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SCALAR CURVATURE FOR MIDDLE PLANES IN ODD-DIMENSIONAL TORSE-FORMING ALMOST RICCI SOLITONS

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Abstract. We derive identities for the scalar curvature of n respectively $(n+1)$-dimensional planes and their orthogonal complements in an $(2n+1)$-dimensional torse-forming almost Ricci soliton. If the torse-forming vector field is an eigenvector of the Ricci endomorphism for a special eigenvalue these identities characterize the almost Ricci soliton case.

Let (M^m, g) be a Riemannian manifold of dimension $m \geq 2$. Let $p \in M$ and the tangent plane $\pi \subseteq T_pM$ spanned by the orthonormal basis $\{u, v\} \in T_pM$. Then the sectional curvature of π is denoted $K(\pi)$ or $K(u \wedge v)$. It represents the Gaussian curvature of the surface: $(\alpha, \beta) \in \mathbb{R}^2 \to \exp_p(\alpha u + \beta v) \in M$. This well-known notion was generalized to arbitrary dimension of the plane sections in [5].

Definition 1. Fix $2 \leq n \leq m$ and $L \subset T_pM$ a n-dimensional plane section with the orthonormal basis $\{e_1, \dots, e_n\}$. The *scalar curvature* of L is:

$$\tau(L) := \sum_{1 \leq i < j \leq n}^{n} K(e_i \wedge e_j). \tag{1}$$

The Singer-Thorpe characterization of 4-dimensional Einstein spaces from [15] is a duality for the usual sectional curvature:

Theorem 1. *(M^4, g) is an Einstein manifold if and only if $K(\pi) = K(\pi^\perp)$ for any plane section π.*

This result was generalized by Chen et. al in [6], see also Proposition 13.1 of [3, p. 254].

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Theorem 2. *(M^{2n}, g) is Einstein if and only if $\tau(L) = \tau(L^{\perp})$ for any n-plane section L.*

The odd-dimensional case was obtained in [7].

Theorem 3. *(M^{2n+1}, g) is Einstein with the corresponding scalar λ if and only if $\tau(L) + \frac{\lambda}{2} = \tau(L^{\perp})$ for any n-plane section L.*

The last two results have been generalized in: [1] for quasi-Einstein manifolds, [8] and [16] for generalized quasi-Einstein manifolds, [9] for super quasi-Einstein manifolds, [12] for mixed super quasi-Einstein manifold, [13] for mixed generalized quasi-Einstein manifold and [14] for pseudo generalized quasi-Einstein manifold. A very recent generalization to arbitrary dimension of L appears in [11].

In this short note we derive a similar result for *almost Ricci solitons* on (M^{2n+1}, g), i.e., pairs $(V, \lambda) \in \mathfrak{X}(M) \times C^{\infty}(M)$ with V a given vector field and λ a smooth real function satisfying:

$$\mathcal{L}_V g + 2\operatorname{Ric} + 2\lambda g = 0. \tag{2}$$

Here Ric is the Ricci tensor field of g and $\mathcal{L}_V$ is the Lie derivative with respect to V. Also, let Q be the $(1,1)$-version of Ric. For V a Killing or homothetical vector field we recover the Einstein manifolds while if λ is a constant then we call (V, λ) as being a *Ricci soliton*. In order to compute explicitely $\mathcal{L}_V$ we add a technical condition regarding V, namely we suppose to be *torse-forming* (see [2]) which means that for any $X \in \mathfrak{X}(M)$ we have for the Levi-Civita connection ∇:

$$\nabla_X V = fX + \gamma(X)V \tag{3}$$

for a smooth function $f \in C^{\infty}(M)$ and a 1-form $\gamma \in \Omega^1(M)$. Note that torse-forming vector fields appear in several areas of differential geometry and physics as is point out in [10]. From (3) it results:

$$\nabla_V V = [f + \gamma(V)]V, \tag{4}$$

which means that the endomorphism ∇V has V as eigenvector with the eigenvalue $f + \gamma(V)$. Our main result is the following characterization of almost Ricci solitons in terms of scalar curvature.

Proposition 1. *Let $(M^{2n+1}, g, V, \lambda)$ be a torse-forming almost Ricci soliton such that V does not have zeros and $n \geq 2$. Let L_1 be an n-plane orthogonal to V and L_2 an $(n+1)$-plane orthogonal to V. Then:*

$$2[\tau(L_1) - \tau(L_1^{\perp})] = \lambda + f + \gamma(V), \quad 2[\tau(L_2) - \tau(L_2^{\perp})] = -\lambda - f + \gamma(V). \tag{5}$$

Conversely, let $(M^{2n+1}, g, V, f, \gamma)$ be a Riemannian manifold endowed with a torse-forming vector field without zeros and $n \geq 2$. Let $\lambda \in C^{\infty}(M)$ such that the identities (5) hold and V is an eigenvalue of Q with the eigenfunction $-\lambda - f - \gamma(V)$. Then (M, g, V, λ) is an almost Ricci soliton.

