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NEW EXAMPLES OF F-PLANAR CURVES IN 3-DIMENSIONAL WARPED PRODUCT MANIFOLDS

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Dedicated to the memory of Academician Professor Mileva Prvanović

Abstract. The notion of F-planar curves generalizes the magnetic curves and implicitly the geodesics. In this note we obtain all F-planar curves in the Euclidean space $\mathbb{E}^3$ with constant coefficients and we prove that they are generalized helices. We give a detailed study for F-planar curves in the warped product $\mathbb{R} \times_f \mathbb{R}^2$.

1. Introduction

A curve $\gamma : [0,1] \to M$ on a Kähler manifold (M, J, g) is called an *H-planar curve* if its velocity vector γ' obeys the equation

$$\nabla_{\gamma'}\gamma' = a(t)\gamma' + b(t)J\gamma',$$

where $a(t)$ and $b(t)$ are some smooth functions of $t \in [0,1]$ and ∇ is the Levi-Civita connection of g. See, e.g., [13]. Being the real representation of geodesics in the space over a complex algebra, the H-planar curves can be treated as natural analogues of the geodesics in the case of complex manifolds.

Let (M, J, g) and (M', J', g') be two Kähler manifolds. A diffeomorphism $f : M \to M'$ is called an *H-projective mapping* if it maps any H-planar curve on M to a H-planar curve on M'. See, e.g., [18]. The H-projective mapping is a generalization of the projective or the geodesic mapping in (pseudo-)Riemannian geometry. More precisely, two Riemannian metrics g and g' are called *projectively equivalent* if their unparametrized geodesics coincide. A typical question is the following: given g, can we find g' in a non-trivial way, i.e., their Levi-Civita connections do not coincide, which is projectively equivalent to g? In [18] the

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authors proved that there are only trivial examples of projectively equivalent Kähler metrics. Hence, the notion of H-projective equivalence appeared. In [19], the H-planar curves are called *holomorphically flat* curves due to their property that the tangent holomorphic plane moved parallelly along the curve (which is planar) remains holomorphically tangent to the curve. The holomorphic projective correspondence is defined (i.e., H-projective equivalence), as well as the H-projective curvature, as a (1,3)-type tensor field invariant under the H-projective transformations. This notion was generalized to curves in almost quaternionic manifolds in [14] as follows: a curve γ on an almost quaternionic manifold (M, J_1, J_2, J_3) with an affine symmetric connection is called 4-planar if its speed γ', being parallelly transported along this curve, remains in the linear 4-dimensional space generated by γ' and $J_a\gamma'$, for $a = 1, 2, 3$. See also [11]. Later on, holomorphic projective mappings were also considered in [15] from equiaffine spaces onto Kählerian spaces. More generally, on an n-dimensional Riemannian manifold (M, g) consider a (1,1)-type tensor field F. A curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called *F-planar* if its speed remains, under parallel translation along the curve γ, in the distribution generated by the vector γ' and $F\gamma'$ along γ. See [10]. This is equivalent to the fact that the tangent vector γ' satisfies

$$\nabla_{\gamma'}\gamma' = a(t)\gamma' + b(t)F\gamma', \quad \text{for all } t \in I, \tag{1.1}$$

where ∇ is the Levi-Civita connection of g and a, b are two functions depending on t. A special case of F-planar curves was recently studied in [2] under the name of *F-geodesics.*

The F-planar curves generalize the magnetic curves and therefore, the geodesics. More precisely, when $F = \Phi$ is a Lorentz force on the manifold and b is a constant, we obtain the magnetic trajectories corresponding to Φ with strength b. In the absence of F, one gets the geodesics.

2. Magnetic Curves on Riemannian Manifolds

Let (M, g) be an n-dimensional Riemannian manifold. A *magnetic field* is a closed 2-form Ω on M and the *Lorentz force* of a magnetic field Ω on (M, g) is a $(1, 1)$-type tensor field Φ given by

$$g(\Phi(X), Y) = \Omega(X, Y), \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

The *magnetic trajectories* of Ω with strength $q \in \mathbb{R}$ are the curves γ on M that satisfy the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = q\ \Phi(\gamma'). \tag{2.1}$$

We notice that magnetic curves are a special case of F-planar curves.

