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EXISTENCE RESULTS AND NUMERICAL SOLUTIONS FOR A MULTI-TERM FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract. We investigate numerical solutions for a multi-term fractional integro-differential equation by using the collocation method. In this way, we handle Alpert's multi-wavelet to procure an approximation solution for the problem. We provide two examples to illustrate our main results.

1. Introduction

As you know, fractional calculus is one of the interest problems in mathematics and engineering and is a beneficial tool for modeling of different types of scientific phenomena (see for example, [13,14] and [15]). The analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigate by many authors ([1,2,5]). In general, it is not easy to derive the analytical solution to most of the fractional differential equations and the numerical solution of fractional differential equations has attached considerable attention from many researchers. During the past decades, an increasing number of numerical schemes are being developed. These methods include finite difference approximation method ([6,21]), fractional linear multi-step method ([9]), collocation method [19,20], the Adomian decomposition method [16,18], variational iteration method [17] and operational matrix method [23].

Recently, the idea of approximating the solution of fractional differential equations by orthogonal family of basis functions have been widely used and the most commonly used orthogonal function are block pulse function, Chebyshev polynomials, Legendre

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and Laguerre polynomials. Wavelets are localized function, which form the basis for $L^2(\mathbb{R})$, so that localized pulse problem can easily be approached and analyzed [11]. They are successfully applied in system analysis, signal analysis, optimal control and many more areas (see [22]). However, wavelets are just another basis set which offers considerable advantage over alternative basis sets and allows us to tackle problems not accessible with conventional numerical methods, these main advantages are discussed in [10]. Different variation of wavelet bases (orthogonal, bi-orthogonal, multiwavelets) have been presented and the design of the corresponding wavelet and scaling function have been addressed [7, 8]. Multiwavelets are generated by more than one scaling function [12]. The advantage of multiwavelets , as extensions from scaler wavelets, and their promising characteristics have resulted in an increasing trend to study them. characteristics such as orthogonality, symmetry, compact support, vanishing moments and simple structure make the multiwavelets useful both in theory and applications. In this paper, we investigate numerical solutions for a multi-term fractional integro-differential equation by using Alpert's multi-wavelets. These multi-wavelets have been constructed in [3] and also have been considered in [25].

The Riemann-Liouville fractional integral of order $\gamma > 0$ for a function f is defined by [13]

$$I^\gamma f(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s)ds.$$

The Caputo fractional derivative of order α for a continuous function f is defined by

$${}^cD^\gamma f(t) = \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(\gamma)} f^{(n)}(s)\, ds,$$

where $n = [\gamma] + 1$ [13].

Lemma 1.1. [21] *Let $\gamma > 0$ and $n = [\gamma] + 1$. The fractional differential equation ${}^cD^\gamma u(t) = 0$ has solution in the form $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and also $I^{\gamma c}D^\gamma u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_0, c_1, \dots, c_n \in \mathbb{R}$ are some constants.*

By following main idea of [4] and [26], we are going to study the existence and uniqueness of solution for the multi-term fractional integro-differential equation

$$ {}^cD^\gamma x(t) = f(t, x(t), I^\beta x(t), {}^cD^\delta x(t)), \tag{1.1}$$

with boundary condition $x(o) + x(1) = a$ and $I^\beta x(\varepsilon) + I^\beta x(\eta) = b \int_0^1 x(s)ds$ in two different ways and under some assumptions, where $1 \le \gamma < 2$, $\beta > 0$, $0 \le \delta < 1$, $\varepsilon > 0$, $\eta > 0$, $a, b \in \mathbb{R}$, $t \in I := [0, 1]$ and $f : I \times \mathbb{R}^3 \to \mathbb{R}$ satisfies some conditions. In fact, the work [26] use the Krasnoselskii's fixed point theorem while we use a new fixed point result. Also, we use different boundary value conditions respect to [26]. In second part of this work, we investigate the problem (1.1) by using the Alpert's multi-wavelet for approximating its solution. In this way, we consider the Banach space $C([0,1],\mathbb{R})$ endowed with the norm

$$\| x-y \|=\sup_{t\in I}|x(t)-y(t)|+\sup_{t\in I}|^cD^{\delta}x(t)-{}^cD^{\delta}y(t)|.$$

Denote by Ψ the family of nondecreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{n=1}^{+\infty}\psi^n<\infty$ for all $t>0$. It is easy to see that $\psi(t)<t$ for all $t>0$. Let(X,d) be a metric space, T a selfmap on X, $\alpha:X\times X\to[0,\infty)$ a map and $\psi\in\Psi$. We say that T is α-ψ-contraction on X whenever $\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y))$ for all $x,y\in X$ [24]. Also, T is called α-admissible whenever $\alpha(x,y)\ge 1$ implies $\alpha(Tx,Ty)\ge 1$ [24]. We need the following result.

