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ON THE $\mathcal{Q}$ CURVATURE TENSOR OF A GENERALIZED SASAKIAN-SPACE-FORM

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Abstract. The object of the present paper is to study ξ-$\mathcal{Q}$ flat, ϕ-$\mathcal{Q}$ flat generalized Sasakian-space-forms. Besides these, we consider generalized Sasakian-space-forms satisfying $P(\xi, X) \cdot \mathcal{Q} = 0$, $\mathcal{Q}(\xi, X) \cdot P = 0$ and $\mathcal{Q}(\xi, X) \cdot \mathcal{Q} = 0$. As a consequence we obtain several important results. Finally, illustrative examples are given.

1. Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine the curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real space-form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$

A Sasakian manifold with constant ϕ-sectional curvature becomes a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame Alegre, Blair and Carriazo introduced the notion of generalized Sasakian-space-form in 2004 [1]. In this connection it should be mentioned that in 1989 Olszak [22] studied generalized complex-space-form and proved its existence. A generalized Sasakian-space-form is defined as follows.

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is a generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the

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curvature tensor R is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \tag{1.1}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1], the authors cited several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. In [19], Kim studied conformally flat generalized Sasakian-space-forms under the assumption that the characteristic vector field ξ is Killing and he classified locally symmetric generalized Sasakian-space-forms. Also he proved some geometric properties of generalized Sasakian-space-forms which depend on the nature of the functions f_1, f_2 and f_3. Generalized Sasakian-space-forms have also been studied in [2–5, 8–12, 14, 15, 20, 24] and many others. In [5], the authors studied trans-Sasakian generalized Sasakian-space-forms and its particular cases. Moreover in [13] the authors studied certain curvature conditions on generalized Sasakian-space-forms. On the other hand recently Mantica and Suh [21] introduced $\mathcal{Q}$ curvature tensor and study its relativistic significance. In the present paper we generalize the results of [13].

The present paper is organized as follows.

After preliminaries in Section 3, we consider ξ-$\mathcal{Q}$ flat generalized Sasakian-space-forms. Section 4 is devoted to study ϕ-$\mathcal{Q}$ flat generalized Sasakian-space-forms. Sections 5, 6 and 7 deal with generalized Sasakian-space-forms satisfying $P(\xi, X) \cdot \mathcal{Q} = 0$, $\mathcal{Q}(\xi, X) \cdot P = 0$ and $\mathcal{Q}(\xi, X) \cdot \mathcal{Q} = 0$ respectively. Finally, illustrative examples are given to verify the results of sections 3 and 4.

2. Preliminaries

In an almost contact metric manifold we have [6–7]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{2.4}$$

$$g(\phi X, \xi) = 0. \tag{2.5}$$

Again for a $(2n+1)$-dimensional generalized Sasakian-space-form we have [1]

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \tag{2.6}$$

$$+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X)], \tag{2.10}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.11}$$

$$S(\xi,\xi) = 2n(f_1 - f_3), \tag{2.12}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.13}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.14}$$

where R, S and r are the curvature tensor, Ricci tensor and scalar curvature of the space-form respectively.

A $(0,p)$-tensor field T on (M,g) is called parallel when it is invariant under parallel translation, i.e., when

$$\nabla T = 0,$$

in particular, if the $(0,4)$-Riemann-Christoffel curvature tensor R is parallel, i.e.,

$$\nabla R = 0,$$

then M is said to be locally symmetric. This property justifies the name given to such manifolds locally they are symmetric with respect to each of their points. If each geodesic symmetry s_p, $p \in M$, is a global isometry of M, then M is called symmetric space. Thus $\nabla R = 0$ for every symmetric space and conversely, every complete and simply connected locally symmetric space is symmetric.

A Riemannian manifold (M^{2n+1}, g) is said to be semisymmetric if its curvature tensor R is satisfies $R(X,Y).R = 0$, $X, Y \in \chi(M)$, where R(X,Y) acts on R as a derivation [11]. Every symmetric space is semisymmetric, but the converse is not true, in general.

Let (M^{2n+1}, g), $n \geq 1$, be a Riemannian manifold. The conformal curvature tensor C is defined by [25]

$$\begin{aligned} C(X,Y)Z = &R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.15}$$

where S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold M. If the scalar curvature r vanishes at each point of the manifold, then the conformal curavture tensor reduces to conharmonic curvature tensor introduced by Ishii [16]. A Riemannian manifold (M^{2n+1}, g), $n \geq 1$, is called conformally flat if the conformal curvature vanishes for $n > 1$. If $n = 1$, then the conformal curvature tensor C vanishes identically. Moreover a manifold of constant sectional curvature is conformally flat. In [19] conformally flat generalized Sasakian space form have been studied by Kim. Also De and Sarkar [8] studied projective curvature tensor of generalized Sasakian-space-forms. Moreover in [23], Shukla et al. studied concircular curvature tensor on generalized Sasakian-space-forms.

