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ON SOME NEW SHARP EMBEDDING THEOREMS FOR NEW WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS IN PRODUCT DOMAINS

R. SHAMOYAN[1] AND O. MIHIĆ[2]

Abstract. We introduce mixed norm analytic spaces in polyball and provide some new sharp embedding theorems for them, extending previously known assertions in the unit disk.

1. Introduction

A complete characterization of positive Borel μ measures in the unit polydisk for which the differentiation operator maps anisotropic weighted space of holomorphic functions with mixed norm into the Lebesgue space $L^q(\mu)$ was obtained in [29]. Later in [30] these results were partially generalized to so called mixed norm spaces in polydisk. We need some definitions.

Let

$$U^n = \{z = (z_1, z_2, \dots, z_n) : |z_j| < 1,\ 1 \leq j \leq n\},$$

be the unit polydisk of n-dimensional complex space $\mathbb{C}^n$, T^n be the Shilov boundary of U^n, $\vec{p} = (p_1, \dots, p_n)$, $0 < p_j < +\infty$, $j = 1, \dots, n$, $\vec{w}(t) = (w_1(t), \dots, w_n(t))$, $t \in (0,1)$, where $w_j(t)$ are positive integrable functions on $(0,1)$. We denote by $A^{\vec{p}}(\vec{w})$ the set of all holomorphic functions in U^n for which

$$\|f\|_{A^{\vec{p}}(\vec{\omega})} = \left(\int_U \left[\cdots \left(\int_U |f(z_1, \dots, z_n)|^{p_1} \omega_1(1 - |z_1|) dm_2(z_1) \right)^{\frac{p_2}{p_1}} \cdots \right]^{\frac{p_n}{p_{n-1}}} \right.$$

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$$\times\, \omega_n(1-|z_n|)dm_2(z_n)\Bigg)^{\frac{1}{p_n}} < +\infty,$$

(see [37]), where m_2 is planar Lebesgue measure on $U := U^1$. Assume further $\vec{\mu} = (\mu_1, \dots, \mu_n)$, where μ_j is the Borel nonnegative finite measure on U, $L^{\vec{p}}(\vec{\mu})$ is related space with mixed norm (see [4], [5]) that is, the space of all measurable functions U^n for which

$$\|f\|_{L^{\vec{p}}(\vec{\mu})} = \left(\int_U \left[\cdots \left(\int_U |f(\zeta_1,\dots,\zeta_n)|^{p_1} d\mu_2(\zeta_1)\right)^{\frac{p_2}{p_1}} \dots\right]^{\frac{p_n}{p_{n-1}}} d\mu_n(\zeta_n)\right)^{\frac{1}{p_n}} < +\infty,$$

$0 < p_i \leq \infty,\ i = 1, \dots, n$, with usual modification for $p_i = \infty$. Such space in $\mathbb{R}^n$ studied by Nikolski and coauthors (see [23]).

In [30] the author obtains a complete characterization of the measure $\vec{\mu}$ for which the operator

$$\overline{D}^m f(z_1,\dots,z_n) = \frac{\partial^{|m|} f(z_1,\dots,z_n)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}, \quad z = (z_1,\dots,z_n) \in U^n,$$

maps $A^{\vec{p}}(\vec{\omega})$ into $L^{\vec{q}}(\vec{\mu})$, where $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n)$, $0 < p_j \leq q_j$, $j = 1, \dots, n$. In addition, they obtain a description of the measures ν on U^n for which the operator $\overline{D}^m$ maps $A^{\vec{p}}(\vec{\omega})$ into $L^q(\nu)$, where $0 < p_j \leq q$, $j = 1, \dots, n$. We will extend in this paper these results to the unit ball case. In connection with these results, we recall that at $n = 1$, $m = 0$ in Hardy classes $H^p(U)$ case corresponding description was obtained in the classical work of L. Carleson [6] and in the case of the Hardy space $H^p(B^k)$ in the ball was obtained by Hörmander in [16]. We also note the work of F. A. Shamoyan [29], where he studied the Hardy space $H^p(U^n)$, there supposed $\vec{m} = (m_1, \dots, m_n)$, $m_j \neq 0$, $j = 1, \dots, n$. The case of weighted Bergman spaces investigated in [25].

Various related assertions (sharp embedding theorems in analytic function spaces) can be seen in [9, 11, 12, 18, 19, 21, 26].

The theory of analytic spaces in unit ball is well-developed by various authors during last decades (see [18, 19, 21] and various references there).

One of the goals of this paper among other things is to define new mixed norm analytic spaces in polyballs and to establish some basic properties of these spaces. We believe this new interesting object can serve as a base for further generalizations and investigations in this active research area. This paper can be seen as direct continuation of our paper in polyball (see [22]). Spaces we mentioned above are closely connected also with so-called multifunctional analytic function spaces on unit ball. Various such connections in analytic and harmonic function spaces were found and mentioned in [3, 19, 34]. We note basic properties of last spaces on product domains are closely connected on the other hand with so-called Trace operator (see [3, 34]). In main part of paper we will turn to study of certain embedding theorems for some new mixed norm analytic classes in unit ball in $\mathbb{C}^n$. We note that in this paper we extend some mentioned theorems from [29] and [30] where they can be seen in context of polydisk and unit disk. Proving embedding theorems in unit ball we heavily use the technique which was developed recently in [1], [2]. Moreover, some results of this paper can be expanded to bounded pseudoconvex domains and even to tube domains based on results of $[1, 2, 35]$ and this will be a topic of our further work. In our embeddings theorems and inequalities for analytic function spaces in unit polyball the so-called Carleson-type measures constantly appear. We add some historical remarks on this important topic now. Carleson measures were introduced by Carleson (see [6]) in his solution of the corona problem in the unit disk of the complex plane, and, since then, have become an important tool in analysis, and an interesting object of study *per se*.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be deferent at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed. The notation $A \asymp B$ means that there is a positive constant C, such that $\frac{B}{C} \leq A \leq CB$.

Let A be a Banach space of holomorphic functions on a domain $D \subset \mathbb{C}^n$, given $p \geq 1$, a finite positive Borel measure μ on D is a Carleson measure of A (for p) if there is a continuous inclusion $A \hookrightarrow L^p(\mu)$, that is there exists a constant $C > 0$ such that

$$\int_U |f(z)|^p d\mu(z) \leq C\|f\|_A^p, \quad \text{for all } f \in A.$$

We shall furthermore say that μ is a vanishing Carleson measure of A if the inclusion $A \hookrightarrow L^p(\mu)$ is compact.

Carleson studied this property (see [6]) taking as Banach space A the Hardy spaces in unit disk Δ $H^p(\Delta)$, and proved that a finite positive Borel measure μ is a Carleson measure of $H^p(\Delta)$ for p if and only if there exists a constant $C > 0$ such that $\mu(S_{\theta_0,h}) \leq Ch$ for all sets

$$S_{\theta_0,h} = \left\{re^{i\theta} \in \Delta : 1-h \leq r < 1, |\theta - \theta_0| < h\right\},$$

(see also [12,25]). In particular the set of Carleson measures of $H^p(\Delta)$ does not depend on p.

In 1975, Hastings [15] (see also [24] and [25]) proved a similar characterization for the Carleson measures of the Bergman $A^p(\Delta)$, still expressed in terms of the sets $S_{\theta_0,h}$. Later Cima and Wogen (see [10]) characterized Carleson measures for Bergman spaces in the unit ball $B_n \subset \mathbb{C}^n$, and Cima and Mercer (see [21]) characterized Carleson measures of Bergman spaces in strongly pseudoconvex domains, showing in particular that the set of Carleson measures of $A^p(D)$ is independent of $p \geq 1$.