Theorem 1.1. [24] *Let (X,d) be a complete metric space, $\alpha:X\times X\to[0,\infty)$ a map, $\psi\in\Psi$ and T an α-admissible and α-ψ-contraction selfmap on X. Assume that there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$ and for each sequence x_n in X with $\alpha(x_n,x_{n+1})\ge 1$ for all n and $x_n\in x$ we have $\alpha(x_n,x)\ge 1$ for all n. Then, T has a fixed point. If for each $x,y\in X$ there exists $z\in X$ such that $\alpha(x,z)\ge 1$ and $\alpha(y,z)\ge 1$, then T has a unique fixed point.*

2. Main Results

Now, we are ready to provide our main results about the existence and uniqueness of solution for the multi-term fractional integro-differential equation (1.1).

Lemma 2.1. *Let $1<\gamma\le 2$, $\beta>0$, $\varepsilon>0$, $\eta>0$, $a,b\in\mathbb{R}$ and $y\in C([0,1],\mathbb{R})$. The unique solution of the boundary value problem ${}^cD^{\gamma}x(t)=y(t)$ with boundary conditions $x(0)+x(1)=a$ and $I^{\beta}x(\varepsilon)+I^{\beta}x(\eta)=b\int_0^1 x(s)ds$ is given by*

$$\begin{aligned}x(t)=&I^{\gamma}y(t)+\frac{1}{\varphi_1-\varphi_2}[\varphi_1(1-t)I^{\gamma}y(1)+b(1-t)I^{\gamma+1}y(1)\\&+(t-1)(I^{\gamma+\beta}y(\eta)+I^{\gamma+\beta}y(\varepsilon))+a\varphi_1(t-1)],\end{aligned}$$

where $\varphi_1=-b+\frac{\varepsilon^{\beta}+\eta^{\beta}}{\Gamma(\beta+1)}$ and $\varphi_2=\frac{-b}{2}+\frac{\varepsilon^{\beta+1}+\eta^{\beta+1}}{\Gamma(\beta+2)}$.

Proof. It is known that the solution of the equation ${}^cD^{\gamma}x(t)=y(t)$ is given by

$$x(t)=I^{\gamma}y(t)+c_0+c_1t, \tag{2.1}$$

where $c_0,c_1\in\mathbb{R}$ are some constants. By using the boundary conditions, we get

$$I^{\gamma}y(1)+2c_0+c_1=a$$

and

$$I^{\gamma+\beta}y(\varepsilon)+I^{\gamma+\beta}y(\eta)+c_0\frac{\varepsilon^{\beta}+\eta^{\beta}}{\Gamma(\beta+1)}+c_1\frac{\varepsilon^{\beta+1}+\eta^{\beta+1}}{\Gamma(\beta+2)}=bI^{\gamma+1}+bc_0+b\frac{c_1}{2}.$$

Therefore, we have

$$c_0=\frac{1}{\varphi_1-\varphi_2}\left(\frac{\varphi_1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}f(s,x(s),I^{\beta}x(s),{}^cD^{\delta}x(s))ds\right.$$

$$
\begin{aligned}
&+ \frac{b}{\Gamma(\gamma+1)} \int_0^1 (1-s)^{\gamma} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \\
&- \frac{1}{\Gamma(\gamma+\beta)} \left(\int_0^{\eta} (\eta-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \right. \\
&\left. + \int_0^{\varepsilon} (\varepsilon-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \right) - a\varphi_1 \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
c_1 =& \frac{1}{\varphi_1-\varphi_2} \left(- \frac{\varphi_1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \right. \\
&- \frac{b}{\Gamma(\gamma+1)} \int_0^1 (1-s)^{\gamma} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \\
&+ \frac{1}{\Gamma(\gamma+\beta)} \left(\int_0^{\eta} (\eta-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \right. \\
&\left. \left. + \int_0^{\varepsilon} (\varepsilon-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s)) ds \right) + a\varphi_1 \right).
\end{aligned}
$$