After the conformal curvature tensor, the projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence betwen each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. The projective curvature tensor is defined by [13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{2.16}$$

where S is the Ricci tensor of M.

A transformation in an $(2n+1)$ dimensional Reimannian manifold M, which transforms every geodesic circle of M into a geodesic circle of M, is said to be a concircular transformation ([18,26]). A concircular transformation is always a conformal transformation [18]. Here, we mean a geodesic circle by a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformation is a generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism. An important invariant of concircular transformation is the concircular curvature tensor $\mathcal{Z}$, defined by [26]

$$\mathcal{Z}(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y], \tag{2.17}$$

for all $X, Y, W \in \chi(M)$, where R is the Reimannian curvature tensor and r is the scalar curvature with respect to the Levi-Civita connection.

In a recent paper Mantica and Suh [21] introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold (M^n, g), $n > 2$, denoted by $\mathcal{Q}$ and defined by

$$\mathcal{Q}(X,Y)W = R(X,Y)W - \frac{\psi}{(n-1)}[g(Y,W)X - g(X,W)Y], \tag{2.18}$$

where ψ is an arbitrary scalar function. Such a tensor $\mathcal{Q}$ is known as Q-curvature tensor. The notion of Q tensor is also suitable to reinterpret some differential structures on a Riemannian manifold. Mantica and Suh [21] have studied pseudo-Q-symmetric Riemannian manifolds. If $\psi = \frac{r}{(2n+1)}$, then Q curvature tensor reduces to concircular curvature tensor.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). At each point $p \in M$, decompose the tangent space

T_pM into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \to \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M.$$

It may be natural to consider the following particular cases.

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to L(\xi_p)$, i.e., the projection of the image of C in $\phi(T_p(M))$ is zero.

(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M))$, i.e., the projection of the image of C in $L(\xi_p)$ is zero. This condition is equivalent to

$$C(X, Y)\xi = 0. \tag{2.19}$$

(3) $C : \phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M)) \to L(\xi_p)$, i.e., when C is restricted to $\phi(T_p(M))\times\phi(T_p(M))\times\phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{2.20}$$

A K-contact manifold satisfying (2.19) and (2.20) are called ξ-conformally flat and ϕ-conformally flat respectively. A K-contact manifold satisfying the cases (1), (2) and (3) are considered in [27], [28] and [29] respectively.

Definition 2.1. A generalized Sasakian-space-form $(M^{(2n+1)}, g)$, $n > 1$, is said to be $\xi\mathcal{Q}$ flat if $\mathcal{Q}(X, Y)\xi = 0$ on M.

Definition 2.2. A generalized Sasakian-space-form $(M^{(2n+1)}, g)$, $n > 1$, is said to be ϕ-$\mathcal{Q}$ flat if $g(\mathcal{Q}(\phi X, \phi Y)\phi Z, \phi W) = 0$ on M.

3. ξ-$\mathcal{Q}$ Flat Generalized Sasakian-Space-Forms

In this section we characterize ξ-$\mathcal{Q}$ flat generalized Sasakian-space-forms.

From (2.18) we have

$$\mathcal{Q}(X, Y)\xi = R(X, Y)\xi - \frac{\psi}{2n}[\eta(Y)X - \eta(X)Y], \tag{3.1}$$

for any for any vector fields X and $Y \in T(M)$. Using (2.9) in (3.1) implies

$$\mathcal{Q}(X, Y)\xi = \left[\frac{\psi}{2n} - (f_1 - f_3)\right][\eta(Y)X - \eta(X)Y]. \tag{3.2}$$

Thus we can state the following.

Theorem 3.1. *A $(2n + 1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ξ-$\mathcal{Q}$ flat if and only if $\psi = 2n(f_1 - f_3)$.*

For $\psi = \frac{r}{2n+1}$, $\mathcal{Q}$ curvature tensor reduces to concircular curvature tensor. Thus in view of Theorem 3.1 and making use of (2.14) we obtain the following.

Corollary 3.1. *A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ξ-concircularly flat if and only if $f_3 = \frac{3}{1-2n} f_2$.*

In [23], Shukla et al. proved that a generalized Sasakian-space-form is concircularly flat if and only if $f_3 = \frac{3}{1-2n} f_2$. Since concircularly flat manifold implies ξ-concircularly flat, therefore from Corollary 3.1 we can mention the following.

Remark 3.1. A (2n+1)-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is concircularly flat if and only if $f_3 = \frac{3}{1-2n} f_2$.

Therefore Theorem 3.1 of [23] is a particular case of Corollary 3.1.

If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form. So $f_1 - f_3 = \frac{c+3}{4} - \frac{c-1}{4} = 1$.