Cima and Mercer's characterization of Carleson measures of Bergman spaces is expressed using interesting generalizations of the sets $S_{\theta_0,h}$. Given $z_0 \in D$ and $0 < r < 1$, let $B_D(z_0, r)$ denote the ball of center z_0 (usual Kobayashi ball which is Bergman ball in the unit ball) and radius $\frac{1}{2}\log\frac{1+r}{1-r}$ for the Kobayashi distance k_D of D (that is, of radius r with respect to the pseudohyperbolic distance $p = \tanh(k_D)$.

Then it is possible to prove (see Luecking [20] for $D = \Delta$, Duren and Weir [13] and Kaptanoğlu [17] for $D = B_n$, and [1, 2] for D strongly pseudoconvex) that a finite positive measure μ is a Carleson measure of $A^p(D)$ for p if and only if (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) < C_r \nu(B_D(z_0, r)),$$

for all $z_0 \in D$. (The proof of this equivalence in [1] relied on Cima and Mercer's characterization [21]).

Thus we will have a new geometrical characterization of Carleson measures of Bergman spaces, and it turns out that this geometrical characterization is very important for the study of the various properties of Toeplitz operators. Given $\theta > 0$, we say that a finite positive Borel measure μ is a (geometric) θ-Carleson measure if for some (and hence all) $0 < r < 1$ there is a constant $c_r > 0$ such that

$$\mu(B_D(z_0, r)) < c_r \nu(B_D(z_0, r))^{\theta},$$

for all $z_0 \in D$. Note a 1-Carleson measures are usual Carleson measures of $A^p(D)$, and we know (see [1, 2]) that θ-Carleson measures are exactly the Carleson measures of suitably weighted Bergman spaces. Note also that when $D = B_n$ a q-Carleson measure in the sense of [17], [39] is a $\left(1 + \frac{q}{n+1}\right)$-Carleson measure in our sense.

In this paper we are however more interested in Carleson type measure for some new Bergman-type mixed norm spaces in product domains (polyballs $B_n \times \cdots \times B_n$).

2. Preliminaries

In this section we introduce notations and provide formulations of several lemmas needed for proofs, some short review of embedding theorems related with our results in the unit ball, will be also discussed.

Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. The boundary of B_n will be denoted by S^n, $S^n = \{z \in \mathbb{C}^n : |z| = 1\}$. Moreover, let $d\nu$ denote the Lebesgue measure on B_n normalized such that $\nu(B_n) = 1$, and let $d\mu$ denote the positive Borel measure. For any $\alpha \in \mathbb{R}$, let $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, for $z \in B_n$. Here, if $\alpha \leq 1$, $c_\alpha = 1$ and $\alpha > -1$, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$, is the normalizing constant so that ν_α has unit total mass. The Bergman metric on B_n is $\beta(z, w) = \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where $\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - <w,z>}$ is the Mobius transformation of B_n that interchanges 0 and z, where $s_z = (1 - |z|)^{\frac{1}{2}}$, P_z is the orthogonal projection into the space spanned by $z \in B_n$, i.e. $P_z w = \frac{<w,z> z}{|z|^2}$, $P_0 w = 0$ and $Q_z = I - P_z$ (see, for example, [39]). Let $D(a, r) = \{z \in B_n : \beta(z, a) < r\}$ denote the Bergman metric ball centered at $a \in B_n$ with radius $r > 0$.

As usual, we denote by $H(B_n)$ the class of all holomorphic functions on B_n. For $0 < p < \infty$ we define the Hardy space $H^p(B_n)$ consist of holomorphic functions f in

B_n such that

$$\|f\|_p^p = \sup_{0<r<1} \int_{S^n} |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

Here $d\sigma$ denotes the surface measure on S^n normalized so that $\sigma(S^n) = 1$.

For every function $f \in H(B_n)$ having a series expansion $f(z) = \sum_{|k|\geq 0} a_k z^k$, we denote the operator of fractional differentiation by

$$\widetilde{D}^\alpha f(z) = \sum_{|k|\geq 0} (|k|+1)^\alpha a_k z^k,$$

where α is any real number.

For a fixed $\alpha > 1$ let $\Gamma_\alpha(\zeta) = \{z \in B_n : |1 - \overline{\zeta} z| < \alpha(1-|z|)\}$ be the admissible approach region vertex is at $\zeta \in S^n$, (so called Lusin cone).

The well known Littlewood-Paley inequality in the unit ball of $\mathbb{C}^n$ for functions $f \in H^p(B_n)$ is the following.

Theorem A. If $2 \leq p < \infty$, $\beta > 0$, then for any $f \in H^p(B_n)$

$$\int_{B_n} |\widetilde{D}^\beta f(z)|^p (1-|z|)^{\beta p - 1} d\nu(z) \leq C \|f\|^p_{H^p(B_n)}. \tag{2.1}$$

Note it is well known

$$\|f\|_{H^p(B_n)} \asymp \int_{S^n} \left(\int_{\Gamma_\delta(\zeta)} |\widetilde{D}^k f(z)|^2 (1-|z|)^{2k-2} d\nu(z) \right)^{\frac{p}{2}} d\sigma(\zeta) \tag{2.2}$$

and

$$\int_{B_n} |f(z)|^p d\mu(z) \asymp \int_{S^n} \left(\int_{\Gamma_\delta(\zeta)} \frac{|f(z)|^p}{(1-|z|)^n} d\mu(z) \right) d\sigma(\zeta), \tag{2.3}$$

for $0 < p < \infty$, where μ is a positive Borel measure on B_n.

Looking at estimates (2.1)-(2.3) it is natural to pose a general problem (see [7–9,11]).

Describe all positive Borel measures μ in the unit ball such that

(2.4)

$$\left(\int_{S^n} \left(\int_{\Gamma_\delta(\zeta)} \frac{|D^\beta f(z)|^p}{(1-|z|)^n} d\mu(z) \right)^{\frac{q}{p}} d\sigma(\zeta) \right)^{\frac{1}{q}} \leq C \left(\int_X \left(\int_{\mathcal{G}} |f(z)|^{q_1} d\nu_\alpha(z) \right)^{\frac{s}{q_1}} dm \right)^{\frac{1}{s}},$$

where $\alpha > -1$, $\beta \geq 0$, $0 < p, q, q_1, s < \infty$, where $\mathcal{G}$ is a subset of S^n or B_n, i.e., $\mathcal{G} = \mathcal{G}(\zeta)$, $\zeta \in S^n$ or $\mathcal{G} = \mathcal{G}(z)$, $z \in B_n$, $X = S^n$ or $X = B_n$ and dm is adequate measure.

For example $\mathcal{G} = Q_r(\zeta) = \{z \in B_n : d(z,\zeta) < r\}$, where d is a non-isotropic metric on S^n, $d(z,w) = |1- <z,w> |^{\frac{1}{2}}$, or $\mathcal{G} = D(z,r)$.

For $z \in B_n$ and $r > 0$ set $D(z,r)$ is called the Bergman metric ball at z, and for $\zeta \in S_n$ and $r > 0$ set $Q_r(\zeta)$ is called the Carleson tube at ζ (see [39]).

We are interested in this paper to similar type sharp embeddings in analytic function spaces but on product domains so called polydomains. Note the simplest case here is the unit polydisk and in this case some sharp embedding theorems are well known in literature (see, for example, [31] and various references there also). We mention that in harmonic function spaces such type results on polydomains and related multifunctional spaces obtained recently in [3].