Proof We follow the technique of [1]. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of L and $\left\{e_{n+1}, \ldots, e_{2n}, e_{2n+1} = \frac{V}{\|V\|}\right\}$ an orthonormal basis on $L^{\perp}$. Also $\operatorname{Ric}(X, X)$ will be denoted $\mathrm{Ric}(X)$. We have:

(6)
$$\begin{cases} \mathrm{Ric}(e_1) = K(e_1 \wedge e_2) + \cdots + K(e_1 \wedge e_{2n+1}) = -\lambda - g(\nabla_{e_1} V, e_1) = -\lambda - f, \\ \vdots \\ \mathrm{Ric}(e_{2n}) = K(e_{2n} \wedge e_1) + \cdots + K(e_{2n} \wedge e_{2n+1}) = -\lambda - g(\nabla_{e_{2n}} V, e_{2n}) = -\lambda - f, \\ \mathrm{Ric}(e_{2n+1}) = K(e_{2n+1} \wedge e_1) + \cdots + K(e_{2n+1} \wedge e_{2n}) = -\lambda - g(\nabla_{e_{2n+1}} V, e_{2n+1}), \end{cases}$$

$$\mathrm{Ric}(e_{2n+1}) = -\lambda - f - \gamma(V).$$

By summing up the first n equation we get:

$$2\tau(L_1) + \sum_{1 \leq n < j \leq 2n+1} K(e_i \wedge e_j) = -n(\lambda + f). \tag{7}$$

Also, by summing up the last $(n+1)$ equations we obtain:

$$2\tau(L_1^\perp) + \sum_{1 \leq n < j \leq 2n+1} K(e_i \wedge e_j) = -(n+1)(\lambda + f) - \gamma(V) \tag{8}$$

and the first claimed relation follows directly. With a similar argument we derive the second claimed identity.

To obtain the converse fix $p \in M$ and $u \in T_pM$ an arbitrary unit vector orthogonal to $V(p)$. Let $\left\{e_1 = u, \ldots, e_{2n}, e_{2n+1} = \frac{V}{\|V\|}(p)\right\}$ be an orthonormal basis of T_pM and consider $L_1 = \mathrm{span}\{e_2, \ldots, e_{n+1}\}$ respectively $L_2 = \mathrm{span}\{e_1, \ldots, e_{n+1}\}$. Then $L_1^\perp = \mathrm{span}\{e_1, e_{n+2}, \ldots, e_{2n+1}\}$ and $L_2^\perp = \mathrm{span}\{e_{n+2}, \ldots, e_{2n+1}\}$. We get:

$$\begin{aligned} \mathrm{Ric}(u) =& [K(e_1 \wedge e_2) + \ldots + K(e_1 \wedge e_{n+1})] + [K(e_1 \wedge e_{n+2}) + \ldots + K(e_1 \wedge e_{2n+1})] \\ =& [\tau(L_2) - \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j)] + [\tau(L_1^\perp) - \sum_{n+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)] \\ =& [\tau(L_2^\perp) - \tau(L_1) - \frac{1}{2}(\lambda + f - \gamma(V))] \\ &+ [\tau(L_1) - \tau(L_2^\perp) - \frac{1}{2}(\lambda + f + \gamma(V))] = -\lambda - f. \end{aligned}$$

From (3) we have:

$$\mathcal{L}_V g(u, u) = 2g(\nabla_u V, u) = 2f.$$

The last two relations yields: $[\mathcal{L}_V g + 2\,\mathrm{Ric} + 2\lambda g]|_{V^\perp} = 0$. From (4) and the hypothesis about Q we derive:

$$(\mathcal{L}_V g + 2\,\mathrm{Ric} + 2\lambda g)(V, V) = 0$$

and the proof is complete. □

Example 1.

I) $f := 1$, i.e., V is a *irrotational* vector field.
II) $f := 0$, i.e., V is a *recurrent* vector field.
III) Let η be the 1-form dual of V with respect to g. If $\gamma = \eta$ then $\gamma(V) = \|V\|^2$.
IV) If V belongs to the annihilator of γ then V is called *torqued* and Ricci solitons of this type are studied in [4].
V) If V is Killing then we recover a half part of Theorem 3.

Open problem. Let V be a fixed vector field on (M, g). We call it *Ricci-sectional vector field* if for any 2-plane π the quantity:

$$K_V^{\mathrm{Ric}}(u, v) := (\mathcal{L}_V g + 2\,\mathrm{Ric})(u, v)$$

does not depends on the basis $\{u, v\}$ of π. Is an open problem to characterize and exemplify this class of vector fields and to connect this family with the theory of (almost) Ricci solitons.

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