI'S INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, we establish a new companion of Ostrowski type inequality for differentiable functions whose first derivatives are bounded, and give its applications to probability density function.

1. Introduction

In 1938, Ostroweski established the following inequality which can be used to estimate the absolute deviation of a function form its integral mean (see [2]).

Theorem 1.1. *Let $f : I \to R$ be a differentiable mapping in the interior I of I, where $I \subset R$ is an interval, and let $a, b \in I$, $a < b$. If $|f'(t)| \leq M, t \in [a, b]$, then we have*

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq \left[\frac{1}{4} + \frac{(x - \frac{(a+b)}{2})^2}{(b-a)^2}\right](b-a)M, \quad x \in [a, b].$$

The following result is well known in literature as Grüss's inequality ([4], p. 296).

Theorem 1.2. [4] *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x \in (a, b)$. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma)$$

and the inequality is sharp in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

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For differentiable mappings whose first derivatives are bounded, Alomari proved two companion inequalities.

Theorem 1.3. [1] *Let $f : I \to R$ be a differentiable mapping in the interior I of I, where $I \subset R$ is an interval, and let $a, b \in I$, $a < b$. If $f' \in L^1[a,b]$ and $\gamma \le f'(t) \le \Gamma$, for all $x \in [a,b]$, then we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{\Gamma-\gamma}{2(b-a)}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2\right], \tag{1.1}$$

for all $x \in [a, \frac{a+b}{2}]$.

Theorem 1.4. [5] *Let the assumptions of Theorem 1.3 hold. Then we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma), \tag{1.2}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

We remark that the bound of (1.1) is better than that of (1.2). In fact, for $x \in [a, \frac{a+b}{2}]$, we have

$$\begin{aligned}\frac{1}{8}(b-a) =& \frac{1}{2}\frac{(x-a+\frac{a+b}{2}-x)^2}{b-a}\\ \ge& \frac{1}{2}\frac{(x-a)^2+\left(\frac{a+b}{2}-x\right)^2}{b-a}.\end{aligned}$$

In this paper, using a different mean, we at first give (1.1) a new proof, and point out that (1.1) is sharp in the sense that the constant $\frac{1}{2}$ cannot be replaced by a smaller one. Secondly, we establish a new companion inequality of Ostrowski type for differentiable mappings whose first derivatives are bounded and consider its applications to probability density function.

2. Main Results

A new proof of Theorem 1.3.

Proof. We define a mapping as

$$k(x,t) = \begin{cases} t-a, & t \in [a,x], \\ t-\frac{a+b}{2}, & (x, a+b-x], \\ t-b, & (a+b-x, b], \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b k(x,t)f'(t)dt = \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt. \tag{2.1}$$

If $t \in [a, b]$ and $x \in [a, \frac{a+b}{2}]$, then from the definition of $k(x, t)$, we have

$$p(x,t) \geq 0, \quad t \in [a,x] \cup \left(\frac{a+b}{2}, a+b-x\right],$$
$$p(x,t) \leq 0, \quad t \in \left[x, \frac{a+b}{2}\right] \cup (a+b-x, b].$$

Hence,

$$\begin{aligned}
(2.2)\qquad & \frac{1}{b-a}\int_a^b p(x,t)f'(t)dt \\
=& \frac{1}{b-a}\left[\int_a^x (t-a)f'(t)dt + \int_x^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)f'(t)dt\right. \\
&+\left.\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)dt + \int_{a+b-x}^b (t-b)f'(t)dt\right] \\
\leq& \frac{\Gamma}{b-a}\left[\int_a^x (t-a)dt + \int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt\right] \\
&+\frac{\gamma}{b-a}\left[\int_x^{\frac{a+b}{2}}\left(t-\frac{a+b}{2}\right)dt + \int_{a+b-x}^b (t-b)dt\right] \\
=& \frac{\Gamma}{2(b-a)}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2\right] \\
&-\frac{\gamma}{2(b-a)}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2\right] \\
=& \frac{\Gamma-\gamma}{2(b-a)}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2\right].
\end{aligned}$$

A similar argument can prove that

$$(2.3)\qquad -\frac{1}{b-a}\int_a^b p(x,t)f'(x) \leq \frac{\Gamma-\gamma}{2(b-a)}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2\right].$$

Form (2.1)–(2.3), we get the inequality (1.1).