The proofs of the following properties the Bergman balls can be found in [39] (see Lemmas 1.24, 2.20, 2.24 and 2.27 in [39]). We need them for proofs for all our main results.

Lemma 2.1. (a) *There exists a positive number $N > 1$ such that, for any $0 < r \le 1$, we can find a sequence $\{v_k\}_{k=1}^{\infty}$ in B_n, to be r-lattice in the Bergman metric of B_n. This means that $B_n = \bigcup_{k=1}^{\infty} D(v_k, r)$, $D\left(v_l, \frac{r}{4}\right) \cap D\left(v_k, \frac{r}{4}\right) = \emptyset$ if $k \neq l$ and each $z \in B_n$ belongs to at most N of the sets $D(v_k, 2r)$*

(b) *For any $r > 0$, there is a constant $C > 0$ so that $\frac{1}{C} \le \left|\frac{1-<z,w>}{1-<z,v>}\right| \le C$, for all $z \in B_n$ and all w, v with $\beta(w, v) < r$.*

(c) *For any $\alpha > -1$ and $r > 0$, $\int_{D(z,r)} (1 - |w|^2)^{\alpha} d\nu(w)$ is comparable with $(1 - |z|^2)^{n+1+\alpha}$ for all $z \in B_n$.*

(d) *Suppose $r > 0$, $p > 0$ and $\alpha > -1$. Then there is a constant $C > 0$ such that $|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p d\nu(w)$, for all $f \in H(B_n)$ and $z \in B_n$.*

For $\alpha > -1$ and $p > 0$, the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(B_n, d\nu_\alpha)$, that is, $A^p_\alpha = L^p(B_n, d\nu_\alpha) \cap H(B_n)$. It is well-known that A^p_α is a closed subspace of $L^p(B_n, d\nu_\alpha)$, (see [39], Chapter 2).

See [31] and [39] for more details of weighted Bergman spaces. Various sharp embedding theorems in the unit ball and their numerous applications were given by many authors in recent years (see, for example, [7–9, 39]). The main purpose of this paper is to provide new estimates and sharp embedding theorems of mentioned type for the unit polyball. Let us finally note that the study of similar to (2.4) embeddings in particular cases in the unit disk started recently in papers of W. Cohn [11] and Z. Wu [38]. We will also study general embeddings like (2.4), but in the polyball and with some restrictions on parameters.

Lemma 2.2 (see [39]). *For each $r > 0$ there exists a positive constant C_r such that*

$$C_r^{-1} \le \frac{1 - |a|^2}{1 - |z|^2} \le C_r, \quad C_r^{-1} \le \frac{1 - |a|^2}{|1- < z, a >} | \le C_r,$$

for all a and z such that $\beta(a, z) < r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Obviously using properties of $\{D(a_k, R)\}$ Bergman balls we will have the following estimates for Bergman space $A^p_\alpha(B_n)$

$$\|f\|^p_{A^p_\alpha} = \int_{B_n} |f(w)|^p \delta^{\alpha}(z) d\nu(w) \asymp \sum_{k=1}^{\infty} \left[\max_{z \in D(a_k,R)} |f(z)|^p\right] \nu_\alpha D(a_k, R)$$

$$\asymp \sum_{k=1}^{\infty} \int_{D(a_k,R)} |f(z)|^p \delta^{\alpha}(z) d\nu(z), \quad 0 < p < \infty,\ \alpha > -1,$$

where $(1-|z|)^\alpha = \delta^\alpha(z),\ \alpha > -1$.

Let now

$$A(p,q,\alpha) = \left\{ f \in H(B_n) : \sum_{k=1}^{\infty} \left(\int_{D(a_k,R)} |f(z)|^p \delta^\alpha(z) d\nu(z) \right)^{\frac{q}{p}} < \infty \right\},$$

where $0 < p, q < \infty$, $\alpha > -1$. These are Banach spaces if $\min\{p,q\} \geq 1$, obviously direct extensions of $A^p_\alpha(B_n)$ spaces in the ball. They were studied in [18, 19, 34].

We will however consider other natural extensions of $A^p_\alpha(B_n)$ to the case of m product domains $B_n \times \cdots \times B_n$ (polyballs).

We consider in this paper analytic spaces on product of balls $B_n^m = B_n \times \cdots \times B_n$. We denote by $H(B_n \times \cdots \times B_n)$ the space of analytic functions (by each variable) on $B_n^m,\ m \in \mathbb{N}$. For $n = 1$ case we have classical case of the unit polydisk (see [31]).

3. Formulations of the Main Results

The main goal of this section is to formulate main results of this paper, which were proved earlier in less general case of unit disk in [30].

Let S be the set of all measurable and positive functions of $L^1(0,1)$ for which there exists numbers M_ω, m_ω, q_ω with $m_\omega, q_\omega \in (0,1]$, that is

$$m_\omega \leq \frac{\omega(\lambda r)}{\omega(r)} \leq M_\omega, \quad r \in (0,1], \lambda \in [q_\omega, 1]. \tag{3.1}$$

Defined on $(0,+\infty)$ functions of this type were studied in detail in [27].

In this section we extend main results of [30]. The main idea is to replace r-lattices of the unit disk heavily used in [30], by r-lattices in the unit ball (see previous section) keeping main steps in old proof of less general case.

Let

$$L^{\vec{p}}(\vec{w},\vec{\nu}) = \Bigg\{ f \in L^1_{loc}(B^n_k) : \left(\int_B \cdots \left(\int_B |f(z_1,\ldots,z_n)|^{p_1} \left(w_1(1-|z_1|) d\nu_1(z_1) \right)^{p_2/p_1} \right) \right.$$
$$\left. \ldots (w_n(1-|z_n|) d\nu_n(z_n))^{1/p_n} < \infty \right\}$$

$0 < p_i < \infty$, $i = 1,\ldots,n$, ν_j, $j = 1,\ldots,n$, be the normalized Lebegues measures on B_k, $A^{\vec{p}}(\vec{w}) = L^{\vec{p}}(\vec{w}) \cap H(B^n_k)$. Replacing $w_j d\nu_j$ by $d\mu_j$ we define similarly the new general space $L^{p_1,\ldots,p_n}(\mu_1,\ldots,\mu_n)$.

Let

$$\widetilde{\mathfrak{D}}^{\vec{\alpha}} f(\vec{z}) = \sum_{|\vec{k}|\geq 0} \prod_{j=1}^{m} (|k_j|+1)^{\alpha_j} a_{k_1\cdots k_m} z_1^{k_1} \cdots z_m^{k_m},$$

where $\sum_{|\vec{k}|\geq 0}$ means $\sum_{k_1\geq 0} \cdots \sum_{k_m \geq 0}$.

We extend in a natural way (as in polydisk case) the definition of differential operator $\widetilde{D}^m$ to differential operators acting on analytic functions defined on product domains for all real α_j, $j = 1,\ldots,m$.

Theorem 3.1. *Let $\{a_k\}$ be r-lattice of B_k. Let $\vec{\omega} = (\omega_1, \dots, \omega_n)$, $\vec{\mu} = (\mu_1, \dots, \mu_n)$, $\omega_j \in S$, $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n) \in \mathbb{R}_+^n$ with $0 < p_j \le q_j$, $j = 1, \dots, n$. Then the following assertions are equivalent:*

1) $\|\tilde{D}^m f\|_{L^{\vec{q}}(\vec{\mu})} \le C(\vec{\mu}) \|f\|_{A^{\vec{p}}(\vec{\omega})}$;

2) $\mu_j(D(a_k, r)) \le c(1 - |a_k|)^{(n+1)\frac{q_j}{p_j} + m_j q_j} [\omega_j(1 - |a_k|)]^{\frac{q_j}{p_j}}$, $j = 1, \dots, n$, $k = 0, 1, 2, \dots$.