Thus we have the following.

Corollary 3.2. *A* $(2n+1)$*-dimensional Sasakian-space-form is* $\xi\mathcal{Q}$ *flat if and only if* $\psi = 2n$.

4. ϕ-$\mathcal{Q}$ Flat Generalized Sasakian-Space-Forms

Suppose M be a $(2n+1)$-dimensional, $n > 1$, ϕ-$\mathcal{Q}$ flat generalized Sasakian-space-form. Then $g(\mathcal{Q}(\phi X, \phi Y)\phi Z, \phi W) = 0$, for any vector fields X, Y, Z and $W \in T(M)$, which implies

$$g(R(\phi X, \phi Y)\phi Z, \phi W) - \frac{\psi}{2n}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)] = 0. \tag{4.1}$$

Using (2.1) in (2.6) we obtain

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) = & f_1[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)] \\ & + f_2[g(\phi X, Z)g(Y, \phi W) - g(\phi Y, Z)g(X, \phi W) \\ & + 2g(\phi X, Y)g(Z, \phi W)]. \end{aligned} \tag{4.2}$$

Using (4.2) in (4.1) yields

$$\begin{aligned} & \left(f_1 - \frac{\psi}{2n}\right)[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)] \\ & + f_2[g(\phi X, Z)g(Y, \phi W) - g(\phi Y, Z)g(X, \phi W) + 2g(\phi X, Y)g(Z, \phi W)] = 0. \end{aligned} \tag{4.3}$$

Contracting Y and Z in (4.3) gives

$$\left(f_1 - \frac{\psi}{2n}\right)[2ng(\phi X, \phi W) - g(\phi^2 X, \phi^2 W)] + f_2[g(\phi X, \phi W) + 2g(\phi X, \phi W)] = 0. \tag{4.4}$$

Using (2.1) in (4.4) we have

$$\left(f_1 - \frac{\psi}{2n}\right)[2ng(\phi X, \phi W) - g(X, W) + \eta(X)\eta(W)] + 3f_2 g(\phi X, \phi W) = 0. \tag{4.5}$$

Again using (2.3) in (4.5) we have

$$\left[(2n-1)\left(f_1 - \frac{\psi}{2n}\right) + 3f_2\right] g(\phi X, \phi W) = 0, \tag{4.6}$$

which yields,

$$f_1 - \frac{\psi}{2n} = -\frac{3f_2}{2n-1}. \tag{4.7}$$

From (4.7) and (4.3) we have

$$\begin{aligned} &-\frac{3f_2}{2n-1}[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)] \\ &+ f_2[g(\phi X,Z)g(Y,\phi W) - g(\phi Y,Z)g(X,\phi W) + 2g(\phi X,Y)g(Z,\phi W)] = 0. \end{aligned} \tag{4.8}$$

Replacing Y by ϕY and using (2.1) in (4.8) implies

$$\begin{aligned} &-\frac{3f_2}{2n-1}[-g(Y,\phi Z)g(\phi X,\phi W) + g(\phi X,\phi Z)g(Y,\phi W)] \\ &+ f_2[g(\phi X,Z)g(\phi Y,\phi W) + g(Y,Z)g(X,\phi W) \\ &- g(X,\phi W)\eta(Y)\eta(Z) + 2g(\phi X,\phi Y)g(Z,YW)] = 0. \end{aligned} \tag{4.9}$$

Substituting Y by ξ in (4.9)

$$f_2 g(X,\phi W)\eta(Z) = 0, \tag{4.10}$$

which implies

$$f_2 = 0.$$

Hence we have the following.

Theorem 4.1. *If a $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ϕ-Ω flat, then $f_2 = 0$.*

In [19], U. K. Kim proved that for a $(2n+1)$-dimensional generalized Sasakian-space-form the following hold.

(i) If $n > 1$, then M is conformally flat if and only if $f_2 = 0$.
(ii) If M is conformally flat and ξ is Killing, then M is locally symmetric and has constant ϕ-sectional curvature.

In the view of the first part of above theorem we have the following.

Corollary 4.1. *In a $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form ϕ-Ω flat and conformally flat are equivalent.*

Again, in view of the second part of the above theorem we have the following.

Corollary 4.2. *A ϕ-Ω flat $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with ξ as a Killing vector field is locally symmetric and has constant ϕ-sectional curvature.*

Suppose $f_2 = 0$. Then from (4.3) we have

$$g(\Omega(\phi X,\phi Y)\phi Z,\phi W) = \left(f_1 - \frac{\psi}{2n}\right)[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{4.11}$$

Substituting $f_2 = 0$ in (4.7) yields

$$f_1 = \frac{\psi}{2n}. \tag{4.12}$$

Combining the equations (4.11) and (4.12) we can state the following.