By substituting c_0 and c_1 in (2.1), we obtain $x(t)$. □

Consider the operator $T : C(I) \to C(I)$ defined by

$$
\begin{aligned}
Tx(t) =& \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s))\, ds \\
&+ \frac{1}{\varphi_1-\varphi_2} \left[\varphi_1(1-t) \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s))\, ds \right. \\
&+ b(1-t) \frac{1}{\Gamma(\gamma+1)} \int_0^1 (1-s)^{\gamma} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s))\, ds \\
&+ (t-1) \left(\frac{1}{\Gamma(\gamma+\beta)} \int_0^{\eta} (\eta-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s))\, ds \right. \\
&\left. \left. + \frac{1}{\Gamma(\gamma+\beta)} \int_0^{\varepsilon} (\varepsilon-s)^{\gamma+\beta-1} f(s, x(s), I^{\beta}x(s), {}^c D^{\delta}x(s))\, ds \right) + a\varphi_1(t-1) \right], \qquad (2.2)
\end{aligned}
$$

for all $t \in [0, 1]$.

Theorem 2.1. *Assume that there exist a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ and a map $\psi \in \Psi$ such that $|f(t, x, p, u) - f(t, y, q, v)| \leq M\psi(|x-y| + |u-v|)$, for all $t > 0$ and $x, y, p, q, u, v \in \mathbb{R}$ with $\xi(x, y) \geq 0$, where $M = \frac{1}{M_1 + M_2} > 0$ is such that*

$$
\frac{1}{\Gamma(\gamma)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma+1)} + \frac{\eta^{\gamma+\beta} + \varepsilon^{\gamma+\beta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta)} < M_1
$$

and

$$
\frac{1}{\Gamma(\gamma-\delta)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma-\delta)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma+1-\delta)}
$$

$$+\frac{\eta^{\gamma+\beta-\delta}+\varepsilon^{\gamma+\beta-\delta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta-\delta)} < M_2.$$

Suppose that there exists $x_0 \in C(I)$ such that $\xi(x_0(t), Tx_0(t)) \geq 0$, for all $t \in I$ and for each $x, y \in C(I)$, with $\xi(x(t), y(t)) \geq 0$ for all $t \in I$, we have $\xi(Tx(t), Ty(t)) \geq 0$ for all $t \in I$. Assume that for each sequence x_n in $C(I)$ with $x_n \to x$ in $C(I)$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$, for all n and $t \in I$, we have $\xi(x_n(t), x(t)) \geq 0$, for all n and $t \in I$. Then the problem (1.1) *has a solution.*

Proof. Let $x, y \in X$ and $t \in I$ be given. Then, we have

$$\begin{aligned}
|Tx(t)-Ty(t)| = &\Bigg|\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))\,ds\\
&+\frac{1}{\varphi_1-\varphi_2}\Bigg[\varphi_1(1-t)\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))\,ds\\
&+b(1-t)\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-s)^{\gamma} f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))\,ds\\
&+(t-1)\frac{1}{\Gamma(\gamma+\beta)}\int_0^\eta (\eta-s)^{\gamma+\beta-1} f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))\,ds\\
&+\frac{1}{\Gamma(\gamma+\beta)}\int_0^\varepsilon (\varepsilon-s)^{\gamma+\beta-1} f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))\,ds)+a\varphi_1(t-1)\Bigg]\\
&-\Bigg[\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))\,ds\\
&+\frac{1}{\varphi_1-\varphi_2}\Bigg[\varphi_1(1-t)\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))\,ds\\
&+b(1-t)\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-s)^{\gamma} f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))\,ds\\
&+(t-1)\frac{1}{\Gamma(\gamma+\beta)}\int_0^\eta (\eta-s)^{\gamma+\beta-1} f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))ds\\
&+\frac{1}{\Gamma(\gamma+\beta)}\int_0^\varepsilon (\varepsilon-s)^{\gamma+\beta-1} f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))\,ds)+a\varphi_1(t-1)\Bigg]\Bigg]\Bigg|\\
\leq&\frac{1}{\Gamma(\gamma)}\int_0^t |t-s|^{\gamma-1}|f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))-f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))|ds\\
&+\frac{1}{|\varphi_1-\varphi_2|}\Bigg[|\varphi_1||(1-t)|\frac{1}{\Gamma(\gamma)}\\
&\times\int_0^1 (1-s)^{\gamma-1}|f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))-f(s,y(s),I^\beta y(s),{}^cD^\delta y(s))|ds\\
&+|b|(1-t)\frac{1}{\Gamma(\gamma+1)}\int_0^1 (1-s)^{\gamma}|f(s,x(s),I^\beta x(s),{}^cD^\delta x(s))
\end{aligned}$$