Since the Lorentz force is skew symmetric, we have

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\nabla_{\gamma'}\gamma', \gamma') = 0,$$

so the magnetic curves (trajectories) have constant speed $v(t) = ||\gamma'|| = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parametrized ($v_0 = 1$), it is called a *normal magnetic curve.*

In the case of a 3-dimensional Riemannian manifold (M, g), 2-forms and vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of the manifold. Therefore, the magnetic fields may be associated with the divergence free vector fields (see e.g. [6]). An important class of magnetic fields, usually called *Killing magnetic fields*, are furnished by Killing vector fields. Recall that a vector field V on M is *Killing* if and only if it satisfies the Killing equation

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0,$$

for every vector fields Y, Z on M, where ∇ is the Levi-Civita connection on M.

Moreover, on the 3-dimensional manifold M, one can define the *cross product* of two vector fields $X, Y \in \mathfrak{X}(M)$ as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \text{for all } Z \in \mathfrak{X}(M).$$

If V is a Killing vector field on M, let $\Omega_V(\cdot,\cdot) = dv_g(V,\cdot,\cdot)$ be the corresponding Killing magnetic field. Then, the Lorentz force of Ω_V is

$$\Phi(X) = V \times X.$$

Consequently, the Lorentz equation (2.1) can be written as

$$\nabla_{\gamma'}\gamma' = q\ V \times \gamma'.$$

The Killing magnetic curves in 3-dimensional manifolds have been intensively studied in the last years. See for example [3, 5, 6, 8, 9, 12, 16, 17].

3. F-planar Curves in a 3-dimensional Euclidean Space

Let $\mathbb{E}^3$ be the Euclidean space and set, as usual, x, y and z the global coordinates. Define the $(1,1)$-type tensor field on $\mathbb{E}^3$ by

$$FX = e_3 \times X, \tag{3.1}$$

for all X tangent to $\mathbb{E}^3$, where $e_3 = (0, 0, 1)$.

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^3$, $\gamma(t) = (x(t), y(t), z(t))$ be a smooth curve satisfying equation (1.1), that is

$$\gamma''(t) = a(t)\gamma'(t) + b(t)F\gamma'(t), \quad \text{for all } t \in I,$$

where a and b are two smooth functions depending on t.

Denote by v the speed of γ, that is $v(t) = \langle\gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}}$. As F is skew-symmetric, we immediately obtain that v satisfies

$$v'(t) = a(t)v(t).$$

We observe that if γ is of constant speed, then a should be zero. Hence, the equation (1.1) may be interpreted as the Lorentz equation for magnetic curves whose strength is variable. Recall that we have the following known situations.

- In the Euclidean plane $\mathbb{E}^2$, this equation generates some aesthetic curves (see, e.g., [20]).

- In the Euclidean space $\mathbb{E}^3$, it is well known that the magnetic curves corresponding to the Killing vector field e_3 and with (constant) strength $q \neq 0$ are circular helices with axis e_3 (see for example [1, 8, 16]).

In the sequel, we study the F-planar curves obtained as the solutions of the equation (3.1) when a and b are real constants, $a \neq 0$, yielding the following classification.

Theorem 3.1. *Let F be the $(1,1)$-type tensor field in $\mathbb{E}^3$ defined by $FX = e_3 \times X$, for any vector field X, where $e_3 = (0,0,1)$. Then, a curve γ in $\mathbb{E}^3$ is a F-planar curve with constant coefficients if and only if it is a generalized helix, explicitly parametrized as: $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^3$, $\gamma(t) = (x(t), y(t), z(t))$, where*

$$\begin{cases} x(t) = x_0 - \dfrac{au_0 + bv_0}{a^2+b^2} + \dfrac{e^{at}}{a^2+b^2}\left[(au_0+bv_0)\cos bt + (bu_0 - av_0)\sin bt\right], \\ y(t) = y_0 - \dfrac{av_0 - bu_0}{a^2+b^2} + \dfrac{e^{at}}{a^2+b^2}\left[(av_0-bu_0)\cos bt + (au_0 + bv_0)\sin bt\right], \\ z(t) = z_0 + \dfrac{w_0}{a}\left(e^{at} - 1\right). \end{cases} \tag{3.2}$$