Theorem 4.2. *A $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is ϕ-$\mathcal{Q}$ flat if and only if $f_2 = 0$.*

5. Generalized Sasakian-Space-Forms Satisfying $P(\xi, X) \cdot \mathcal{Q} = 0$

In this section we characterize generalized Sasakian-space-forms satisfying $P(\xi, X) \cdot \mathcal{Q} = 0$, where P is the projective curvature tensor. Suppose a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form satisfies $(P(\xi, X)\cdot\mathcal{Q})(Y, Z)U = 0$, for any vector fields X, Y, Z and $U \in T(M)$. Then we have

$$P(\xi, X)\mathcal{Q}(Y, Z)U - \mathcal{Q}(P(\xi, X)Y, Z)U - \mathcal{Q}(Y, P(\xi, X)Z)U - \mathcal{Q}(Y, Z)P(\xi, X)U = 0. \tag{5.1}$$

Using (2.10) and (2.16) we have

$$\begin{aligned} P(\xi, X)\mathcal{Q}(Y, Z)U = &\left[\frac{f_3 - 3f_2}{2n} - f_3\right][g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)\eta(X)\xi] \\ &- \frac{\psi}{2n}\left[\frac{f_3 - 3f_2}{2n} - f_3\right] g(Z, U)[g(X, Y)\xi - \eta(X)\eta(Y)\xi] \\ &+ \frac{\psi}{2n}\left[\frac{f_3 - 3f_2}{2n} - f_3\right] g(Y, U)[g(X, Z)\xi - \eta(X)\eta(Z)\xi]. \end{aligned} \tag{5.2}$$

Again using (2.10), (2.16) and (2.18) we obtain

$$\begin{aligned} \mathcal{Q}(P(\xi, X)Y, Z)U = &\left[\frac{f_3 - 3f_2}{2n} - f_3\right]\left[(f_1 - f_3) - \frac{\psi}{2n}\right] \\ &\times [g(X, Y) - \eta(X)\eta(Y)][g(Z, U)\xi - \eta(U)Z]. \end{aligned} \tag{5.3}$$

Making use of (2.10) and (2.16) we have

$$\begin{aligned} \mathcal{Q}(Y, P(\xi, X)Z)U = &\left[\frac{f_3 - 3f_2}{2n} - f_3\right]\left[(f_1 - f_3) - \frac{\psi}{2n}\right] \\ &\times [g(X, Z) - \eta(X)\eta(Z)][\eta(U)Y - g(Y, U)\xi]. \end{aligned} \tag{5.4}$$

Similarly using (2.10), (2.16) and (2.18) we have

$$\begin{aligned} \mathcal{Q}(Y, Z)P(\xi, X)U = &\left[\frac{f_3 - 3f_2}{2n} - f_3\right]\left[(f_1 - f_3) - \frac{\psi}{2n}\right] \\ &\times [g(X, U) - \eta(X)\eta(U)][\eta(Z)Y - \eta(Y)Z]. \end{aligned} \tag{5.5}$$

Substituting (5.2)-(5.5) in (5.1) yields

$$\left[\frac{f_3 - 3f_2}{2n} - f_3\right][g(X, R(Y, Z)U)\xi - \eta(X)\eta(R(Y, Z)U)\xi]$$

$$- \frac{\psi}{2n}\left[\frac{f_3-3f_2}{2n}-f_3\right]g(Z,U)[g(X,Y)\xi-\eta(X)\eta(Y)\xi]$$
$$+ \frac{\psi}{2n}\left[\frac{f_3-3f_2}{2n}-f_3\right]g(Y,U)[g(X,Z)\xi-\eta(X)\eta(Z)\xi]-\left[\frac{f_3-3f_2}{2n}-f_3\right]$$
$$\times\left[(f_1-f_3)-\frac{\psi}{2n}\right][g(X,Y)-\eta(X)\eta(Y)][g(Z,U)\xi-\eta(U)Z]-\left[\frac{f_3-3f_2}{2n}-f_3\right]$$
$$\times\left[(f_1-f_3)-\frac{\psi}{2n}\right][g(X,Z)-\eta(X)\eta(Z)][\eta(U)Y-g(Y,U)\xi]-\left[\frac{f_3-3f_2}{2n}-f_3\right]$$
(5.6)
$$\times\,[(f_1-f_3)-\frac{\psi}{2n}][g(X,U)-\eta(X)\eta(U)][\eta(Z)Y-\eta(Y)Z]=0.$$