Remark 3.1. In the case of the unit disk Theorem 3.1 can be seen in [30].

In the case of measures ν defined on $B_k^n = B_k \times \cdots \times B_k$ there is following result.

Theorem 3.2. *Let $p_j \le q < +\infty$, $\tilde{\nu}$ be the Borel nonnegative measure on B_k^n, $\vec{\omega} = (\omega_1, \dots, \omega_n)$, $\omega_j \in S$, $j = 1, \dots, n$, $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$. Then the following assertion are equivalent:*

1) $\left(\int_{B_k^n} \left| \widetilde{D}^m f(z) \right|^q d\tilde{\nu}(z) \right)^{\frac{1}{q}} \le C \|f\|_{A^{\vec{p}}(\vec{\omega})}$;

2) $\tilde{\nu}(D(a_{k_1}, r) \times \cdots \times D(a_{k_n}, r)) \le c \prod_{j=1}^n (1 - |a_{k_j}|)^{(n+1)\frac{q}{p_j} + m_j q} [\omega_j(1 - |a_{k_j}|)]^{\frac{q}{p_j}}$.

Remark 3.2. In the case of the unit disk Theorem 3.2 can be seen in [30].

To prove Theorems 3.1 and 3.2 we need some auxiliary results for the proof. All preliminaries below needed for the proof are classical assertions for unit disk, polydisk for particular values of parameters. Moreover even the general case (general version of these assertions) of arbitrary p_j, $j = 1, \dots, n$, can be seen in the case of the unit disk in [30] and [37]. We provide same type results in the unit ball based on properties of r lattices in the unit ball (see [39]). Proofs are similar and will be omitted (we readers refer to [30] and [37]). Some lemmas are valid even in context of bounded strongly pseudoconvex domains with smooth boundary under some condition on Bergman kernel.

These lemmas are interesting also as separate assertions on these new mixed norm spaces in polyballs we defined and study in this paper.

Lemma 3.1. *Let $f \in A^{\vec{p}}(\vec{\omega})$, $0 < p_j < +\infty$, $\omega_j \in S$, $j = 1, \dots, n$. Then the following estimate holds*

$$\sum_{k_n=0}^{+\infty} \left(\max_{\zeta_n \in D(a_{k_n}, r)} \left(\cdots \sum_{k_2=0}^{+\infty} \left(\max_{\zeta_2 \in D(a_{k_2}, r)} \left(\sum_{k_1=0}^{+\infty} \left(\max_{\zeta_1 \in D(a_{k_1}, r)} |f(\zeta_1, \dots, \zeta_n)|^{p_1} \right.\right.\right.\right.\right.\right.$$
$$\times \omega_1 \left(|D(a_{k_1}, r)|^{\frac{1}{n+1}} \right) |D(a_{k_1}, r)| \Bigg)^{\frac{p_2}{p_1}} \omega_2 \left(|D(a_{k_2}, r)|^{\frac{1}{n+1}} \right) |D(a_{k_2}, r)| \Bigg)^{\frac{p_3}{p_2}} \cdots \Bigg)^{\frac{p_n}{p_{n-1}}}$$
$$\times |D(a_{k_n}, r)| \Big) \le C(\vec{\omega}, \vec{p}) \|f\|_{A^{\vec{p}}(\vec{\omega})},$$

where $|D(a_{k_j}, r)|$ Lebesgues measure of $D(a_{k_j}, r)$, $j = 1, \dots, n$.

Remark 3.3. Lemma 3.1 is valid also in more general situation when our function is subharmonic by each variable (n-subharmonic) in polyball.

Note, this Lemma is valid when f is n-subharmonic in product domains. Also, this Lemma 3.1 for $p_j = p$, $j = 1, \dots, n$, is valid even in pseudoconvex domains (see [2] for $p_j = p$, $j = 1, \dots, n$ and for spaces with standard weights). The general case for $\vec{p} = (p_1, \dots, p_m)$ can be seen in [30,36,37] in polydisk and with same proof in polyball.

The following lemma is known for $n = 1$ in disk and polydisk (see [30,31]).

Lemma 3.2. *In the context of the previous lemma we have the estimate*

$$|f(z_1, \dots, z_n)| \le C \frac{\|f\|_{A^{\vec{p}}(\vec{\omega})}}{\prod_{j=1}^n (1 - |z_j|)^{\frac{n+1}{p_j}} \omega_j^{\frac{1}{p_j}}(1 - |z_j|)}, \quad z = (z_1, \dots, z_n) \in B_k^n.$$

The proof of Lemma 3.2 is based on application of one functional result and use induction by variables.

This lemma is well known for particular values of parameters in case of the unit polydisk (see [30,31,37] and various references there).

In the future, we need an integral representation of the class $A^{\vec{p}}(\vec{\omega})$.

We first add some facts on ω function from S class (see, for example, [27,28,37]). We note that if $\omega_j \in S$ then ω_j admits the representation $\omega_j(t) = \exp\left(\eta_j(t) + \int_t^1 \frac{\varepsilon_j(u)}{u} du\right)$, $t \in (0,1)$, where η_j, ε_j is bounded measurable functions on $(0,1)$, while

$$\frac{\ln m_{\omega_j}}{\ln \frac{1}{q_{\omega_j}}} \le \varepsilon_j(u) \le \frac{\ln M_{\omega_j}}{\ln \frac{1}{q_{\omega_j}}},$$

where m_{ω_j}, M_{ω_j}, q_{ω_j} is the number corresponding to the functions ω_j in the estimates (3.1) (see [27,28,37]). Assuming that $\alpha_{\omega_j} = \frac{\ln m_{\omega_j}}{\ln \frac{1}{q_{\omega_j}}}$, $\beta_{\omega_j} = \frac{\ln M_{\omega_j}}{\ln \frac{1}{q_{\omega_j}}}$, where $\alpha_{\omega_j} > -1$, $0 < \beta_{\omega_j} < 1$, thus, without limiting the generality, also we assume that $\eta(x) \equiv 0$, $x \in (0,1)$. Also, $m_{\omega_j} < \frac{\omega_j(\lambda r)}{\omega_j(r)} < M_{\omega_j}$, $r \in (0,1)$, $\lambda \in (q_{\omega_j}, 1]$, $q_{\omega_j}, m_{\omega_j} \in (0,1)$, $M_{\omega_j} > 0$ and if $\omega_j \in S$, then $\omega_j(t) \in [t^{-\alpha_{\omega_j}}, t^{-\beta_{\omega_j}}]$, where $t \in (0,1)$.

We introduce the kernel $D_\alpha(\zeta, z)$, one-dimensional analogues of which were introduced by M. M. Dzhrbashyan in the work [14]:

$$D_\alpha(\zeta, z) = \prod_{j=1}^n \frac{\alpha_j + 1}{\pi} \frac{(1 - |\zeta|^2)^\alpha}{\left(1 - \bar{\zeta}_j z_j\right)^{\alpha + n + 1}},$$

$\zeta = (\zeta_1, \dots, \zeta_n)$, $z = (z_1, \dots, z_n) \in B_k^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$.

Lemmas 3.3, 3.4, 3.5 are known in polydisk (see [28,30,37]). Proofs are similar for polyballs.