Here $a \neq 0$, $b, x_0, y_0, z_0, u_0, v_0, w_0 \in \mathbb{R}$.

Proof. Using the global coordinates on $\mathbb{E}^3$, the tangent vector to the curve γ, namely $\gamma'(t) = (x'(t), y'(t), z'(t))$, can be replaced in equation (3.1), which takes the form

$$\begin{cases} x'' = ax' - by', \\ y'' = ay' + bx', \\ z'' = az'. \end{cases} \tag{3.3}$$

If we set the initial data for γ as $\gamma(0) = (x_0, y_0, z_0)$ and $\gamma'(0) = (u_0, v_0, w_0)$, then the system of ordinary differential equations (3.3) has the solution (3.2) given in the theorem. Moreover, the expression (3.2) of γ represents the parametrization of a generalized helix, i.e., the ratio between torsion and curvature is constant. More precisely, using the classical formulas for the curvature and the torsion of a curve γ, we obtain

$$\kappa = \frac{|b|\sqrt{x'(t)^2 + y'(t)^2}}{v(t)^2} \quad \text{and} \quad \tau = \frac{bz'(t)}{v(t)^2}.$$

Replacing the expression of the coordinate function $z(t)$ from (3.2) and using the fact that $v(t)$ is the speed of γ, we immediately get $\frac{\tau}{\kappa} = \pm\frac{w_0}{\sqrt{1-w_0^2}} = \text{const.}$

Conversely, by straightforward computations, we find that the curve γ parametrized by (3.2) is indeed a F-planar curve in $\mathbb{E}^3$. □

Remark 3.1. Unlike magnetic curves, which have constant curvature and torsion, F-planar curves in $\mathbb{E}^3$ are not, in general, helices, but, however, they are generalized helices. Moreover, we notice the next particular situations.

- The curve γ is a line if either $b = 0$ or $u_0 = v_0 = 0$.
- Suppose that $b \neq 0$ and $u_0^2 + v_0^2 \neq 0$. Then γ is planar if and only if $w_0 = 0$.

Let us emphasise that if φ is a transformation in $\mathbb{R}^3$ of the form

$$\varphi(x,y,z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, \alpha z + \beta),$$

where θ, α and β are real constants, then any F-planar curve of $\mathbb{E}^3$ is mapped to a F-planar curve of $\mathbb{E}^3$.

We conclude this section proposing the following

Problem. *Find all transformations of* $\mathbb{R}^3$ *which map F-planar curves to F-planar curves, where $FX = e_3 \times X$.*

See also [7].

4. F-planar Curves in a 3-dimensional Warped Product Manifold

Let $(N, \bar{g})$ be a 2-dimensional Riemannian manifold and consider the warped product $M = \mathbb{R} \times_f N$, where $f : \mathbb{R} \to (0, \infty)$ is the warping function. Then, the metric on M is given by

$$g = dz^2 + f^2(z)\bar{g},$$

where z is the global coordinate on $\mathbb{R}$. The notion of warped product or, more generally, warped bundle was introduced by Bishop and O'Neill in [4] in order to construct a large variety of manifolds of negative curvature.

Denoting by $\bar{\nabla}$ and ∇ the Levi-Civita connections on N and M respectively, we recall the following formulas:

$$\begin{cases} \nabla_X Y = \bar{\nabla}_X Y - \dfrac{f'}{f} g(X,Y)\xi, \\ \nabla_X \xi = \nabla_\xi X = \dfrac{f'}{f} X, \\ \nabla_\xi \xi = 0, \end{cases} \tag{4.1}$$

for any X, Y tangent to N, where we set $\xi = \frac{\partial}{\partial z}$.