Taking inner product with ξ in (5.6) implies

$$\left[\frac{f_3-3f_2}{2n}-f_3\right][g(X,R(Y,Z)U)-\eta(X)\eta(R(Y,Z)U)]-\frac{\psi}{2n}\left[\frac{f_3-3f_2}{2n}-f_3\right]$$
$$\times\, g(Z,U)[g(X,Y)-\eta(X)\eta(Y)]+\frac{\psi}{2n}\left[\frac{f_3-3f_2}{2n}-f_3\right]g(Y,U)[g(X,Z)-\eta(X)\eta(Z)]$$
$$-\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[(f_1-f_3)-\frac{\psi}{2n}\right][g(X,Y)-\eta(X)\eta(Y)][g(Z,U)-\eta(U)\eta(Z)]$$
$$-\left[\frac{f_3-3f_2}{2n}-f_3\right]\left[(f_1-f_3)-\frac{\psi}{2n}\right][g(X,Z)-\eta(X)\eta(Z)]$$
(5.7)
$$\times\,[\eta(U)\eta(Y)-g(Y,U)]=0.$$

Putting $X=Y=e_i$, where $\{e_i,\xi\}$, $1\le i\le 2n$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\frac{(1-2n)f_3-3f_2}{2n}\Big[S(Z,U)-2n(f_1-f_3)g(Z,U)$$
$$+2n\left(f_1-f_3-\frac{\psi}{2n}\right)\eta(U)\eta(Z)\Big]=0. \tag{5.8}$$

Therefore, either

$$(1-2n)f_3-3f_2=0$$

or

$$S(Z,U)=2n(f_1-f_3)g(Z,U)-2n\left(f_1-f_3-\frac{\psi}{2n}\right)\eta(U)\eta(Z). \tag{5.9}$$

In the second case, comparing this equation with (2.7) for Z and U orthogonal to ξ we get

$$2n(f_1-f_3)=2nf_1+3f_2-f_3. \tag{5.10}$$

It follows that

$$f_3 = \frac{3}{1-2n} f_2. \tag{5.11}$$

Then as $f_3 = \frac{3}{1-2n} f_2$, from (2.7), $S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y)$ and M is an Einstein manifold. Conversely, if $\frac{f_2 - 3f_2}{2n} - f_3 = 0$, that is, $f_3 = \frac{(1-2n)}{3} f_2$, then in view of equations (5.2)-(5.5) we have $P(\xi, X) \cdot \mathcal{Q} = 0$.

In view of the above results we can state the following.

Theorem 5.1. *A* $(2n+1)$*-dimensional* $n > 1$ *generalized Sasakian-space-form satisfies* $P(\xi, X) \cdot \mathcal{Q} = 0$ *if and only if* $f_3 = \frac{3}{(1-2n)} f_2$. *In such a case it is an Einstein manifold.*

For $\psi = \frac{r}{(2n+1)}$, the $\mathcal{Q}$ curvature tensor reduces to the concircular curvature tensor. Now putting $Z = U = e_i$ in (5.9), where $\{e_i, \xi\}$, $1 \le i \le 2n$, is the orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get $r = 2n(2n+1)(f_1 - f_3) - 2n\left[(f_1 - f_3) - \frac{\psi}{2n}\right]$. Using $\psi = \frac{r}{(2n+1)}$, then $(2n+1)\psi = 2n(2n+1)(f_1 - f_3) - 2n(f_1 - f_3) + \psi$, which implies $\frac{\psi}{2n} = (f_1 - f_3)$ and hence the manifold reduces to an Einstein manifold.

Thus we have following.

Theorem 5.2. *A* $(2n+1)$*-dimensional* $n > 1$ *generalized Sasakian-space-form satisfies* $P(\xi, X) \cdot \mathcal{Z} = 0$ *if and only if* $f_3 = \frac{3}{(1-2n)} f_2$. *In such a case it is an Einstein manifold.*

Remark 5.1. The above theorem has been proved by De and Yildiz in [13].

6. Generalized Sasakian-Space-Forms Satisfying $\mathcal{Q}(\xi, X) \cdot P = 0$

Suppose a $(2n+1)$-dimensional $n > 1$ generalized Sasakian-space-form satisfies $(\mathcal{Q}(\xi, X) \cdot P)(Y,Z)U = 0$, for any vector fields X, Y, Z and $U \in T(M)$. Then

$$\mathcal{Q}(\xi, X)P(Y,Z)U - P(\mathcal{Q}(\xi, X)Y, Z)U - P(Y, \mathcal{Q}(\xi, X)Z)U - P(Y,Z)\mathcal{Q}(\xi, X)U = 0. \tag{6.1}$$

Now using (2.10), (2.16) and (2.18) we have

$$\begin{aligned} \mathcal{Q}(\xi, X)P(Y,Z)U &= \left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X, R(Y,Z)U)\xi - \eta(R(Y,Z)U)X] \\ &\quad - \frac{1}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X,Y)\xi - \eta(Y)X]S(Z,U) \\ &\quad + \frac{1}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X,Z)\xi - \eta(Z)X]S(Y,U). \end{aligned} \tag{6.2}$$