Lemma 3.3. *Let*

$$f \in A^{\vec{p}}(\vec{\omega}),\ \vec{\omega} = (\omega_1, \dots, \omega_n),\ \alpha_j > \frac{\alpha_{\omega_j} + n + 1}{p_j} - 1, \quad j = 1, \dots, n. \tag{3.2}$$

Then the following representation holds

$$f(z)=\int_{B_k^n} D_{\overrightarrow{\alpha}}(\vec{\zeta},z)f(\zeta)d\nu(\xi_1)\cdots d\nu(\xi_n),\quad z\in B_k^n. \tag{3.3}$$

Sketch of proof. Using Lemma 3.2, we obtain

$$|f(z_1,\dots,z_n)|\le C\frac{\|f\|_{A^{\vec{p}}(\vec{\omega})}}{\prod_{j=1}^n (1-|z_j|)^{\frac{n+1}{p_j}}\,\omega_j^{\frac{1}{p_j}}(1-|z_j|)},\quad z=(z_1,\dots,z_n)\in B_k^n.$$

We obtain that the space $A^{\vec{p}}(\vec{\omega})$ is embedded in $A^1(\overrightarrow{\alpha})$, where $A^1(\overrightarrow{\alpha})$ coincides with the class $A^{\vec{p}}(\vec{\omega})$ at $\omega_j(t)=t^{\alpha_j}$, $j=1,\dots,n$, $\vec{p}=(p_1,\dots,p_n)$.

Using the result in ball by each variable, we obtain that f admits a representation (3.3). The lemma is proved. □

The following lemma is proved in [37] in the unit polydisk. The general case use same ideas based on same estimates related with r-lattices but in the unit ball case.

Lemma 3.4. *Let $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $\alpha_j>\frac{\alpha_{\omega_j}+n+1}{p_j}-1$, $\vec{\omega}=(\omega_1,\dots,\omega_n)$, $\omega_j\in S$, $j=1,\dots,n$. Then the operator*

$$T_\alpha(f)(z)=\int_{B_k^n}\left|D_{\overrightarrow{\alpha}}(\vec{\zeta},z)\right||f(\zeta)|\,d\nu(\xi_1)\dots d\nu(\xi_n),\quad z\in B_k^n, \tag{3.4}$$

maps the space $A^{\vec{p}}(\vec{\omega})$ into $L^{\vec{p}}(\vec{\omega})$, where $L^{\vec{p}}(\vec{\omega})$ means the class $L^{\vec{p}}(\vec{\mu})$ with $d\mu_j=\omega_j(1-|\zeta_j|)d\nu(\zeta_j)$, $\zeta_j\in B_k$, $j=1,\dots,n$.

Lemma 3.5. *Let $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $\vec{\omega}=(\omega_1,\dots,\omega_n)$, $\omega_j\in S$, $j=1,\dots,n$, $m=(m_1,\dots,m_n)\in\mathbb{Z}_+^n$. Then the following estimate holds*

$$\begin{aligned}&\Bigg(\int_{B_k}\omega_n(1-|\zeta_n|)(1-|\zeta_n|)^{m_np_n}\Bigg[\cdots\Bigg(\int_{B_k}\omega_n(1-|\zeta_2|)(1-|\zeta_2|)^{m_2p_2}\\&\times\left[\int_{B_k}|\widetilde{D}^m f(\zeta_1,\dots,\zeta_n)|^{p_1}\omega_1(1-|\zeta_1|)(1-|\zeta_1|)^{m_1p_1}d\nu(\zeta_1)\right]^{\frac{p_2}{p_1}}\\&\times d\nu(\zeta_2)\Bigg)^{\frac{p_3}{p_2}}\cdots\Bigg]^{\frac{p_n}{p_{n-1}}}d\nu(\zeta_n)\Bigg)^{\frac{1}{p_n}}\le C(m,\vec{\omega},\vec{p})\|f\|_{A^{\vec{p}}(\vec{\omega})}.\end{aligned}$$

Sketch of proof. We prove the lemma for $n=2$, since $n>2$ there are similar arguments. Let $\alpha_j>\frac{\alpha_{\omega_j}+n+1}{p_j}-1$, $j=1,\dots,n+1$, then by Lemma 3.3 f admits the integral representation

$$f(z_1,z_2)=C(\alpha)\int_{B_k^2}\frac{(1-|\zeta_1|^2)^{\alpha_1}(1-|\zeta_2|^2)^{\alpha_2}}{(1-\overline{\zeta}_1z_1)^{\alpha_1+n+1}(1-\overline{\zeta}_2z_2)^{\alpha_2+n+1}}f(\zeta_1,\zeta_2)d\nu(\zeta_1)d\nu(\zeta_2).$$

Consequently,

$$\begin{aligned}&\left|\widetilde{D}^{m_1,m_2}f(z_1,z_2)\right|\\&\le C(\alpha)\int_{B_k^2}\frac{(1-|\zeta_1|^2)^{\alpha_1}(1-|\zeta_2|^2)^{\alpha_2}}{\left|(1-\overline{\zeta}_1z_1)^{\alpha_1+m_1+n+1}\right|\left|(1-\overline{\zeta}_2z_2)^{\alpha_2+m_2+n+1}\right|}|f(\zeta_1,\zeta_2)|\,d\nu(\zeta_1)d\nu(\zeta_2).\end{aligned}$$

Therefore,

$$\begin{aligned}&(1-|z_1|^2)^{m_1}(1-|z_2|^2)^{m_2}\left|\widetilde{D}^m f(z_1,z_2)\right|\\ \le& C(\alpha,m)\int_{B_k^2}\frac{(1-|\zeta_1|^2)^{\alpha_1}(1-|\zeta_2|^2)^{\alpha_2}}{\left|(1-\overline{\zeta}_1 z_1)^{\alpha_1+n+1}\right|\left|(1-\overline{\zeta}_2 z_2)^{\alpha_2+n+1}\right|}\,|f(\zeta_1,\zeta_2)|\,d\nu(\zeta_1)d\nu(\zeta_2),\end{aligned}$$

where $z_1, z_2 \in B_k^2$. Using Lemma 3.4, we obtain assertion of Lemma 3.5. □

The following lemma is also important for this paper. We provide the simplest model of the unit disk case (the case of more general domains can be covered similarly based on basic results on subharmonic functions in general bounded domains in higher dimension). We remind the reader we denote by D_1 the unit disk in $\mathbb{C}$ and by D_1^2 the bidisk (a product of two disks). Note even a little bit more general version of this lemma with the same proof is valid for $|F|^p \cdot |G|^q$, where p and q are positive and where both functions are analytic in bidisk.

We denote by dm_2 Lebegues measure on D_1.

Lemma 3.6. *Let $F \in H\left(D_1^2\right)$, $F = f_1 \cdot f_2$. Then*

$$\psi_\beta^\alpha(z_2) = \left(\int_{D_1} |F(z_1,z_2)|^{p_1}\cdot(1-|z_1|)^\alpha\, dm_2(z_1)\right)^\beta,$$

$\beta \ge 0$, $\alpha > -1$, $z_2 \in D_1$ is subharmonic function in D_1, where p_1 is an arbitrary positive number.

Sketch of the proof. For the proof of Lemma (3.6) we will use basic facts on subharmonic function spaces. Let $D_r = \{z_1 : |z_1| < r\}$, $\delta > 0$. We show first

$$\psi_r(z_2) = \lg\int_{D_r}\left(|F(\zeta_1,z_2)|+\delta\right)^\alpha dm_2(\zeta_1)$$

is subharmonic for all $\alpha \ge 0$, where lg is a logarithm of function. Then we have by known properties of subharmonic functions that the following function $\psi(z_2) = \lim_{r\to 1-0}\psi_r(z_2)$ is also subharmonic.