$$
\begin{aligned}
&- f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))ds + |t-1|\frac{1}{\Gamma(\gamma+\beta)}\\
&\times \int_0^{\eta}(\eta-s)^{\gamma+\beta-1}|f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s)) - f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))|ds\\
&+ \frac{1}{\Gamma(\gamma+\beta)}\\
&\left.\times \int_0^{\varepsilon}(\varepsilon-s)^{\gamma+\beta-1}|f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s)) - f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))|\, ds\right]\\
\leq &\left(\frac{1}{\Gamma(\gamma)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma+1)} + \frac{\eta^{\gamma+\beta}+\varepsilon^{\gamma+\beta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta)}\right)\\
&\times |f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s)) - f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))|\\
<&M_1 \sup_{t\in I}|f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s)) - f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))|
\end{aligned}
$$

and

$$
\begin{aligned}
&|{}^{c}D^{\delta}Tx(t) - {}^{c}D^{\delta}Ty(t)| = \left|\frac{1}{\Gamma(\gamma-\delta)}\int_0^t (t-s)^{\gamma-\delta-1} f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s))\, ds\right.\\
&+ \frac{1}{\varphi_1-\varphi_2}\left[\varphi_1(1-t)\frac{1}{\Gamma(\gamma-\delta)}\int_0^1 (1-s)^{\gamma-\delta-1} f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s))\, ds\right.\\
&+ b(1-t)\frac{1}{\Gamma(\gamma-\delta+1)}\int_0^1 (1-s)^{\gamma-\delta} f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s))\, ds\\
&+ (t-1)\frac{1}{\Gamma(\gamma+\beta-\delta)}\int_0^{\eta} (\eta-s)^{\gamma+\beta-\delta-1} f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s))\, ds\\
&\left.+ \frac{1}{\Gamma(\gamma+\beta-\delta)}\int_0^{\varepsilon} (\varepsilon-s)^{\gamma+\beta-\delta-1} f(s,x(s),I^{\beta}x(s),^{c}D^{\delta}x(s))\, ds) + a\varphi_1(t-1)\right]\\
&- \left[\frac{1}{\Gamma(\gamma-\delta)}\int_0^t (t-s)^{\gamma-\delta-1} f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))\, ds\right.\\
&+ \frac{1}{\varphi_1-\varphi_2}\left[\varphi_1(1-t)\frac{1}{\Gamma(\gamma-\delta)}\int_0^1 (1-s)^{\gamma-\delta-1} f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))\, ds\right.\\
&+ b(1-t)\frac{1}{\Gamma(\gamma-\delta+1)}\int_0^1 (1-s)^{\gamma-\delta} f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))\, ds\\
&+ (t-1)\frac{1}{\Gamma(\gamma+\beta-\delta)}\int_0^{\eta} (\eta-s)^{\gamma+\beta-\delta-1} f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))\, ds\\
&\left.\left.\left.+ \frac{1}{\Gamma(\gamma+\beta-\delta)}\int_0^{\varepsilon} (\varepsilon-s)^{\gamma+\beta-\delta-1} f(s,y(s),I^{\beta}y(s),^{c}D^{\delta}y(s))\, ds) + a\varphi_1(t-1)\right]\right]\right|\\
\leq &\frac{1}{\Gamma(\gamma-\delta)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^t |t-s|^{\gamma-\delta-1}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\beta y(s))|ds \\
&+ \frac{1}{|\varphi_1-\varphi_2|}\Bigg[|\varphi_1||(1-t)|\frac{1}{\Gamma(\gamma-\delta)} \\
&\times \int_0^1 (1-s)^{\gamma-\delta-1}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))|ds \\
&+ |b|(1-t)\frac{1}{\Gamma(\gamma-\delta+1)} \\
&\times \int_0^1 (1-s)^{\gamma-\delta}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))ds| \\
&+ |t-1|\frac{1}{\Gamma(\gamma+\beta-\delta)} \\
&\times \int_0^\eta (\eta-s)^{\gamma+\beta-\delta-1}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))|ds \\
&+ \frac{1}{\Gamma(\gamma+\beta-\delta)}\int_0^\varepsilon (\varepsilon-s)^{\gamma+\beta-\delta-1}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) \\
&- f(s,y(s),I^\beta y(s),^c D^\delta y(s))|ds\Bigg] \\
\leq &\Bigg(\frac{1}{\Gamma(\gamma-\delta)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma-\delta)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma-\delta+1)} \\
&+\frac{\eta^{\gamma+\beta-\delta}+\varepsilon^{\gamma+\beta-\delta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta-\delta)}\Bigg) \\
&\times |f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))| \\
<&M_2 \sup_{t\in I}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))|.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&|Tx(t)-Ty(t)| + |^c D^\delta Tx(t) -^c D^\delta Ty(t)| \\
<&(M_1+M_2)\sup_{t\in I}|f(s,x(s),I^\beta x(s),^c D^\delta x(s)) - f(s,y(s),I^\beta y(s),^c D^\delta y(s))| \\
=&\psi(|x-y| + |^c D^\delta x -^c D^\delta y|)
\end{aligned}
$$