Using (2.10), (2.16) and (2.18) we obtain

$$P(\mathcal{Q}(\xi, X)Y, Z)U = \left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X,Y)P(\xi, Z)U - \eta(Y)P(X,Z)U]. \tag{6.3}$$

Again using (2.10), (2.16) and (2.18) we get

$$P(Y,\mathcal{Q}(\xi,X)Z)U=\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,Z)P(Y,\xi)U-\eta(Z)P(Y,X)U]. \tag{6.4}$$

Finally, using (2.10), (2.16) and (2.18), we have

$$P(Y,Z)\mathcal{Q}(\xi,X)U=\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,U)P(Y,Z)\xi-\eta(U)P(Y,Z)X]. \tag{6.5}$$

Using (6.2)-(6.5) in (6.1) yields

$$\begin{aligned}
&\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)X]\\
&-\frac{1}{2n}\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,Y)\xi-\eta(Y)X]S(Z,U)\\
&+\frac{1}{2n}[f_1-f_3-\frac{\psi}{2n}][g(X,Z)\xi-\eta(Z)X]S(Y,U)\\
&-\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,Y)P(\xi,Z)U-\eta(Y)P(X,Z)U]\\
&-\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,Z)P(Y,\xi)U-\eta(Z)P(Y,X)U]\\
&-\left[f_1-f_3-\frac{\psi}{2n}\right][g(X,U)P(Y,Z)\xi-\eta(U)P(Y,Z)X]=0.
\end{aligned} \tag{6.6}$$

Putting $X=Y=e_i$, where $\{e_i,\xi\}$, $1\le i\le 2n$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we have

$$\begin{aligned}
&\left[f_1-f_3-\frac{\psi}{2n}\right][(f_1-f_3)\{\eta(Z)g(U,W)-g(Z,U)\eta(W)\}\\
&-\frac{1}{2n}\{S(Z,U)\eta(W)+\eta(Z)S(U,W)\}\\
&-(2n-1)(f_1-f_3)\{g(Z,U)\eta(W)-\eta(U)g(Z,W)\}\\
&+\frac{2n-1}{2n}S(Z,U)\eta(W)-(2n-1)(f_1-f_3)\eta(U)g(Z,W)\\
&+\eta(U)S(Z,W)+\frac{1}{2n}S(Z,W)\eta(U)-\frac{r}{2n}g(Z,W)\eta(U)]=0.
\end{aligned} \tag{6.7}$$

Putting $W=\xi$ in (6.7) we have

$$\begin{aligned}
&\left[f_1-f_3-\frac{\psi}{2n}\right]\left[\left(1-\frac{1}{n}\right)S(Z,U)-2(n-1)(f_1-f_3)g(Z,U)\right.\\
&\left.-\left\{(2n+1)(f_1-f_3)+\frac{r}{2n}\right\}\eta(Z)\eta(U)\right]=0.
\end{aligned} \tag{6.8}$$

Therefore, either $f_1 - f_3 - \frac{\psi}{2n} = 0$ or

$$\left(1 - \frac{1}{n}\right) S(Z,U) - 2(n-1)(f_1 - f_3)g(Z,U)$$
$$- \left\{(2n+1)(f_1 - f_3) + \frac{r}{2n}\right\} \eta(Z)\eta(U) = 0. \tag{6.9}$$

In the second case, substituting $U = Z = \xi$ we get $-(2n+1)(f_1 - f_3) = \frac{r}{2n}$ and hence $\left(1 - \frac{1}{n}\right) S(Z,U) = 2(n-1)(f_1 - f_3)g(Z,U)$. In this case M is an Einstein manifold.

Conversely, if $f_1 - f_3 - \frac{\psi}{2n} = 0$, then in view of equation (6.2)-(6.5) we have $\mathcal{Q}(\xi, X) \cdot P = 0$.

Thus we can state the following.

Theorem 6.1. *A* $(2n+1)$*-dimensional* $n > 1$ *generalized Sasakian-space-form satisfies* $\mathcal{Q}(\xi, X) \cdot P = 0$ *if and only if* $f_1 - f_3 - \frac{\psi}{2n} = 0$*. In such a case* M *is an Einstein manifold.*

In particular, if $\psi = \frac{r}{(2n+1)}$, then the $\mathcal{Q}$ curvature tensor reduces to the concircular curvature tensor. Thus in view of (2.14), $f_1 - f_3 - \frac{\psi}{2n} = 0$ reduces to $3f_2 + (2n-1)f_3 = 0$. Thus we can state the following.

Theorem 6.2. *A* $(2n+1)$*-dimensional* $n > 1$ *generalized Sasakian-space-form satisfies* $\mathcal{Q}(\xi, X) \cdot \mathcal{Z} = 0$ *if and only if* $f_3 = \frac{3}{2n-1} f_2$*. In such a case* M *is an Einstein manifold.*

Remark 6.1. The above theorem has been proved by De and Yildiz in [13].