Form the proof, let

$$f(x) = \begin{cases} \Gamma(t-a), & t \in [a,x], \\ \Gamma(x-a) + \gamma(t-x), & \left(x, \frac{a+b}{2}\right], \\ \Gamma\left(t+x-\frac{3a+b}{2}\right) + \gamma\left(\frac{a+b}{2}-x\right), & \left(\frac{a+b}{2}, a+b-x\right], \\ \Gamma\frac{b-a}{2} + \gamma\left(t-\frac{a+b}{2}\right), & (a+b-x, b], \end{cases}$$

then the inequality (1.1) holds equality. Thus the constant 1/2 is sharp in the sense that it cannot be replaced by a smaller one. □

Theorem 2.1. *Let $f : I \to R$ be a differentiable mapping in the interior I of I, where $I \subset R$ is an interval, and let $a, b \in I$, $a < b$. If $f' \in L^1[a, b]$ and $\gamma \le f'(t) \le \Gamma$, for all $t \in [a, b]$, then we have*

$$\left|\frac{1}{4}\left(f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \tag{2.4}$$
$$\le \begin{cases} \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2} - x\right)^2 - \left(\frac{3a+b}{4} - x\right)^2\right)(\Gamma - \gamma), & a \le x \le \frac{3a+b}{4}, \\ \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2} - x\right)^2 + \left(\frac{3a+b}{4} - x\right)^2\right)(\Gamma - \gamma), & \frac{3a+b}{4} \le x \le \frac{a+b}{2}, \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. We define the function given by

$$k(x,t) = \begin{cases} t - a, & t \in \left[a, \dfrac{a+x}{2}\right], \\ t - \dfrac{3a+b}{4}, & \left(\dfrac{a+x}{2}, x\right], \\ t - \dfrac{a+b}{2}, & (x, a+b-x], \\ t - \dfrac{a+3b}{4}, & \left(a+b-x, \dfrac{a+2b-x}{2}\right], \\ t - b, & \left(\dfrac{a+2b-x}{2}, b\right] \end{cases}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Integrating by parts, we can state that

$$\frac{1}{b-a}\int_a^b k(x,t)f'(t)dt \tag{2.5}$$
$$= \frac{1}{4}\left(f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right) - \frac{1}{b-a}\int_a^b f(t)dt.$$

We also have

$$\frac{1}{b-a}\int_a^b k(x,t)dt = 0. \tag{2.6}$$

Let $C = \frac{\Gamma+\gamma}{2}$. From (2.5) and (2.6), we have

$$\frac{1}{b-a}\int_a^b k(x,t)(f'(t) - C)dt \tag{2.7}$$
$$= \frac{1}{4}\left(f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right) - \frac{1}{b-a}\int_a^b f(t)dt.$$

Hence, we have

$$\begin{aligned}(2.8)\qquad &\left|\frac{1}{b-a}\int_a^b k(x,t)(f'(t)-C)dt\right| \\ &\le \max_{t\in[a,b]}|f'(t)-C|\frac{1}{b-a}\int_a^b |k(x,t)|dt.\end{aligned}$$

Noting

$$(2.9)\qquad \max_{t\in[a,b]}|f'(t)-C| \le \frac{\Gamma-\gamma}{2}.$$

If $x \in \left[a, \frac{3a+b}{4}\right]$, we have

$$\begin{aligned}(2.10)\qquad &\int_a^b |k(x,t)|dt \\ &= \int_a^{\frac{a+x}{2}}(t-a)dt + \int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt + \int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)dt \\ &\quad + \int_{\frac{a+x}{2}}^{x}\left(\frac{3a+b}{4}-t\right)dt + \int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dt + \int_{\frac{a+2b-x}{2}}^{b}(b-t)dt \\ &= \left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2}-x\right)^2 - \left(\frac{3a+b}{4}-x\right)^2\right).\end{aligned}$$

