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# FRACTIONAL DIFFERENCE EQUATIONS OF VOLTERRA TYPE

JAGAN MOHAN JONNALAGADDA

Abstract. In this article, we devote our attention to study a few qualitative properties of linear fractional nabla difference equations of Volterra type. Under appropriate assumptions, we examine the existence, uniqueness, boundedness and stability of the solutions by means of the resolvent kernel. Examples are provided to demonstrate the applicability of established results.

## 1. Introduction

The theory of implicit and explicit Volterra difference equations has gained attention due to its applicability in various fields of science and engineering. These equations arise in the investigation of discretization methods for Volterra integrodifferential equations. For a detailed discussion on this topic, we refer [4, 7, 17, 28].

Fractional calculus [22] is a new branch of mathematics that deals with the generalization of differentiation and integration to arbitrary order. It represents a natural instrument to model non-local phenomena either in space or time. In various problems of science and engineering, fractional differential equations have been proved to be valuable tools in modelling many phenomena. In recent times, the theory and applications of fractional integrodifferential equations have been the focus of many studies by virtue of its dense aspects in diverse fields such as signal processing, mechanics, econometrics, fluid dynamics, nuclear reactor dynamics, acoustic waves and electromagnetics. However, it is quite difficult to obtain the closed form solutions for various classes of fractional integrodifferential equations. But, the development of high speed digital computing machinery has allowed the use of fractional nabla difference equations as approximations to their continuous counterparts. Therefore, many

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effective methods for obtaining numerical solutions of fractional integrodifferential equations have been presented recently [3, 6, 8, 13, 15, 16, 18–20, 23–27, 29, 30].

On the other hand, the idea of fractional nabla difference is very recent. The combined efforts of a number of researchers during the past two decades laid a fairly strong basic theory of fractional nabla difference equations [5]. In spite of the existence of a substantial mathematical theory of fractional nabla difference equations, the theory of Volterra type fractional nabla difference equations is not yet initiated in parallel. Motivated by the necessity to study the latter qualitatively, in this article, we consider a particular class of Volterra fractional nabla difference equations of convolution type which assumes the form

$$\left(\nabla_0^\alpha u\right)(t) = \mu u(t) + \sum_{s=0}^{t} b(t-s)u(s), \quad t \in \mathbb{N}_1, \tag{1.1}$$

$$\left(\nabla_{0*}^\alpha u\right)(t) = \mu u(t) + \sum_{s=0}^{t} b(t-s)u(s), \quad t \in \mathbb{N}_1, \tag{1.2}$$

where $\nabla_0^\alpha$, $\nabla_{0*}^\alpha$ are the $\alpha^{\text{th}}$-order Riemann-Liouville and Caputo type nabla difference operators, respectively, $u, b : \mathbb{N}_0 \to \mathbb{R}$ and $\alpha, \mu \in \mathbb{R}$ such that $0 < \alpha < 1$.

For this purpose, we analyse the alternative form of (1.1) and (1.2) which turns out to be a Volterra difference equation of convolution type of the form

$$u(t) = f(t) + \sum_{s=0}^{t} k(t-s;\lambda)u(s), \quad t \in \mathbb{N}_0, \tag{1.3}$$

where $f$, $k : \mathbb{N}_0 \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. For simplicity, we choose the convolution kernel

$$k(t;\lambda) = \frac{\lambda (t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \quad t \in \mathbb{N}_0. \tag{1.4}$$

The significance of (1.3)-(1.4) lies in its association with fractional nabla calculus as described follows.

Let $u(0) = u_0$. Consider a linear fractional nabla difference equation of Riemann-Liouville type

$$\left(\nabla_0^\alpha u\right)(t) = \lambda u(t), \quad 0 < \alpha < 1, \lambda \neq 1, t \in \mathbb{N}_1. \tag{1.5}$$

Then, $u(t)$ is a solution of (1.5) if and only if

$$u(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(0) + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_0. \tag{1.6}$$

We observe that (1.6) belongs to the class of (1.3)-(1.4) with

$$f(t) = (1-\lambda)\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(0).$$

If we consider a fractional nabla difference equation of Caputo type of the form

$$\left(\nabla_{0*}^\alpha u\right)(t) = \lambda u(t), \quad 0 < \alpha < 1, \lambda \neq 1, t \in \mathbb{N}_1, \tag{1.7}$$

then $u(t)$ is a solution of (1.7) if and only if

$$u(t) = u(0) + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_0. \tag{1.8}$$

Clearly, (1.8) also belongs to the class of (1.3)-(1.4) with

$$f(t) = \left[1 - \lambda \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right] u(0).$$

Consequently, the qualitative properties of solutions of (1.5) and (1.7) are similar to that of (1.3)-(1.4).