7. Generalized Sasakian-Space-Forms Satisfying $\mathcal{Q}(\xi, X) \cdot \mathcal{Q} = 0$

Suppose a $(2n + 1)$-dimensional $n > 1$ generalized Sasakian-space-form satisfies $(\mathcal{Q}(\xi, X) \cdot \mathcal{Q})(Y, Z)U = 0$, for any vector fields X, Y, Z and $U \in T(M)$. Then

$$\mathcal{Q}(\xi, X)\mathcal{Q}(Y, Z)U - \mathcal{Q}(\mathcal{Q}(\xi, X)Y, Z)U - \mathcal{Q}(Y, \mathcal{Q}(\xi, X)Z)U - \mathcal{Q}(Y, Z)\mathcal{Q}(\xi, X)U = 0. \tag{7.1}$$

Now using (2.10), (2.16) and (2.18) we get

$$\mathcal{Q}(\xi, X)\mathcal{Q}(Y, Z)U = \left[f_1 - f_3 - \frac{\psi}{2n}\right] [g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X]$$
$$- \frac{\psi}{2n} \left[f_1 - f_3 - \frac{\psi}{2n}\right] g(Z, U)[g(X, Y)\xi - \eta(Y)X]$$
$$+ \frac{\psi}{2n} \left[f_1 - f_3 - \frac{\psi}{2n}\right] g(Y, U)[g(X, Z)\xi - \eta(Z)X]. \tag{7.2}$$

Using (2.10), (2.16) and (2.18) in (7.2) we have

$$\mathcal{Q}(\mathcal{Q}(\xi, X)Y, Z)U = \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X, Y)[g(X, Z)\xi - \eta(U)Z]$$

$$-\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(Y)Q(X,Z)U. \tag{7.3}$$

Again using (2.10), (2.16) and (2.18) in (7.3) we obtain

$$\begin{aligned} \Omega(Y,\Omega(\xi,X)Z)U = &\left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,Z)[\eta(U)Y - g(Y,U)\xi] \\ &-\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(Z)\Omega(Y,X)U. \end{aligned} \tag{7.4}$$

Finally, using (2.10), (2.16) and (2.18) in (7.4) we get

$$\begin{aligned} \Omega(Y,Z)\Omega(\xi,X)U = &\left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,U)[\eta(Y)X - \eta(X)Y] \\ &-\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(U)\Omega(Y,Z)X. \end{aligned} \tag{7.5}$$

Substituting (7.2)-(7.5) in (7.1) we have

$$\begin{aligned} &\left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X,R(Y,Z)U)\xi - \eta(R(Y,Z)U)X] \\ &-\frac{\psi}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right]g(Z,U)[g(X,Y)\xi - \eta(Y)X] + \frac{\psi}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right] \\ &\times g(Y,U)[g(X,Z)\xi - \eta(Z)X] - \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,Y)[g(X,Z)\xi - \eta(U)Z] \\ &+\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(Y)\Omega(X,Z)U - \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,Z)[\eta(U)Y - g(Y,U)\xi] \\ &+\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(Z)\Omega(Y,X)U - \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,U)[\eta(Y)X - \eta(X)Y] \\ &+\left[f_1 - f_3 - \frac{\psi}{2n}\right]\eta(U)\Omega(Y,Z)X = 0. \end{aligned} \tag{7.6}$$

Taking inner product with ξ in (7.6) we obtain

$$\begin{aligned} &\left[f_1 - f_3 - \frac{\psi}{2n}\right][g(X,R(Y,Z)U) - \eta(R(Y,Z)U)\eta(X)] \\ &-\frac{\psi}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right]g(Z,U)[g(X,Y) - \eta(Y)\eta(X)] + \frac{\psi}{2n}\left[f_1 - f_3 - \frac{\psi}{2n}\right] \\ &\times g(Y,U)[g(X,Z) - \eta(Z)\eta(X)] - \left[f_1 - f_3 - \frac{\psi}{2n}\right]^2 g(X,Y)[g(U,Z) - \eta(U)\eta(Z)] \end{aligned}$$

$$+\left[f_1-f_3-\frac{\psi}{2n}\right]\eta(Y)\eta(\mathcal{Q}(X,Z)U)-\left[f_1-f_3-\frac{\psi}{2n}\right]^2 g(X,Z)[\eta(U)\eta(Y)-g(Y,U)]$$

(7.7)

$$+\left[f_1-f_3-\frac{\psi}{2n}\right]\eta(Z)\eta(\mathcal{Q}(Y,X)U)+\left[f_1-f_3-\frac{\psi}{2n}\right]\eta(U)\eta(\mathcal{Q}(Y,Z)X)=0.$$