and so $\| Tx-Ty \| \leq \psi(\| x-y \|)$, for all x,y and $t\in I$ with $\xi(x(t),y(t))\geq 0$. Now, define the map $\alpha : C(I)\times C(I)\to [0,\infty)$ by $\alpha(x,y)=1$ whenever $\xi(x(t),y(t))\geq 0$, for all t and $\alpha(x,y)=0$ otherwise. This implies that $\alpha(x,y)d(Tx,Ty)\leq \psi(d(x,y))$ for all $x,y\in C(I)$ and so T is α-ψ-contraction. One can easily to check that T is α-admissible and there exists $x_0\in C(I)$ such that $\alpha(x_0,Tx_0)\geq 1$. Now by using Theorem 1.1, T has a fixed point which is a solution for the problem (1.1). □

By using different assumptions, we review again the problem (1.1) in next result.

Theorem 2.2. *Assume that* $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and there exists a constant* $K > 0$ *such that*

$$|f(t,x,p,y) - f(t,u,q,v)| \leq K(|x-u| + |y-v|),$$

for all $x, y, p, q, u, v \in \mathbb{R}$ *and* $t \in [0,1]$. *Then the problem* (1.1) *has a unique solution whenever* $\Omega < 1$, *where*

$$\begin{aligned}\Omega =& K\Bigg[\left(\frac{1}{\Gamma(\gamma)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma+1)} + \frac{\eta^{\gamma+\beta}+\varepsilon^{\gamma+\beta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta)}\right) \\ &+ \left(\frac{1}{\Gamma(\gamma-\delta)} + \frac{|\varphi_1|}{|\varphi_1-\varphi_2|\Gamma(\gamma-\delta)} + \frac{|b|}{|\varphi_1-\varphi_2|\Gamma(\gamma-\delta+1)} \right. \\ &+ \left. \frac{\eta^{\gamma+\beta-\delta}+\varepsilon^{\gamma+\beta-\delta}}{|\varphi_1-\varphi_2|\Gamma(\gamma+\beta-\delta)}\right)\Bigg].\end{aligned}$$

3. Alpert's Multi-Wavelet Bases

In this section, we reviewed a class of wavelet basis that constructed by Alpert for $L^2[0,1]$ ([3]). We present a brief review of Alpert's multiwavelet ([25]).

For function $\phi^m \in L^2(\mathbb{R})$, $m = 0, 1, \dots, r$, consider a reference subspace V_0 be generated as the L^2-closure of the linear span of the integer translation of ϕ^m:

$$V_0 = \overline{\langle \phi^m(.-k) : \ k \in \mathbb{Z}, \ m = 0, 1, \dots, r \rangle}$$

and, let other subspace

$$V_j = \overline{\langle \phi^m_{j,k} = \phi^m(2^j x - k) : \ k \in \mathbb{Z}, \ m = 0, 1, \dots, r, \ j \in \mathbb{Z} \rangle}.$$

Definition 3.1. Function $\phi^m \in L^2(\mathbb{R})$, are said to generate a multiresolution analysis (MRA), if they generate a nested sequence of closed subspace V_j that satisfies:

1. $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$;
2. $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;
3. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
4. $f(x) \in V_j \Leftrightarrow f(x + 2^{-j}) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$;
5. $\{\phi^m(.-k)\}_{k\in\mathbb{Z}}$ is an Riesz basis of V_0.