Denoting by J the usual complex structure on N, we can naturally extend it to a $(1,1)$-type tensor field on M, as follows:

$$FX = JX \text{ for any } X \text{ tangent to } N \text{ and } F\xi = 0.$$

It is straightforward to check that F is not, in general, compatible with the metric g.

Let now $\gamma : I \to M$ be a F-planar curve on M, that is, the equation (1.1) is satisfied. Denoting by $\bar{\gamma}$ its projection on N, the curve γ may be written as $\gamma(t) = \big(z(t), \bar{\gamma}(t)\big)$, for all $t \in I$. We have the velocity of γ, as follows $\dot{\gamma} = \dot{z}\partial_z + \dot{\bar{\gamma}}$. Then, we compute the acceleration of γ $\nabla_{\dot{\gamma}}\dot{\gamma} = \bar{\nabla}_{\dot{\bar{\gamma}}}\dot{\bar{\gamma}} + \ddot{z}\partial_z + \dot{z}\nabla_{\dot{\bar{\gamma}}}\partial_z + \dot{z}^2\nabla_{\partial_z}\partial_z + \dot{z}\nabla_{\partial_z}\dot{\bar{\gamma}}$. Now, on one hand, using formula (4.1) we obtain:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \bar{\nabla}_{\dot{\bar{\gamma}}}\dot{\bar{\gamma}} + 2\dot{z}\frac{f'}{f}\dot{\bar{\gamma}} + \left(\ddot{z} + \frac{f'}{f} g(\dot{\bar{\gamma}}, \dot{\bar{\gamma}})\right)\partial_z. \tag{4.2}$$

On the other hand, the equation (1.1) becomes:

$$\nabla_{\dot\gamma}\dot\gamma = a(t)\left(\dot z\partial_z + \dot{\bar\gamma}\right) + b(t)J\dot{\bar\gamma}. \tag{4.3}$$

Combining (4.2) and (4.3) and identifying the vectors tangent to $\mathbb{R}$, respectively to N, and finally using the fact that $\dot\gamma$ has unit length, i.e., $\dot z^2 + f^2 g(\dot{\bar\gamma},\dot{\bar\gamma}) = 1$, we get

$$\begin{cases} \ddot z(t) - \dfrac{f'(z(t))}{f(z(t))}\left(1-\dot z^2(t)\right) - a(t)\dot z(t) = 0,\\ \bar\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} = \left[a(t) - 2\dot z(t)\dfrac{f'(z(t))}{f(z(t))}\right]\dot{\bar\gamma} + b(t)J\dot{\bar\gamma}. \end{cases}$$

The second equation shows that $\bar\gamma$ is an H-planar curve on N.

In the sequel we give a detailed study when N is the Euclidean 2-plane, proving the next classification result.

Theorem 4.1. *Let F be a $(1,1)$-type tensor field on the warped product manifold $\mathbb{R}\times_f\mathbb{R}^2$ defined as $FX = JX$ for any X tangent to $\mathbb{R}^2$ and $F\partial_z = 0$, where J is the natural complex structure on $\mathbb{R}^2$. Then, a curve γ is a F-planar curve with constant coefficients in the warped product manifold $\mathbb{R}\times_f\mathbb{R}^2$, where $f:\mathbb{R}\to(0,\infty)$ is the warping function, if and only if it is given by one of the following cases:*

(a) *a line parallel to z-axis;*
(b) *$\gamma : I\subseteq\mathbb{R}\to\mathbb{E}^3$, $\gamma(t) = (x(t),y(t),z(t))$, where*

$$\begin{cases} x(t) = x_0 + c_1 I_1(t) + c_2 I_2(t),\\ y(t) = y_0 + c_1 I_2(t) - c_2 I_1(t),\\ z(t) = G^{-1}(t-t_0). \end{cases} \tag{4.4}$$

Here $G(z) = \int_{z_0}^{z}\dfrac{f(u)}{\sqrt{f^2(u)-\lambda^2}}du$, $I_1(t) = \int_{t_0}^{t}\dfrac{\cos(bu)}{f^2(z(u))}du$, $I_2(t) = \int_{t_0}^{t}\dfrac{\sin(bu)}{f^2(z(u))}du$, such that $\lambda, c_1, c_2\in\mathbb{R}$ satisfy $c_1^2 + c_2^2 = \lambda^2$. Moreover, $x_0, y_0, t_0, b\in\mathbb{R}$.