The present article is organized as follows: Section 2 contains preliminaries on fractional nabla calculus. In section 3, we construct the resolvent kernel associated with (1.3)-(1.4) using discrete Laplace transform. In Section 4, we establish sufficient conditions on boundedness and stability properties of solutions of (1.3)-(1.4). As an application, we study a few qualitative properties of the following linear nonhomogeneous fractional nabla difference equations in Section 5

$$\left(\nabla_0^\alpha u\right)(t) = \lambda u(t) + g(t), \quad t \in \mathbb{N}_1, \tag{1.9}$$

$$\left(\nabla_{0*}^\alpha u\right)(t) = \lambda u(t) + g(t), \quad t \in \mathbb{N}_1, \tag{1.10}$$

where $g : \mathbb{N}_0 \to \mathbb{R}$. Using the results obtained in Sections 3 and 4, we establish sufficient conditions on the qualitative behaviour of (1.1) and (1.2) in Section 6.

## 2. Preliminaries

Throughout this article, we use the following notations, definitions and known results of fractional nabla calculus [1, 5, 7, 14]. Denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For any $a$, $b \in \mathbb{R}$ such that $a < b$, define $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$. Assume that empty sums and products are taken to be 0 and 1, respectively.

**Definition 2.1** (Gamma Function)**.** For any $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds, \quad t > 0,$$
$$\Gamma(t+1) = t\Gamma(t).$$

**Definition 2.2** (Rising Factorial Function)**.** For any $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $\alpha \in \mathbb{R}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of rising factorial functions.

**Theorem 2.1.** *Assume that the following factorial functions are well defined:*

(a) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$;
(b) *if* $t \le r$ *then* $t^{\overline{\alpha}} \le r^{\overline{\alpha}}$;
(c) *if* $\alpha < t \le r$ *then* $r^{\overline{-\alpha}} \le t^{\overline{-\alpha}}$;
(d) $(t+1)^{\alpha-1} \le (t+1)^{\overline{\alpha-1}} \le t^{\alpha-1}$, $0 \le \alpha \le 1$;
(e) $(t+b)^{\overline{a-b}} = t^{a-b}\left[1 + O\left(\frac{1}{t}\right)\right]$, $|t| \to \infty$.

**Definition 2.3.** Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The $\alpha^{\text{th}}$-order backward (nabla) sum of $u$ is given by

$$\left(\nabla_a^{-\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s+1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

**Definition 2.4.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

(a) (Nabla Difference) The first order backward (nabla) difference of $u$ is defined by

$$(\nabla u)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the $N^{th}$-order nabla difference of $u$ is defined recursively by

$$\left(\nabla^N u\right)(t) = \left(\nabla\left(\nabla^{N-1} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

(b) (R-L Fractional Nabla Difference) The Riemann-Liouville type $\alpha^{\text{th}}$-order nabla difference of $u$ is given by

$$\left(\nabla_a^{\alpha} u\right)(t) = \left(\nabla^N \left(\nabla_a^{-(N-\alpha)} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

(c) (Caputo Fractional Nabla Difference) The Caputo type $\alpha^{\text{th}}$-order nabla difference of $u$ is given by

$$\left(\nabla_{a*}^{\alpha} u\right)(t) = \left(\nabla_a^{-(N-\alpha)} \left(\nabla^N u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

**Theorem 2.2** (Power Rule)**.** *Let* $\alpha \in \mathbb{R}^+$ *and* $\nu \in \mathbb{R}$. *Assume that the following factorial functions are well defined*

$$\nabla_a^{-\alpha}(t-a+1)^{\overline{\nu}} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}(t-a+1)^{\overline{\nu+\alpha}}.$$

**Theorem 2.3.** [9] *For any* $\alpha > 0$, *the following equality holds:*

$$\nabla_{a+1}^{-\alpha} \nabla u(t) = \nabla \nabla_a^{-\alpha} u(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(a).$$

Mittag-Leffler [22] and Agarwal [22] introduced the one and two parameter Mittag-Leffler functions which play a very important role in the theory of fractional calculus.