To show this we note that if $D_r = \bigcup_{k=1}^{m_n} \triangle_k$ where $\triangle_k$ is any decomposition of D_r circle such that $\operatorname{diam}(\triangle_k) \le \frac{1}{n}$,

$$u_n(z_2) = \lg\left\{\sum_{k=1}^{n}(|F(\zeta_k,z_2)|+\delta)^\alpha|\Delta_k|\right\},$$

$|\triangle_k|$ is a Lebesgue measure of $\triangle_k$, then $u_n(z_2)$ is uniformly tending to $\psi_r(z_2)$ on D_r. So now to show the subharmonicity of $\psi_r(z_2)$, we show the subharmonicity of $u_n(z_2)$.

We have

$$\begin{aligned}\lg(|F(\zeta_k,z_2)|+\delta)^\alpha|\Delta_k| =& \lg(|F(\zeta_k,z_k)|+\delta)^\alpha + \lg|\Delta_k|\\ =&\alpha\lg(|F(\zeta_k,z_k)|+\delta)+\lg|\Delta_k|.\end{aligned}$$

Since both function $\lg(|F(\zeta_k,z_2)|)$ and $\lg\delta$ are subharmonic

$$\alpha\lg(|F(\zeta_k,z_k)|+\delta)+\lg|\Delta_k|, \quad \alpha \ge 0,$$

is also subharmonic. Hence $\lg(|F(\zeta_k, z_2)| + \delta)^\alpha |\Delta_k|$ is subharmonic.

Hence $u_n(z_2)$ is also subharmonic by known properties of subharmonic functions $\beta\psi_r(z_2)$, $\quad z_2 \in D_1$, hence and $\psi_\beta(z_2) = \lim_{r\to 1-0} \psi_r(z_2)$ are subharmonic and the limit is attained uniformly in D_1. Hence we have that

$$V_{\alpha,\beta,\delta}(z_2) = \exp(\psi_\beta(z_2)) = \left(\int_D (|F(z_1, z_2)| + \delta)^\alpha dm_2(z_1))\right)^\beta, \quad z_2 \in D_1,$$

is also subharmonic.

Obviously $V_{\alpha,\beta,\delta} \searrow V_{\alpha,\beta,0}(z_2)$, when $\delta \to 0$, $\mathbf{V}_{\alpha,\beta,\delta_1} \geq V_{\alpha,\beta,\delta_2}$, $\delta_1 < \delta_2$ ($V_{\alpha,\beta,\delta}$ is decreasing to $V_{\alpha,\beta,0}$). Hence using again known properties of subharmonic functions

$$\psi_{\alpha,\beta}(z_2) = \left(\int_{D_1} |F(z_1, z_2)|^\alpha dm_2(z_1)\right)^\beta$$

is subharmonic. So we proved this lemma. $\square$

Remark 3.4. Note it is enough to only assume in our proof that $|F(z)|$ is subharmonic and our assertion is still valid.

4. Proofs of Main Theorems

In this section we provide proofs of our main results.

Proof of Theorem 3.1. The biball case ($n = 2$) is typical for the proof of the general case and we will restrict ourselves for this less general case. First we note that the implication 2) $\Rightarrow$ 1) can be checked by the standard way using standard test function and properties of r-lattices of ball (see [30, 39] for similar arguments in embedding theorem and Lemma (2.1)).

$$e_z(\zeta) = \prod_{j=1}^n \frac{1}{(1 - \zeta_j z_j)^{\beta_j+1}}, \quad z = (z_1, \dots, z_n),\ \zeta = (\zeta_1, \dots, \zeta_n) \in B_k^n,$$

for sufficiently large β_j, (see [4, 18, 19, 30]).

Therefore, we turn to the proof of the implication 1) $\Rightarrow$ 2). Presenting the unit ball as the union of dyadic Bergman balls (see Lemma 2.1), we obtain

$$\begin{aligned} I(f, z_2) &= \left(\int_{B_k} \left|\widetilde{D}^m f(z_1, z_2)\right|^{q_1} d\mu_1(z_1)\right)^{\frac{1}{q_1}} \\ &\leq C \left(\sum_{k_1=0}^\infty \max_{z_1 \in D(a_k, r)} \left\{\left|\widetilde{D}^m f(z_1, z_2)\right|^{q_1} (1 - |z_1|)^{m_1 q_1 + \frac{(n+1)q_1}{p_1}} w_1^{\frac{q_1}{p_1}}(1 - |z_1|)\right\}\right)^{\frac{1}{q_1}}. \end{aligned}$$

Taking into account that $\frac{p_1}{q_1} \leq 1$, we have

$$I(f, z_2) \leq C \left(\sum_{k_1=0}^\infty \max_{z_1 \in D(a_{k_1}, r)} \left\{\left|\widetilde{D}^m f(z_1, z_2)\right|^{p_1} (1 - |z_1|)^{m_1 p_1 + n + 1} w_1(1 - |z_1|)\right\}\right)^{\frac{1}{p_1}}.$$

Now, applying Lemma (3.1) to the function $D^m f(z_1, z_2)$ for fixed $z_2 \in B_k$, we obtain the estimate

$$I(f, z_2) \le C \left(\int_{B_k} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1 - |z_1|)(1 - |z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{1}{p_1}}.$$

Raising to the power q_2 both sides of the last inequality and integrating over μ_2, we obtain

$$\begin{aligned}
&\left(\int_{B_k} (I(f, z_2)^{q_2} d\mu_2(z_2)) \right)^{\frac{1}{q_2}} \\
\le& C \left(\int_{B_k} \left(\int_{B_k} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1 - |z_1|)(1 - |z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{q_2}{p_1}} d\mu_2(z_2) \right)^{\frac{1}{q_2}}.
\end{aligned}$$

Again using partition of the ball at dyadic balls and take into account (3.4):

$$\begin{aligned}
&\left(\int_{B_k} (I(f, z_2)^{q_2} d\mu_2(z_2)) \right)^{\frac{1}{q_2}} \\
\le& C_1 \left(\sum_{k_2=0}^{\infty} \max_{z_2 \in D(a_{k_2}, r)} \left(\int_{B_k} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1 - |z_1|)(1 - |z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{q_2}{p_1}} \right. \\
&\left. \times \mu_2(D(a_{k_2}, r)) \right)^{\frac{1}{q_2}} \\
\le& C_2 \left(\sum_{k_2=0}^{\infty} \max_{z_2 \in D(a_{k_2}, r)} \left(\int_{B_k} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1 - |z_1|)(1 - |z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{q_2}{p_1}} \right. \\
&\left. \times w_2^{\frac{q_2}{p_2}}(1 - |z_2|)(1 - |z_1|)^{m_2 q_2 + (n+1)\frac{q_2}{p_2}} \right)^{\frac{1}{q_2}}.
\end{aligned}$$

Considering equality $\frac{1}{q_2} = \frac{p_2}{q_2} \frac{1}{p_2}$ and $\frac{p_2}{q_2} \le 1$, we have the estimate

$$\begin{aligned}
&\left(\int_{B_k} (I(f, z_2)^{q_2} d\mu_2(z_2)) \right)^{\frac{1}{q_2}} \\
\le& C_2 \left(\sum_{k_2=0}^{\infty} \max_{z_2 \in D(a_{k_2}, r)} \left(\left(\int_{B_k} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1 - |z_1|)(1 - |z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{p_2}{p_1}} \right. \right. \\
&\left. \left. \times w_2(1 - |z_2|)(1 - |z_1|)^{m_2 q_2 + n + 1} \right) \right)^{\frac{1}{q_2}}.
\end{aligned}$$

To prove the theorem it remains to apply Lemma 3.1 and Lemma 3.5.