Putting $X=Y=e_i$, where $\{e_i,\xi\}$, $1\leq i\leq 2n$, is the orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we have

(7.8)

$$\left[f_1-f_3-\frac{\psi}{2n}\right][S(Z,U)-2n(f_1-f_3)g(Z,U)+2\left(f_1-f_3-\frac{\psi}{2n}\right)\eta(Z)\eta(U)]=0.$$

Therefore, either $f_1-f_3-\frac{\psi}{2n}=0$ or

$$S(Z,U)=2n(f_1-f_3)g(Z,U)-2\left(f_1-f_3-\frac{\psi}{2n}\right)\eta(Z)\eta(U).$$

In the second case, comparing this equation with (2.7) for Z and U orthogonal to ξ, we get

$$2n(f_1-f_3)=2nf_1+3f_2-f_3. \tag{7.9}$$

It follows that

$$f_3=\frac{3}{1-2n}f_2. \tag{7.10}$$

So comparing the expression $f_3=\frac{3}{1-2n}f_2$ with (2.8) we get

$$S(\xi,\xi)=2n(f_1-f_3)=2n(f_1-f_3)-2\left(f_1-f_3-\frac{\psi}{2n}\right). \tag{7.11}$$

So also in this case $f_1-f_3-\frac{\psi}{2n}=0$. Again if $f_3=\frac{3}{1-2n}f_2$, then from (2.7), $S(X,Y)=(2nf_1+3f_2-f_3)g(X,Y)$ and M is an Einstein manifold.

Conversely, if $f_3=\frac{3}{1-2n}f_2$, then in view of equations (7.2)-(7.5) we have $\mathcal{Q}(\xi,X)\cdot\mathcal{Q}=0$. This leads to the following.

Theorem 7.1. *A* $(2n+1)$*-dimensional* $n>1$ *generalized Sasakian-space-form satisfies* $\mathcal{Q}(\xi,X)\cdot\mathcal{Q}=0$ *if and only if* $f_1-f_3-\frac{\psi}{2n}=0$. *In such a case it is an Einstein manifold.*

In particular, if $\psi=\frac{r}{(2n+1)}$, then the $\mathcal{Q}$ curvature tensor reduces to the concircular curvature tensor. Thus in view of (2.14), $f_1-f_3-\frac{\psi}{2n}=0$ reduces to $3f_2+(2n-1)f_3=0$. Thus we can state the following.

Theorem 7.2. *A* $(2n+1)$*-dimensional* $n>1$ *generalized Sasakian-space-form satisfies* $\mathcal{Z}(\xi,X)\cdot\mathcal{Z}=0$ *if and only if* $f_3=\frac{3}{1-2n}f_2$. *In such a case it is an Einstein manifold.*

Remark 7.1. The above theorem has been proved by De and Yildiz in [13].

8. Example

Example 8.1. Let $N(a,b)$ be a generalized complex space-form, then the warped product $M = \mathbb{R} \times_f N$ endowed with the almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ [1] with

$$f_1 = \frac{a - (f')^2}{f^2}, \quad f_2 = \frac{b}{f^2}, \quad f_3 = \frac{a - (f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$, and f' denotes the derivative of f with respect to t. If we choose $a = 2$, $b = 0$ and $f(t) = t$ with $t \neq 0$, then $f_1 = \frac{1}{t^2}$, $f_2 = 0$ and $f_3 = \frac{1}{t^2}$,

$$\begin{aligned} R(X,Y)Z =& \frac{1}{t^2}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{8.1}$$

From (8.1) it follows that

$$R(X,Y)\xi = 0. \tag{8.2}$$

Moreover in this case $\psi = 2n(f_1 - f_3)$ will be $\psi = 2n\left(\frac{1}{t^2} - \frac{1}{t^2}\right) = 0$. Thus from (3.1) we get

$$\mathcal{Q}(X,Y)\xi = R(X,Y)\xi - \frac{\psi}{2n}[\eta(Y)X - \eta(X)Y] = 0. \tag{8.3}$$

Thus the generalized Sasakian-space-form is $\xi\mathcal{Q}$ flat if and only if $\psi = 2n\left(\frac{1}{t^2} - \frac{1}{t^2}\right) = 0$. Hence, Theorem 3.1 is verified.

Example 8.2. In [1], it was shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$, and f' denotes the derivative of f with respect to t. If we choose $m = 4$ and $f(t) = e^t$, then $M(f_1, f_2, f_3)$ is a 5-dimensional conformally flat generalized Sasakian-space-form with $f_1 = -1$, $f_2 = 0$ and $f_3 = 0$. Therefore, the generalized Sasakian-space-form is ϕ-$\mathcal{Q}$ flat. Hence, Theorem 4.1 is verified.

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