If $x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$, we have

$$\begin{aligned}(2.11)\qquad &\int_a^b |\, k(x,t)\,|\, dt = \int_a^{\frac{a+x}{2}}(t-a)dt + \int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\ &+ \int_{\frac{3a+b}{4}}^{x}\left(t-\frac{3a+b}{4}\right)dt + \int_{a+b-x}^{\frac{a+3b}{4}}\left(\frac{a+3b}{4}-t\right)dt + \int_{\frac{3a+b}{4}}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)dt \\ &+ \int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dt + \int_{\frac{a+2b-x}{2}}^{b}(b-t)dt + \int_{\frac{a+x}{2}}^{\frac{a+3b}{4}}\left(\frac{3a+b}{4}-t\right)dt \\ &= \left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2}-x\right)^2 + \left(\frac{3a+b}{4}-x\right)^2\right).\end{aligned}$$

We have

$$\begin{aligned}(2.12)\qquad &\left|\frac{1}{b-a}\int_a^b k(x,t)(f'(t)-C)dt\right| \\ &\le \begin{cases} \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2}-x\right)^2 - \left(\frac{3a+b}{4}-x\right)^2\right)(\Gamma-\gamma), & a \le x \le \frac{3a+b}{4}, \\ \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2 + \frac{5}{4}\left(\frac{a+b}{2}-x\right)^2 + \left(\frac{3a+b}{4}-x\right)^2\right)(\Gamma-\gamma), & \frac{3a+b}{4} \le x \le \frac{a+b}{2}. \end{cases}\end{aligned}$$

Form (2.7) and (2.12), we get (2.4). This completes the proof. □

Corollary 2.1. *In inequality* (2.4)*, choosing*

(1) $x = a$*, we have the same results of* [5]*:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma);$$

(2) $x = \frac{3a+b}{4}$*, we have*

$$\left|\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le\frac{3}{64}(b-a)(\Gamma-\gamma);$$

(3) $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le\frac{1}{16}(b-a)(\Gamma-\gamma).$$

3. Applications to Probability Density Function

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f : [a,b] \to [0,1]$ with the cumulative distribution function $F(x) = Pr(X \le x) = \int_a^x f(t)dt$.

Theorem 3.1. *With the above assumptions and f being as in Theorem* 2.1*, we have*

$$\left|\tfrac{1}{4}\left(F(x)+F(a+b-x)+F\left(\tfrac{a+x}{2}\right)+F\left(\tfrac{a+2b-x}{2}\right)\right) - \tfrac{b-E(x)}{b-a}\right| \tag{3.1}$$
$$\le\begin{cases} \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2+\frac{5}{4}\left(\frac{a+b}{2}-x\right)^2-\left(\frac{3a+b}{4}-x\right)^2\right)(\Gamma-\gamma), & a\le x\le\frac{3a+b}{4}], \\ \frac{1}{2(b-a)}\left(\left(\frac{x-a}{2}\right)^2+\frac{5}{4}\left(\frac{a+b}{2}-x\right)^2+\left(\frac{3a+b}{4}-x\right)^2\right)(\Gamma-\gamma), & \frac{3a+b}{4}\le x\le\frac{a+b}{2}, \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$*, where* $E(X)$ *is the expectation of* X*.*

Proof. Let $f = F$ in (2.4), and taking into account

$$E(X) = \int_a^b tdF(t) = b - \int_a^b F(t)ft,$$

we obtain (3.1). ☐

Corollary 3.1. *In Theorem* 3.1*, putting* $x = \frac{3a+b}{4}$ *we have*

$$\left|\frac{1}{4}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)+F\left(\frac{7a+b}{8}\right)+F\left(\frac{a+7b}{8}\right)\right] - \frac{b-E(X)}{b-a}\right|$$
$$\le\frac{3(b-a)}{64}(\Gamma-\gamma).$$

Corollary 3.2. *In Theorem* 3.1*, if* $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{1}{4}\left[2F\left(\frac{a+b}{2}\right)+F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|$$
$$\leq\frac{(b-a)}{16}(\Gamma-\gamma).$$

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