Proof. Let M be $\mathbb{R}^3$ with global coordinates x, y and z endowed with the following Riemannian metric

$$g = dz^2 + f^2(z)\left(dx^2+dy^2\right),$$

where $f:\mathbb{R}\to(0,\infty)$ is a smooth function. Define the $(1,1)$-type tensor field F on M as follows

$$F\partial_x = \partial_y,\quad F\partial_y = -\partial_x,\quad F\partial_z = 0,$$

where we put $\partial_x = \frac{\partial}{\partial x}$, and so on. Note that F is skew-symmetric, that is, Ω defined by

$$\Omega(X,Y) = g(FX,Y),$$

for all X, Y tangent to M, is a 2-form. Moreover, it is easy to show that Ω is closed if and only if f is constant.

Let us consider a smooth curve γ on M satisfying the equation (1.1). As F is skew-symmetric, γ has constant speed if and only if the function a vanishes identically.

From now on we take $a = 0$ and b a real constant. Without loss of generality, we suppose that γ is parametrized by arc-length, that is $g(\dot\gamma, \dot\gamma) = 1$, where $\dot\gamma = \frac{d\gamma}{dt}$. This condition can be expressed as

$$\dot z^2(t) + f^2(z(t))\left(\dot x^2(t) + \dot y^2(t)\right) = 1, \quad \text{for all } t \in \mathbb{R}. \tag{4.5}$$

Equation (1.1) becomes

$$\begin{cases} \ddot x + 2\dot x \dot z \dfrac{f'(z)}{f(z)} = -b\dot y, \\ \ddot y + 2\dot y \dot z \dfrac{f'(z)}{f(z)} = \ b\dot x, \\ \ddot z - (\dot x^2 + \dot y^2) f'(z) f(z) = 0. \end{cases} \tag{4.6}$$

Combining equation (4.5) with the third equation in (4.6) we obtain

$$\ddot z - \frac{f'(z)}{f(z)}(1 - \dot z^2) = 0.$$

This implies the following

$$\frac{d}{dt}\left[\left(1 - \dot z^2\right) f^2(z)\right] = 0.$$

Hence, there exists a constant $\lambda \geq 0$ such that

$$1 - \dot z^2 = \frac{\lambda^2}{f^2(z)}. \tag{4.7}$$

Note that $\lambda = 0$ is equivalent to $\dot z = \pm 1$, case when the curve γ is a line parallel to the z-axis, proving case (a) of the theorem.

In the sequel this case is excluded.

Let us define the following function

$$G(z) = \int_{z_0}^{z} \frac{f(u)}{\sqrt{f^2(u) - \lambda^2}} du, \tag{4.8}$$

which is strictly increasing, hence bijective.

From (4.8) and using (4.7), we compute $\frac{d}{dt} G(z(t)) = \pm 1$. Thus, we find (up to orientation)

$$z(t) = G^{-1}(t - t_0), \quad t_0 \in \mathbb{R}. \tag{4.9}$$

Let us consider the following two functions

$$A(t) = \dot x(t) f^2(z(t)) \quad \text{and} \quad B(t) = \dot y(t) f^2(z(t)). \tag{4.10}$$

The first two equations in (4.6) can be rewritten as

$$\frac{dA}{dt} = -bB(t), \quad \frac{dB}{dt} = bA(t).$$

We obtain

$$\begin{cases} A(t) = c_1 \cos(bt) + c_2 \sin(bt), \\ B(t) = c_1 \sin(bt) - c_2 \cos(bt), \end{cases} \tag{4.11}$$

where c_1 and c_2 are real constants. Since on one hand $A^2 + B^2 = c_1^2 + c_2^2$ and on the other hand $A^2 + B^2 = (\dot{x}^2 + \dot{y}^2) f^2(z(t)) = (1 - \dot{z}^2) f^2(z(t))$, we must have, in virtue of (4.7), that $c_1^2 + c_2^2 = \lambda^2$.