**Definition 2.5.** [22] The one and two parameter Mittag-Leffler functions are defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha$, $\beta > 0$ and $t \in \mathbb{R}$.

Recently, Wang et al. [12] have obtained the following estimations on $E_\alpha$, $E_{\alpha,\alpha}$ and $E_{\alpha,\alpha+\beta}$.

**Lemma 2.1.** [12] *Let* $0 < \alpha, \beta < 1$. *The functions* $E_\alpha$, $E_{\alpha,\alpha}$ *and* $E_{\alpha,\alpha+\beta}$ *are non-negative and for any* $\lambda > 0$ *and* $t \in [0, T]$, $T > 0$,

$$E_\alpha(-\lambda t^\alpha) \leq 1, \quad E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}, \quad E_{\alpha,\alpha+\beta}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha+\beta)}.$$

Atsushu Nagai [2] and Atici & Eloe [10] defined the one and two parameter nabla Mittag-Leffler functions of fractional nabla calculus as follows.

**Definition 2.6.** [2,10] The one and two parameter nabla Mittag-Leffler functions are defined by

$$F_\alpha(\lambda, t^{\overline{\alpha}}) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)}, \quad F_{\alpha,\beta}(\lambda, t^{\overline{\alpha}}) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\overline{\alpha k}}}{\Gamma(\alpha k + \beta)},$$

where $\alpha$, $\beta > 0$, $|\lambda| < 1$ and $t \in \mathbb{N}_0$.

Analogous to Lemma 2.1, here we obtain the estimates on $F_\alpha$, $F_{\alpha,\alpha}$ and $F_{\alpha,\beta}$.

**Lemma 2.2.** *Let* $0 < \alpha, \beta < 1$. *The functions* $F_\alpha$, $F_{\alpha,\alpha}$ *and* $F_{\alpha,\beta}$ *are non-negative and for any* $0 < \lambda < 1$ *and* $t \in \mathbb{N}_0$,

$$F_\alpha(-\lambda, t^{\overline{\alpha}}) \leq 1, \quad F_{\alpha,\alpha}(-\lambda, t^{\overline{\alpha}}) \leq \frac{1}{\Gamma(\alpha)}, \quad F_{\alpha,\alpha+\beta}(-\lambda, t^{\overline{\alpha}}) \leq \frac{1}{\Gamma(\alpha+\beta)}.$$

*Proof.* Clearly $F_\alpha(-\lambda, 0^{\overline{\alpha}}) = 1$. For $t \in \mathbb{N}_1$, consider

$$\begin{aligned}
F_\alpha(-\lambda, t^{\overline{\alpha}}) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)} \\
&= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k + 1)} \frac{\Gamma(t + \alpha k)}{\Gamma(t)} \\
&= \frac{1}{\Gamma(t)} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k + 1)} \Big[ \int_0^\infty e^{-s} s^{(t+\alpha k)-1} ds \Big] \\
&= \frac{1}{\Gamma(t)} \int_0^\infty e^{-s} s^{t-1} \Big[ \sum_{k=0}^{\infty} \frac{(-\lambda s^\alpha)^k}{\Gamma(\alpha k + 1)} \Big] ds \\
&= \frac{1}{\Gamma(t)} \lim_{T \to \infty} \int_0^T e^{-s} s^{t-1} E_\alpha(-\lambda s^\alpha) ds \\
&\leq \frac{1}{\Gamma(t)} \int_0^\infty e^{-s} s^{t-1} ds \\
&= 1.
\end{aligned}$$

Similarly, we can prove the other results. Hence the proof. ☐

Acar & Atici [21] studied exponential functions of fractional nabla calculus along with some relations to the nabla Mittag-Leffler functions.

**Definition 2.7.** [21] The exponential function of fractional nabla calculus is defined by

$$\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = (1-\lambda)\sum_{k=0}^{\infty} \frac{\lambda^k (t+1)^{\overline{(k+1)\alpha-1}}}{\Gamma((k+1)\alpha)},$$

where $0 < \alpha < 1$, $|\lambda| < 1$ and $t \in \mathbb{N}_0$.

We observe the following properties of exponential functions from the literature [21].