If ϕ^m generate an MRA, thus ϕ^m are nominated scaling functions. When the different integer translates of ϕ^m are orthogonal, namely:

$$\phi^m(\cdot - k) \perp \phi^{m'}(\cdot - k'), \quad \text{for } m \neq m', k \neq k',$$

the scaling function are named orthogonal scaling functions. Because the subspace V_j are nested, there exist complementary orthogonal subspaces W_j such that

$$V_{j+1} = V_j \bigoplus W_j, \quad j \in \mathbb{Z},$$

This gives an orthogonal decomposition of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = \bigoplus_{j\in F} V_j = \bigoplus_{j\in F} W_j.$$

Definition 3.2. Functions $\psi^m \in L^2(\mathbb{R})$ are called wavelets, if they supply the complementary orthogonal subspaces W_j of an MRA, namely:

$$W_j = \overline{\langle \psi^m_{j,k} = \psi^m(2^j x - k) : \ k \in \mathbb{Z}, \ m = 0, 1, \ldots, m, \ j \in \mathbb{Z} \rangle}.$$

Evidently, $\psi^m_{j,k} \perp \psi'^{m'}_{j',k'}$ for $j \neq j'$, $k \neq k'$, $m \neq m'$, thus ψ^m are called orthonormal wavelets.

Alpert's multiwavelets systems with multiplicity r consist of $r+1$ scaling functions and $r+1$ wavelets. The rth order scaling functions are the $r+1$ functions $\phi^0(x), \ldots, \phi^r(x)$ where $\phi^i(x)$ is a polynomial of ith order and allϕ's form orthonormal basis, for $i = 0, 1, \ldots, r$,

$$\phi^i(x) = \sum_{k=0}^{i} a_{ik} x^k, \quad \text{for } a_{ik} \geq 0, \int_0^1 \phi^i(x)\phi^k(x) = \delta_{i,k}. \tag{3.1}$$

The two-scale relation for scaling functions of order r, are in the form:

$$\phi_i(x) = \sum_{k=0}^{r} c_{i,j}\phi^j(2x) + \sum_{k=0}^{r} c_{i,r+j+1}\phi^j(2x-1). \tag{3.2}$$

The cofficients $\{c\}$ are assigned uniquely by substituting equatin (3.1) in to (3.2).

The two-scale relations for the rth order multiwavelets are in the form:

$$\psi^i(x) = \sum_{k=0}^{r} d_{i,j}\phi^j(2x) + \sum_{k=0}^{r} d_{i,r+j+1}\phi^j(2x-1).$$

In order to find $2(r+1)^2$ unknown cofficients $\{d_{i,j}\}$, we applying the followin $2r(r+1)$ vanish moment condition and $2r(r+1)$ orthonormal conditions.

1. Vanishing moments:

$$\int_0^1 \psi^i(x) x^j = 0, \quad \text{for } i = 0, 1, \ldots, r, j = 0, 1, \ldots, i+r.$$

2. Orthonormality

$$\int_0^1 \psi^i(x)\psi^j(x) = \delta_{i,j}, \quad \text{for } i, j = 0, 1, \ldots, r.$$

For example, a basis for space V_1^2 is given by

$$V_1^2 = \{\phi^0(t) = 1, \phi^1(t) = \sqrt{3}(2t-1), \psi^0(t), \psi^1(t)\},$$

where $\psi^0(t) = \sqrt{3}(-4t+1)$ whenever $0 < t < \frac{1}{2}$, $\psi^0(t) = \sqrt{3}(4t-3)$ whenever $\frac{1}{2} < t < 1$ and $\psi^1(t) = 6t-1$ whenever $0 < t < \frac{1}{2}$, $\psi^1(t) = 6t-5$ whenever $\frac{1}{2} < t < 1$.

In the following, we represent a theorem that show this approximation is convergence.

Theorem 3.1. *Suppose Q_m^k be the orthonormal projection of $L^2[0,1]$ on to V_m^k. If $f \in C^k[0,1]$, thus:*

$$\| Q_m^k f - f \| \leq 2^{-mk} \frac{2}{4^k k!} \sup_{x \in [0,1]} |f^{(k)}(x)|.$$