At this point, formulas (4.10) and (4.11) suggest us to set the following two functions:

$$I_1(t) = \int_{t_0}^{t} \frac{\cos(bu)}{f^2(z(u))} du \quad \text{and} \quad I_2(t) = \int_{t_0}^{t} \frac{\sin(bu)}{f^2(z(u))} du.$$

We obtain now the first two components of the curve γ

$$\begin{cases} x(t) = x_0 + c_1 I_1(t) + c_2 I_2(t), \\ y(t) = y_0 + c_1 I_2(t) - c_2 I_1(t). \end{cases} \tag{4.12}$$

Combining (4.9) and (4.12), case (b) of the theorem is proved.

Conversely, one can check by straightforward computations that the curves described in the theorem are indeed F-planar curves in $\mathbb{R} \times_f \mathbb{R}^2$. $\square$

Remark 4.1. A special situation of case (b) in Theorem 4.1 is obtained when $\dot{z} = 0$ (on an open interval), which leads to $z = z_0$ (constant), $f'(z_0) = 0$ and

$$\begin{cases} x(t) = x_0 + \dfrac{1}{bf(z_0)} \left[\sin(bt - \mu) - \sin(bt_0 - \mu)\right], \\ y(t) = y_0 - \dfrac{1}{bf(z_0)} \left[\cos(bt - \mu) - \cos(bt_0 - \mu)\right], \end{cases}$$

where μ is a real constant. Thus, the curve γ is a horizontal circle.

In the following we give some examples of metrics obtained for different values of f and we draw the corresponding F-planar curves.

1. If $f = 1$, that is g is the Euclidean metric, the 2-form Ω is closed and hence the F-planar curve γ is a magnetic curve with strength $b \in \mathbb{R}$. The general solution (4.9) and (4.12) yields the known result that γ is a cylindrical helix parametrized by

$$\begin{cases} x(t) = x_0 + \dfrac{\lambda}{b} \left[\sin(bt - \mu) - \sin(bt_0 - \mu)\right], \\ y(t) = y_0 - \dfrac{\lambda}{b} \left[\cos(bt - \mu) - \cos(bt_0 - \mu)\right], \\ z(t) = z_0 + \sqrt{1 - \lambda^2}(t - t_0), \quad t \in \mathbb{R}, \end{cases}$$

where $\lambda \in [0, 1]$ and $\mu \in \mathbb{R}$ are constants. Obviously, the degenerate situations are obtained when $\lambda = 0$ (a vertical line) and when $\lambda = 1$ (an horizontal circle).

2. If $f(z) = z$, $z > 0$, the metric g is known as a *cone metric*. We obtain

$$z(t) = \sqrt{\lambda^2 + (t - t_1)^2}$$

for a certain constant t_1 depending on t_0 and z_0.

In order to draw the curve, we set $\lambda = 1$, $t_1 = 0$, $c_1 = 1$ and $c_2 = 0$. See Figure 1, for $t \in (-1, 1)$.