**Theorem 2.4.** [21] *Let $0 < \alpha < 1$, $|\lambda| < 1$ and $t \in \mathbb{N}_0$. Then we have the following:*

(a) $\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \geq 0$;
(b) $\hat{e}_{\alpha,\alpha}(\lambda, 0^{\overline{\alpha}}) = 1$;
(c) $\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = (1-\lambda)(t+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t+\alpha)^{\overline{\alpha}})$.

Atici & Eloe [10] introduced the discrete Laplace transform, known as $N$-transform, to solve fractional nabla difference equations.

**Definition 2.8.** [10] For any $u : \mathbb{N}_a \to \mathbb{R}$, the $N$-transform of $u$ is defined by

$$N_a\Big[u(t)\Big] = \sum_{j=a}^{\infty} u(j)(1-z)^{j-1},$$

for each $z \in \mathbb{C}$ for which the series converges.

**Definition 2.9.** [11] Let $u : \mathbb{N}_a \to \mathbb{R}$. $u$ is said to be of exponential order $r$, $r > 0$, if there exists a constant $A > 0$ such that

$$\Big|u(t)\Big| \leq A r^{-t},$$

for sufficiently large $t \in \mathbb{N}_a$.

**Theorem 2.5.** [11] *Suppose $u$ is of exponential order $r$, $r > 0$. Then, $N_a\left[u(t)\right]$ exists for each $z$ lies inside the open ball $B_1(r) = \{z \in \mathbb{C} : |1-z| < r\}$.*

**Definition 2.10.** [10] For any $u$, $v : \mathbb{N}_a \to \mathbb{R}$, the convolution of $u$ and $v$ is defined by

$$\left(u *_a v\right)(t) = \sum_{s=a}^{t} u(t+a-s+1)v(s).$$

**Theorem 2.6.** [10] *We observe the following properties of $N$-transform.*

(a) $N_a\left[\left(u *_a v\right)(t)\right] = N_1\left[u(t+a)\right] N_a\left[v(t)\right]$.
(b) $N_a\left[(t-a+1)^{\overline{\alpha}}\right] = (1-z)^{a-1}\frac{\Gamma(\alpha+1)}{z^{\alpha+1}}$, $|1-z| < 1$, $\alpha \in \mathbb{R} \setminus \{\ldots, -3, -2, -1\}$.
(c) $N_a\left[(t-a+1)^{\overline{\alpha}}\mu^{-t}\right] = (1-z)^{a-1}\frac{\mu^{\alpha+1-a}\Gamma(\alpha+1)}{(z+\mu-1)^{\alpha+1}}$, $|1-z| < \mu$.
(d) $N_a\left[\left(\nabla_a^{-\alpha} u\right)(t)\right] = z^{-\alpha} N_a\left[u(t)\right]$, $\alpha > 0$.
(e) $N_a\left[\left(\nabla_a^{\alpha} u\right)(t)\right] = z^{\alpha} N_a\left[u(t)\right]$, $0 < \alpha < 1$.
(f) $N_{a+1}\left[\left(\nabla_a^{\alpha} u\right)(t)\right] = z^{\alpha} N_a\left[u(t)\right] - (1-z)^{a-1}u(a)$, $0 < \alpha < 1$.
(g) $N_a\left[F_{\alpha}(\lambda, (t-a+1)^{\overline{\alpha}})\right] = \frac{(1-z)^{a-1}z^{\alpha-1}}{(z^{\alpha}-\lambda)}$, $|z^{\alpha}| > |\lambda|$.
(h) $N_a\left[\hat{e}_{\alpha,\alpha}(\lambda, (t-a)^{\overline{\alpha}})\right] = \frac{(1-z)^{a-1}(1-\lambda)}{(z^{\alpha}-\lambda)}$, $|z^{\alpha}| > |\lambda|$.

## 3. Resolvent Kernel of (1.3)–(1.4)

In this section, we obtain the resolvent kernel associated with (1.3)-(1.4), using $N$-transform.