Let further as above $\nu_\alpha(z) = (1 - |z|)^\alpha d\nu(z)$, $\alpha > -1$.

Note now the amount of variables is not important at all when we talk about the necessity of the condition on measure, namely the proof of the general case in this part is the same as in $m = 2$ case which we provided above. We simply must repeat same arguments concerning the other implication we apply induction. To prove this it is enough to consider only a particular case of spaces with zero smoothness (spaces without fractional $\mathcal{D}^m$ derivatives and spaces with ordinary $(1-|z|)^\alpha$ weights).

Note for $p_3 \le q_3$ and for any positive subharmonic $\widetilde{\phi}$ function we have

$$\left(\int_{B_k} |\widetilde{\phi}(z_3)|^{q_3} d\mu_3(z_3)\right)^{\frac{1}{q_3}} \le \tilde{c}_1 \left(\int_{B_k} |\widetilde{\phi}(z_3)|^{p_3} d\nu_{\alpha_3}(z_3)^{p_3}|\right)^{\frac{1}{p_3}}.$$

Simply, since our theorem is valid for simplest $m=1$ case. Then we also note that

$$\begin{aligned}&\left(\int_{B_k}\left(\int_{B_k} |f(z_1,z_2,z_3)|^{q_1} d\mu_1(z_1)\right)^{\frac{q_2}{q_1}} d\mu_2(z_2)\right)^{\frac{1}{p_2}}\\ \le&\tilde{c}\phi(z_3) = \tilde{c}\left(\int_{B_k}\left(\int_{B_k} |f(z_1,z_2,z_3)|^{p_1} d\nu_{\alpha_1}(z_1)\right)^{\frac{p_2}{p_1}} d\nu_{\alpha_2}(z_2)\right)^{\frac{1}{p_2}},\end{aligned}$$

for a constant $\tilde{c}$ and for fixed $z_3 \in B_k$ as we proved above already for all $p_j \le q_j$, $j=1,2$. As we see from discussion above ϕ function is subharmonic.

Note now combining both estimates we get what we need, so our theorem based on $m=1,2$ cases is now proved for $m=3$. Add hence using now induction for all m.

Theorem 3.1 is proved. □

Proof of Theorem 3.2. As in the proof of Theorem 3.1, the implication 2) ⇒ 1) is verified in a standard way, so its proof is omitted. It is based on properties of r-lattices in the ball and similar arguments can be seen in [30,39] for various embedding theorems proved there.

We proceed to the proof of 1) ⇒ 2). Again, we prove the theorem for $n=2$, since $n>2$ there are similar arguments. First suppose that

$$p_2 \le p_1 \le q. \tag{4.1}$$

Using the arguments used in the proof of Theorem 3.1, we have

$$\begin{aligned}I(f)=&\left(\int_{B_k^2}\left|\widetilde{D}^m f(z_1,z_2)\right|^q d\widetilde{\nu}(z_1,z_2)\right)^{\frac{1}{q}}\\ \le&C\Bigg(\int_{B_k^2}\left|\widetilde{D}^m f(z_1,z_2)\right|^q w_1^{\frac{q}{p_1}}(1-|z_1|)w_2^{\frac{q}{p_2}}(1-|z_2|)(1-|z_1|)^{m_1q+(n+1)\frac{q}{p_1}-(n+1)}\\ &\times(1-|z_2|)^{m_2q+(n+1)\frac{q}{p_2}-(n+1)}d\nu(z_1)d\nu(z_2)\Bigg)^{\frac{1}{q}}.\end{aligned}$$

Using Lemma 3.6 and taking into account that $p_1 \le q$, we obtain

$$\begin{aligned}I\le&C\Bigg(\int_{B_k} w_2^{\frac{p_1}{p_2}}(1-|z_2|)(1-|z_2|)^{m_2p_1+(n+1)\frac{p_1}{p_2}-(n+1)}\\ &\times\left(\int_{B_k^2}\left|\widetilde{D}^m f(z_1,z_2)\right|^q w_1^{\frac{q}{p_1}}(1-|z_1|)(1-|z_1|)^{m_1q+(n+1)\frac{q}{p_1}-(n+1)}d\nu(z_1)\right)^{\frac{p_1}{q}}d\nu(z_2)\Bigg)^{\frac{1}{p_1}}.\end{aligned}$$

Again, using similar arguments to the inner integral, we obtain the estimate (see Lemma 3.6)

$$\begin{aligned} I(f) \leq & C \left(\int_{B_k} w_2^{\frac{p_1}{p_2}}(1-|z_2|)(1-|z_2|)^{m_2 p_1 + (n+1)\frac{p_1}{p_2} - (n+1)} \right. \\ & \left. \times \left(\int_{B_k^2} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1-|z_1|)(1-|z_1|)^{m_1 p_1} d\nu(z_1) \right) d\nu(z_2) \right)^{\frac{1}{p_1}}. \end{aligned}$$

Taking into account that $\frac{p_2}{p_1} \leq 1$, as above, we obtain using the fact that the function in brackets is subharmonic

$$\begin{aligned} I(f) \leq & C \left(\int_{B_k} w_2(1-|z_2|)(1-|z_2|)^{m_2 p_2} \right. \\ & \left. \times \left(\int_{B_k^2} \left| \widetilde{D}^m f(z_1, z_2) \right|^{p_1} w_1(1-|z_1|)(1-|z_1|)^{m_1 p_1} d\nu(z_1) \right)^{\frac{p_2}{p_1}} d\nu(z_2) \right)^{\frac{1}{p_2}}. \end{aligned}$$

It remains to apply Lemma 3.5. The theorem is proved under the condition (4.1).

Now we turn to the case

$$p_1 \leq p_2 \leq q. \tag{4.2}$$

If $(z_{k_1}, z_{k_2}) \in D(a_{k_1}, r) \times D(a_{k_2}, r)$ then (see [36])

$$\begin{aligned} \left| \widetilde{D}^m f(z_{k_1}, z_{k_2}) \right|^q \leq & \frac{C}{(1-|z_{k_1}|)^{m_1 q + (n+1)\frac{q}{p_1}} (1-|z_{k_2}|)^{m_2 q}} \\ & \times \left(\int_{D^*(a_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} d\nu(\zeta_1) \right)^{\frac{q}{p_1}}, \end{aligned}$$

where $D^*(a_{k_1}, r)$ is expansion of the dyadic ball at the same center $D(a_{k_1}, r)$, (see [39]). Therefore, as in the proof of the first part:

$$\begin{aligned} I(f) = & \left(\int_{B_k^2} \left| \widetilde{D}^m f(z_1, z_2) \right|^q d\widetilde{\nu}(z_1, z_2) \right)^{\frac{1}{q}} \\ \leq & \left(\sum_{k_2 \geq 0} \sum_{k_1 \geq 0} \left| \widetilde{D}^m f(z_{k_1}, z_{k_2}) \right|^q \widetilde{\nu}(D(a_{k_1}, r) \times D(a_{k_2}, r)) \right)^{\frac{1}{q}} \\ \leq & C_1 \left(\sum_{k_2 \geq 0} \sum_{k_1 \geq 0} w_1^{\frac{q}{p_1}}(1-|z_{k_1}|) w_2^{\frac{q}{p_2}}(1-|z_{k_2}|)(1-|z_{k_2}|)^{(n+1)\frac{q}{p_2}} \right. \\ & \left. \times \left(\int_{D^*(a_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} d\nu(\zeta_1) \right)^{\frac{q}{p_1}} \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq C_2 \left(\sum_{k_2 \geq 0} \sum_{k_1 \geq 0} w_2^{\frac{q}{p_2}} (1-|z_{k_2}|)(1-|z_{k_2}|)^{(n+1)\frac{q}{p_2}} \right.$$
$$\left. \times \left(\int_{D^*(a_{k_1},r)} |f(\zeta_1, z_{k_2})|^{p_1} \, w_1(1-|\zeta_1|) d\nu(\zeta_1) \right)^{\frac{q}{p_1}} \right)^{\frac{1}{q}}.$$