Any function $x(t)$ which is square integrable in the interval $[0,1]$ can be expanded in to the scaling function as [3, 25]:

$$x(t) \approx \sum_{k=0}^{2^J-1} \sum_{m=0}^{r} c_{J,k} \phi_{J,k}^m(t) = C^T \Phi_J(t)$$

and the corresponding wavelet functions as:

$$x(t) \approx \sum_{m=0}^{r} \{c_{0,0}^m \phi_{0,0}^m(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(t)\} = D^T \Psi_J(t), \tag{3.3}$$

where

$$c_{J,k}^m = \int_0^1 x(t) \phi_{J,k}^m(t) dt, \quad d_{j,k}^m = \int_0^1 x(t) \psi_{j,k}^m(t) dt$$

and C, D are $n \times 1$, $(n = (r+1)2^J)$ matrices, given by

$$\begin{aligned} C =&[c_{J,0}^0, \ldots, c_{J,0}^r, \ldots, c_{J,2^J-1}^0, \ldots, c_{J,2^J-1}^r]^T, \\ D =&[c_{0,0}^0, \ldots, c_{0,0}^r, d_{0,0}^0, \ldots, d_{0,0}^r, \ldots, d_{J-1,0}^0, \ldots, d_{J-1,0}^r, \ldots, d_{J-1,2^{J-1}-1}^0, \\ &\ldots, d_{J-1,2^{J-1}-1}^r]^T. \end{aligned}$$

If we want to approximate the $x(t)$ by (3.3), for simplicity, we write (3.3) as

$$x(t) \approx \sum_{k'=0}^{r} d_{k'} \psi_{k'}(t) = D^T \Psi(t).$$

Lemma 3.1. *Suppose $\alpha > 0$ be given. Then for $k = 0, 1, \ldots, [\alpha] - 1$ we have:*

$$^cD^\alpha(x^k) = 0$$

and for $k \geq [\alpha]$ we have

$$^cD^\alpha(x^k) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}.$$

Hence, for the Riemann-Liouville fractional integral, we get

$$I^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{k+\alpha}.$$

The approximations $x(t)$ is sought in the form of the truncated series:

$$x(t) \simeq \sum_{k'=0}^{r} d_{k'} \psi_{k'}(t), \tag{3.4}$$

where $\psi_{k'}(t)$ are Alpert's multi-wavelet basis.

By substitution of formule (3.4) in the problem (1.1) and by applying lemma (3.1), we obtain

$$\sum_{k'=0}^{r} d_{k'} {}^cD^\gamma(\psi_{k'}(t)) = f\left(t, \sum_{k'=0}^{r} d_{k'} \psi_{k'}(t), \sum_{k'=0}^{r} d_{k'} I^\alpha(\psi_i(t)), \sum_{k'=0}^{r} d_{k'} {}^cD^\delta(\psi_{k'}(t))\right).$$

Since, ${}^cD^{\gamma}(x^k) = 0$ for $k = 0, 1, \ldots, [\gamma] - 1$, we get

(3.5)
$$\sum_{k'=\lceil\gamma\rceil}^{r} d_{k'}{}^cD^{\gamma}(\psi_{k'}(t)) = f\left(t, \sum_{k'=0}^{r} d_{k'}\psi_{k'}(t), \sum_{k'=0}^{r} d_{k'}I^{\alpha}(\psi_{k'}(t)), \sum_{k'=\lceil\delta\rceil}^{r} d_{k'}{}^cD^{\delta}(\psi_{k'}(t))\right).$$

Now, we collocate equation (3.5) at $(r - \lceil\gamma\rceil)$ point t_p as

$$\sum_{k'=\lceil\gamma\rceil}^{r} d_{k'}{}^cD^{\gamma}(\psi_{k'}(t_p)) = f\left(t, \sum_{k'=0}^{r} d_{k'}\psi_{k'}(t_p), \sum_{k'=0}^{r} d_{k'}I^{\alpha}(\psi_{k'}(t_p)), \sum_{k'=\lceil\delta\rceil}^{r} d_{k'}{}^cD^{\delta}(\psi_{k'}(t_p))\right).$$

By using the boundary conditions, we obtain

$$\sum_{k'=0}^{r} d_i(\psi_{k'}(0) + \psi_{k'}(1)) = a$$

and

$$I^{\beta}\left(\sum_{k'=0}^{r} d_{k'}\psi_{k'}(\varepsilon)\right) + I^{\beta}\left(\sum_{k'=0}^{r} d_{k'}\psi_{k'}(\eta)\right) = b\int_0^1 \left(\sum_{k'=0}^{r} d_{k'}\psi_{k'}(s)\right)\, ds.$$

We can find $\lceil\gamma\rceil + 1$ equations. Consequently, we have $(r+1)$ nonlinear equation which can be solved, for unknown coefficients d_j by using the Newton's iterative method.

Here, we provide two computational examples.