**Definition 3.1.** [28] The resolvent kernel $r(t;\lambda)$ for the kernel $k(t;\lambda)$ in (1.3) is defined as the solution of the summation equation

$$r(t) = -k(t) + \sum_{s=0}^{t} r(t-s;\lambda)k(s), \quad t \in \mathbb{N}_0. \tag{3.1}$$

The equivalent form of (3.1) is given by

$$r(t) = -k(t) + \sum_{s=0}^{t} k(t-s;\lambda)r(s), \quad t \in \mathbb{N}_0.$$

Clearly $r : \mathbb{N}_0 \to \mathbb{R}$. The condition $\lambda \neq 1$ guarantees the existence of unique solution $r(t;\lambda)$ of (3.1). Further, the solution of (1.3) is given by

$$u(t) = f(t) - \sum_{s=0}^{t} r(t-s;\lambda)f(s), \quad t \in \mathbb{N}_0. \tag{3.2}$$

Now we express the $N$-transform of $r(t;\lambda)$ in terms of the N-transform of $k(t;\lambda)$. Applying the $N_0$-transform to (1.3), we get

$$N_0\left[u(t)\right] = N_0\left[f(t)\right] + N_1\left[k(t-1)\right] N_0\left[u(t)\right]. \tag{3.3}$$

Applying the $N_0$-transform to (3.2), we get

$$N_0\left[u(t)\right] = N_0\left[f(t)\right] - N_1\left[r(t-1)\right] N_0\left[f(t)\right]. \tag{3.4}$$

Eliminating $N_0\left[u(t)\right]$ and $N_0\left[f(t)\right]$ from (3.3) and (3.4), we obtain

$$N_1\Big[r(t-1)\Big] = \frac{N_1\left[k(t-1)\right]}{N_1\left[k(t-1)\right]-1}. \tag{3.5}$$

Next, we determine $r(t;\lambda)$ from (3.5). Applying inverse $N_1$-transform to (3.5), we get

$$\begin{aligned} r(t-1;\lambda) &= N_1^{-1}\left[\frac{N_1\left[k(t-1)\right]}{N_1\left[k(t-1)\right]-1}\right] \\ &= -\lambda N_1^{-1}\left[\frac{1}{z^{\alpha}-\lambda}\right] \\ &= -\frac{\lambda}{(1-\lambda)}\hat{e}_{\alpha,\alpha}(\lambda,(t-1)^{\overline{\alpha}}), \end{aligned}$$

implies

$$r(t;\lambda) = -\frac{\lambda}{(1-\lambda)}\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}), \quad |\lambda|<1, t \in \mathbb{N}_0. \tag{3.6}$$

Thus, a necessary and sufficient condition for the existence and unicity of the solution of (1.3)-(1.4) is that $|\lambda| < 1$. Finally, using (3.6) in (3.2), the unique solution of (1.3)-(1.4) is given by

$$u(t) = f(t) + \frac{\lambda}{(1-\lambda)} \sum_{s=0}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) f(s), \quad |\lambda| < 1, t \in \mathbb{N}_0. \tag{3.7}$$

## 4. Boundedness and Stability of (1.3)-(1.4)

This section deals with boundedness and stability properties of (1.3)-(1.4) in $l^\infty$, the Banach space comprising of bounded sequences of real numbers with respect to the supremum norm defined by

$$\|u\|_\infty = \sup_{t\in\mathbb{N}_0} |u(t)|,$$

for any $u = \{u(t)\}_{t\in\mathbb{N}_0} \in l^\infty$. Let $f \in l^\infty$. From (3.7), we have

$$\begin{aligned}
|u(t)| &\le |f(t)| + \frac{|\lambda|}{(1-\lambda)} \sum_{s=0}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) |f(s)| \\
&\le \|f\|_\infty + \frac{|\lambda|}{(1-\lambda)} \|f\|_\infty \sum_{s=0}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) \\
&= \|f\|_\infty + |\lambda| \|f\|_\infty \sum_{s=0}^{t} (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t-s+\alpha)^{\overline{\alpha}}) \\
&= \|f\|_\infty + |\lambda| \|f\|_\infty \sum_{s=0}^{t} (t-s+1)^{\overline{\alpha-1}} \sum_{k=0}^{\infty} \frac{\lambda^k (t-s+\alpha)^{\overline{\alpha k}}}{\Gamma(\alpha k+\alpha)} \\
&= \|f\|_\infty + |\lambda| \|f\|_\infty \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \sum_{s=0}^{t} (t-s+1)^{\overline{\alpha-1}} (t-s+\alpha)^{\overline{\alpha k}} \\
&= \|f\|_\infty + |\lambda| \|f\|_\infty \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \sum_{s=0}^{t} (t-s+1)^{\overline{\alpha k+\alpha-1}} \\
&= \|f\|_\infty + |\lambda| \|f\|_\infty \sum_{k=0}^{\infty} \frac{\lambda^k (t+1)^{\overline{\alpha k+\alpha}}}{\Gamma(\alpha k+\alpha+1)} \\
&= \|f\|_\infty + \frac{|\lambda|}{\lambda} \|f\|_\infty \Big[ F_\alpha(\lambda, (t+1)^{\overline{\alpha}}) - 1 \Big]. \qquad (4.1)
\end{aligned}$$

We make the following observations from (4.1).