Using the inequality (4.2), we obtain

$$I^{p_2} \leq C_2^{p_2} \left(\sum_{k_2 \geq 0} \sum_{k_1 \geq 0} w_2(1-|z_{k_2}|)(1-|z_{k_2}|)^{n+1} \right.$$
$$\left. \times \left(\int_{D^*(a_{k_1},r)} |f(\zeta_1, z_{k_2})|^{p_1} \, w_1(1-|\zeta_1|) d\nu(\zeta_1) \right)^{\frac{p_2}{p_1}} \right).$$

Now apply Lemma 3.1, we have

$$I^{p_2}(f) \leq C_2^{p_2} \int_{B_k} \left(\sum_{k_1 \geq 0} \left(\int_{D^*(a_{k_1},r)} |f(\zeta_1, z_2)|^{p_1} \, w_1(1-|\zeta_1|) d\nu(\zeta_1) \right)^{\frac{p_2}{p_1}} \right) \times w_2(1-|\zeta_2|) d\nu(\zeta_2). \tag{4.3}$$

By the condition (4.2) $\frac{p_2}{p_1} = \alpha \geq 1$. Hence

$$\sum_{k=0}^{\infty} b_k^{\alpha} \leq \left(\sum_{k=0}^{\infty} b_k \right)^{\alpha}, \quad \text{for all } b_k \geq 0, \ k \in \mathbb{N}.$$

Therefore, by (4.3) we obtain

$$I^{p_2}(f) \leq C_2^{p_2} \int_{B_k} \left(\sum_{k_1 \geq 0} \left(\int_{D^*(a_{k_1},r)} |f(\zeta_1, z_2)|^{p_1} \, w_1(1-|\zeta_1|) d\nu(\zeta_1) \right)^{\frac{p_2}{p_1}} \right)$$
$$\times w_2(1-|\zeta_2|) d\nu(\zeta_2)$$
$$\leq C_3 \int_{B_k} \left(\int_{B_k} |f(z_1, z_2)|^{p_1} \, w_1(1-|z_1|) d\nu(z_1) \right)^{\frac{p_2}{p_1}} w_2(1-|\zeta_2|) d\nu(\zeta_2).$$

Note now it is easy to see the necessity of condition on measure is valid for any m (we discussed shortly $m = 2$ case above) and the proof is simply repetition of arguments of $m = 2$ case which was given above, and no new idea is needed here. Let us turn to the proof of other implication, for this theorem. Again, we consider only a particular case of spaces with zero smoothness (spaces without fractional $\mathcal{D}^m$ derivatives and spaces with ordinary $(1-|z|)^{\alpha}$ weights). We simply modify arguments based on induction we provided at the end of proof of previous theorem. Note that we have proved above that for $m = 2$ (we below denote by $\delta(z) = (1-|z|)$)

$$I(f) = \left(\int_{B_k^2} |f(z_1, z_2)|^q d\tilde{\nu}(z_1, z_2) \right)^{\frac{1}{q}}$$

$$\le c\left(\int_{B_k^2}|f(z_1,z_2)|^q\delta(z_1)^{\alpha_1\frac{q}{p_1}}\delta(z_2)^{\alpha_2\frac{q}{p_2}}(\delta(z_1))^{(\widetilde{n}+1)\frac{q}{p_1}}(\delta(z_2))^{(\widetilde{n}+1)\frac{q}{p_2}}d\nu(z_1)d\nu(z_2)\right)^{\frac{1}{q}}$$
$$=cJ.$$

Based on properties of r-lattice the same proof can be given by repetition of arguments for any $m=1,2,\ldots$. Than we showed

$$J\le\tilde{c}\left(\int_{B_k}\left(\int_{B_k}|f(z_1,z_2)|^{p_1}\delta(z_1)^{\alpha_1}d\nu(z_1)\right)^{\frac{p_2}{p_1}}\delta(\xi_2)^{\alpha_2}d\nu(\xi_2)\right)^{\frac{1}{p_2}}.$$

The question is how to show that

$$\int_{B_k^3}|f(z_1,z_2,z_3)|^q\left(\prod_{j=1}^{3}(\delta(z_j))^{\alpha_j\frac{q}{p_j}+(n+1)q(\frac{1}{p_j})}\right)d\nu(z_j)$$
$$\le c\left(\int_{B_k}\left(\int_{B_k}\left(\int_{B_k}|f(z_1,z_2,z_3)|^{p_1}(\delta(z_1))^{\alpha_1}d\nu(z_1)\right)^{\frac{p_2}{p_1}}\delta(\xi_2)^{\alpha_2}d\nu(\xi_2)\right)^{\frac{p_3}{p_2}}\right.$$
$$\left.\times\delta(\xi_3)^{\alpha_3}d\nu(\xi_3)\right)^{\frac{1}{p_3}},$$

when we have this estimate for $m=1,2$. We use induction. We have first for $m=2$

$$\left(\int_{B_k^2}|f(z_1,z_2,z_3)|^q)\delta^{\tau_1}(z_1)\delta^{\tau_2}(z_2)d\nu(z_1)d\nu(z_2)\right)$$
$$\le c\int_{B_k}\left(\int_{B_k}|f(z_1,z_2,z_3|^{p_1}\delta(z_1)^{\alpha_1}d\nu(z_1)\right)^{\frac{p_2}{p_1}}(\delta(\xi_2)^{\alpha_2})d\nu(\xi_2))^{\frac{1}{p_2}}=c(\phi(z_3)),$$

then we have also that

$$\int_{B_k}|\phi(z_3)|^q\left(\delta^{\tau_3}(z_3)\right)d\nu(z_3)\le c\left(\int_{B_k}|\phi(z_3)|^{p_3}\delta(z_3)^{\alpha_3}d\nu(z_3)\right)^{\frac{1}{p_3}},$$

since ϕ is subharmonic (see discussion above) and the result is valid for $m=1$ case. It remains to combine this two estimates and use induction to set the result for all $m=1,2,3$.

The theorem is completely proved. □

Remark 4.1. We finally note analogues of these results based on same approaches are valid with some restrictions in tubular domains and in pseudoconvex domains with smooth boundary. Proofs are again based on properties of r-lattices in these domains (see, for example, [1,2] for lattices in pseudoconvex domains). We refer to [32,33] for some analogues results in tubular and pseudoconvex domains.

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[1]Department of Mathematics,
Bryansk State Technical University,
Bryansk, Russia
Email address: rshamoyan@gmail.com

[2]Faculty of Organizational Sciences,
University of Belgrade,
Serbia
Email address: oliveradj@fon.rs