Example 3.1. Consider the problem

$${}^cD^{\frac{3}{2}}x(t) = f(t) + 0.29633\left(x(t) + \int_0^t \int_0^{\vartheta_1} x(\vartheta)d\vartheta d\vartheta_1\right),$$

with boundary conditions $I^{(2)}x\left(\frac{1}{2}\right) + I^{(2)}x\left(\frac{1}{3}\right) = 0.01871113\int_0^1 x(s)ds$ and $x(0) + x(1) = 1$, where $f(t) = \frac{4}{\Gamma(\frac{1}{2})}\sqrt{t} - 0.29633\left(t^2 + \frac{t^4}{12}\right)$. Put $\gamma = \frac{3}{2}$, $\beta = 2$, $\delta = 0$, $a = 1$, $b = 0.01871113$, $\varepsilon = \frac{1}{2}$, $\eta = \frac{1}{3}$, $\varphi_1 = 0.161844425$ and $\varphi_2 = 0.017650607$. Consider the maps $\psi(t) = \dfrac{t}{2}$ for all $t \geq 0$ and $\xi(t, s) = t$ for all $t, s \in \mathbb{R}$. If $M_1 = M_2 = 2.72158$, then $M = 0.1837168$.

Let T be the defined operator in (2.2). One can check that $Tx(t) = 1.69765x(t) + 0.723468\left(1 - \sqrt{x(t)}\right)x(t)^{\frac{1}{4}}$ whenever $x \in C(I)$, with $x(t) \geq 0$ for all $t \in I$, $|f(t, x, p, u) - f(t, y, q, v)| \leq M\psi(|x - y| + |u - v|)$ for all $t > 0$ and $x, y \in \mathbb{R}$ with $\xi(x, y) = x \geq 0$ and $f \in C(I \times \mathbb{R}^2, \mathbb{R})$. Now by using Theorem 2.1, we conclude that the problem has a solution. In fact, $x(t) = t^2$ is a solution for the problem. Finally by choosing the collocation points from equally spaced of subdivisions, we obtain the Table 1.

Example 3.2. Consider the fractional integro-differential problem

$${}^cD^{\frac{5}{4}}x(t) = f(t) + 0.027(x(t) + I^{(3)}x(t)), \tag{3.6}$$

with boundary conditions $I^{(3)}x\left(\frac{1}{2}\right) + I^{(3)}x\left(\frac{1}{4}\right) = 0.0000130208\int_0^1 x(s)ds$ and $x(0) + x(1) = 1$, where $f(t) = \frac{6}{\Gamma(\frac{11}{4})}t^{\frac{7}{4}} - 0.027\left(t^3 + \frac{t^6}{120}\right)$. Now, put $\gamma = \frac{5}{4}$, $\beta = 3$, $a = 1$,

Table 1. $x(t) = t^2$

t_i	The coefficient value	Error with Alpert multiwavelet V_0^8
0	0.333333	$5.16164e - 06$
0.142	0.288676	$6.50827e - 06$
0.285	0.745356	$5.91668e - 07$
0.428	$-1.64297e - 09$	$4.00517e - 07$
0.571	$3.17629e - 06$	$1.59465e - 07$
0.714	$-2.42931e - 06$	$2.96244e - 06$
0.857	$-2.64283 - 06$	$1.69162e - 07$
1	$2.08127e - 06$	$9.52886e - 06$

$b = 0.000130208$, $\varepsilon = \frac{1}{2}$, $\eta = \frac{1}{4}$, $\varphi_1 = 0.02342448$ and $\varphi_2 = 0.00054704$. But, $f(t, x, I^\beta x) = \frac{6}{\Gamma(\frac{11}{4})} t^{\frac{7}{4}} - 0.027 \left(t^3 + \frac{t^6}{120} - x(t) - I^\beta x(t) \right)$. If $K = 0.027$, then $\Omega = 0.12278349 < 1$. Now by using Theorem 2.2, one can find that the problem (3.6) has a unique solution. In fact, the exact solution of the problem is $x(t) = t^3$. Finally by choosing the collocation points from equally spaced of subdivisions, we obtain the Table 2.

Table 2. $x(t) = t^3$

t_i	The coefficient value	Error with Alpert multiwavelet V_0^8
0	0.269146	$1.01052e - 06$
0.142	0.273465	$1.9791e - 07$
0.285	0.1064475	$2.65242e - 06$
0.428	0.0221049	$3.14649e - 06$
0.571	$1.93177e - 5$	$3.58952e - 07$
0.714	$1.06004e - 5$	$3.89971e - 07$
0.857	$3.97509e - 6$	$5.40425e - 07$
1	$6.17575e - 8$	$1.95951e - 06$

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