(a) If $\lambda \in (-1, 0]$ then $\|u\|_\infty \le 2\|f\|_\infty$. So, the unique solution of (1.3)-(1.4) is bounded in $l^\infty$ if $\lambda \in (-1, 0]$.

(b) The unique solution of (1.3)-(1.4) is uniformly stable in $l^\infty$ if $\lambda \in (-1, 0]$.

We also define

$$l_0^\infty = \left\{ u : u \in l^\infty \text{ such that } \lim_{t\to\infty} u(t) = 0 \right\}.$$

Then, $l_0^\infty$ is also a Banach space with respect to the supremum norm. Now, assume that $-1 < \lambda \leq 0$ and $f \in l_0^\infty$. Then, for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$|f(t)| < \epsilon, \quad t \in \mathbb{N}_N.$$

From (3.7), we have

$$\begin{aligned}
|u(t)| =& \Big| f(t) + \frac{\lambda}{(1-\lambda)} \sum_{s=0}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) f(s) \Big| \\
=& \Big| f(t) + \frac{\lambda}{(1-\lambda)} \sum_{s=0}^{N-1} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) f(s) \\
& + \frac{\lambda}{(1-\lambda)} \sum_{s=N}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) f(s) \Big| \\
\leq& |f(t)| + \frac{|\lambda|}{(1-\lambda)} \sum_{s=0}^{N-1} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) |f(s)| \\
& + \frac{|\lambda|}{(1-\lambda)} \sum_{s=N}^{t} \hat{e}_{\alpha,\alpha}(\lambda, (t-s)^{\overline{\alpha}}) |f(s)| \\
\leq& |f(t)| + |\lambda| \|f\|_\infty \sum_{s=0}^{N-1} (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t-s+\alpha)^{\overline{\alpha}}) \\
& + \varepsilon |\lambda| \sum_{s=N}^{t} (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t-s+\alpha)^{\overline{\alpha}}) \\
\leq& |f(t)| + |\lambda| \|f\|_\infty \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{N-1} (t-s+1)^{\overline{\alpha-1}} \\
& + \varepsilon |\lambda| \sum_{s=0}^{t} (t-s+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(\lambda, (t-s+\alpha)^{\overline{\alpha}}) \\
\leq& |f(t)| + |\lambda| \|f\|_\infty \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{N-1} \left(\frac{t}{N}\right)^{\alpha-1} (N-1-s+1)^{\overline{\alpha-1}} \\
& + \varepsilon \frac{|\lambda|}{\lambda} \Big[ F_\alpha(\lambda, (t+1)^{\overline{\alpha}}) - 1 \Big] \\
=& |f(t)| + \Big( \frac{|\lambda| \|f\|_\infty N^{\overline{\alpha}} N^{1-\alpha}}{\Gamma(\alpha+1)} \Big) t^{\alpha-1} + \varepsilon \frac{|\lambda|}{\lambda} \Big[ F_\alpha(\lambda, (t+1)^{\overline{\alpha}}) - 1 \Big].
\end{aligned}$$

Consequently, we get

$$\lim_{t\to\infty} u(t) = 0.$$

Based on the above discussion, we frame the following result.

**Theorem 4.1.** *Assume that $-1 < \lambda \leq 0$ and $f \in l_0^\infty$. Then, the unique solution of* (1.3)-(1.4) *is uniformly asymptotically stable in $l_0^\infty$.*

## 5. Applications

In this section, we discuss a few qualitative properties of solutions of (1.5), (1.7), (1.9) and (1.10) using the main results established in previous section.

Consider (1.5). Clearly,

$$\sup_{t\in\mathbb{N}_0} |f(t)| = (1-\lambda)|u(0)| \text{ and } \lim_{t\to\infty} f(t) = 0.$$

Thus, we have the following result.

**Corollary 5.1.** *Assume that* $-1 < \lambda \leq 0$. *Then, all solutions of* (1.5) *are bounded and asymptotically stable.*

The following example illustrates this fact.