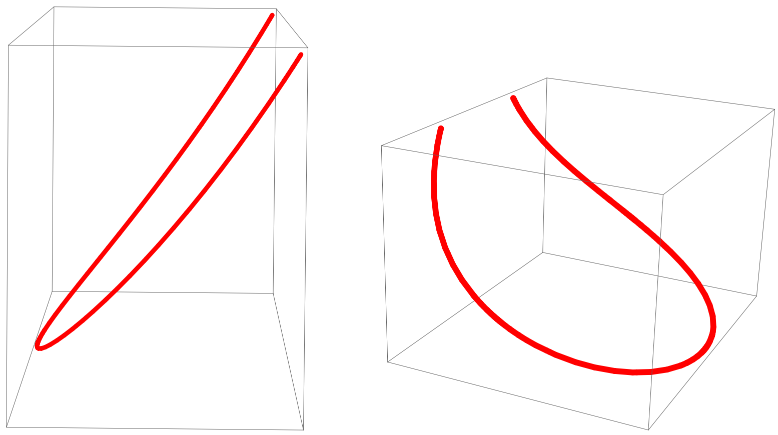

FIGURE 1. $b = 1$ (left), $b = 3$ (right)

3. Let us consider now $f(z) = \frac{1}{z}$, $z > 0$. We obtain

$$z(t) = \sin(t - t_1), \quad t_1 \in \mathbb{R}.$$

As before, we set $\lambda = 1$, $t_1 = 0$, $c_1 = 1$ and $c_2 = 0$ and we draw the curve in the interval $(0, \pi)$, for different values of b. See Figure 2 and Figure 3.

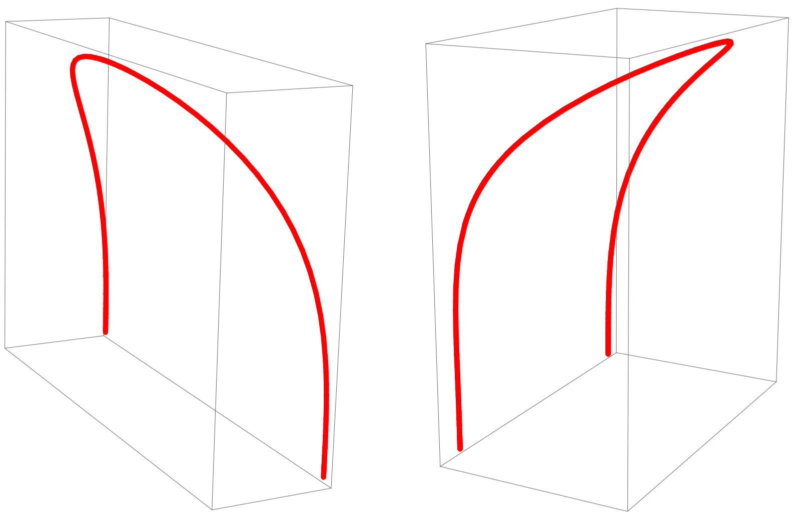

FIGURE 2. $b = 1$ (left), $b = 2$ (right)

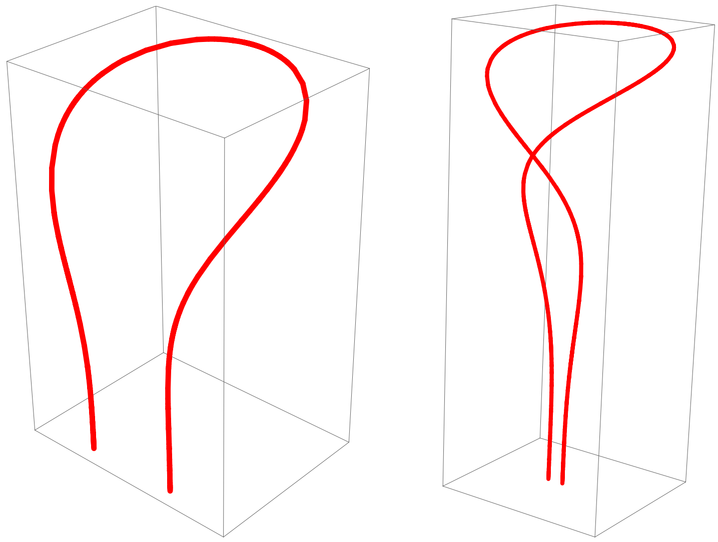

FIGURE 3. $b = 3$ (left), $b = 5$ (right)

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