*Example* 5.1. Consider the linear fractional nabla difference equation

$$\left(\nabla_0^{0.5} u\right)(t) = (-0.5)u(t), \quad t \in \mathbb{N}_1. \tag{5.1}$$

**Solution:** Here $\alpha = 0.5 \in (0,1)$ and $\lambda = -0.5 \in (-1,0]$. Thus, by Corollary 5.1, all solutions of (5.1) are bounded and asymptotically stable. A particular solution of (5.1) for $u(0) = 1$ is shown in Figure 1.

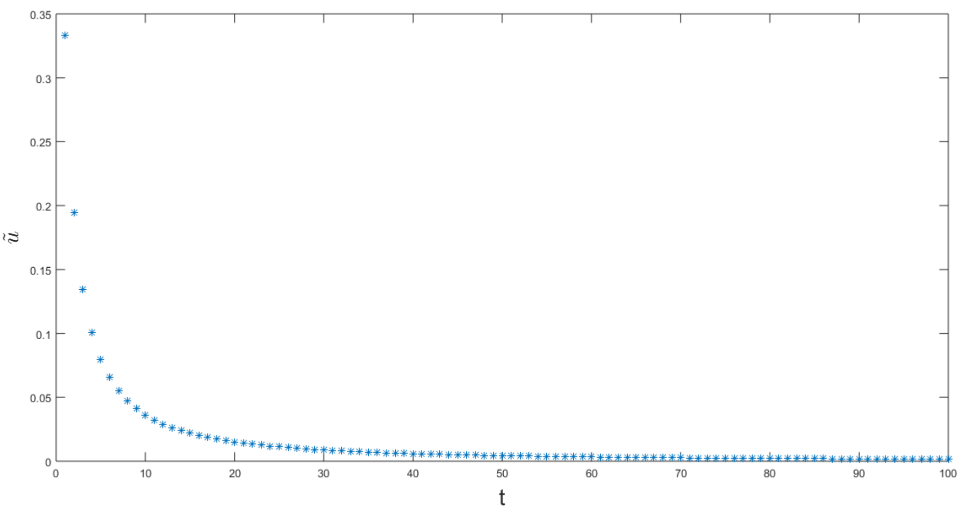


Figure 1.

Next, consider (1.7). Clearly,

$$\sup_{t\in\mathbb{N}_0} |f(t)| = |u(0)| \text{ and } \lim_{t\to\infty} f(t) = u(0).$$

Thus, we have the following result.

**Corollary 5.2.** *Assume that* $-1 < \lambda \leq 0$. *Then, all solutions of* (1.7) *are bounded and stable. Only, the zero solution of* (1.7) *is asymptotically stable.*

Next, consider (1.9). Clearly, $u(t)$ is a solution of (1.9) if and only if

$$u(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}[\lambda u(s) + g(s)], \quad t \in \mathbb{N}_0. \tag{5.2}$$

We observe that (5.2) belongs to the class of (1.3)-(1.4) with

$$f(t) = (1-\lambda)\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s).$$

Hence, we have the following results.

**Corollary 5.3.** *Assume that* $-1 < \lambda \leq 0$ *and*

$$\sup_{t\in\mathbb{N}_0}\left|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right| < \infty.$$

*Then, all solutions of* (1.9) *are bounded and stable. Further, if*

$$\lim_{t\to\infty}\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right] = 0$$

*holds, then all solutions of* (1.9) *are asymptotically stable.*

The following example demonstrates this corollary.

*Example* 5.2. The solution of

$$\left(\nabla_0^{0.5}u\right) = (-0.5)u + \frac{1}{(t+1)^{0.75}}, \quad t \in \mathbb{N}_1. \tag{5.3}$$

Here $\alpha = 0.5 \in (0,1)$ and $\lambda = -0.5 \in (-1,0]$. Since $(t+1)^{\alpha-1} \leq (t+1)^{\overline{\alpha-1}}$ for $0 \leq \alpha \leq 1$, we have

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s) &= \frac{1}{\Gamma(0.5)}\sum_{s=1}^{t}(t-s+1)^{\overline{-0.5}}(s+1)^{-0.75}\\ &\leq \frac{1}{\Gamma(0.5)}\sum_{s=1}^{t}(t-s+1)^{\overline{-0.5}}(s+1)^{\overline{-0.75}}\\ &= \nabla_0^{-0.5}(t+1)^{\overline{-0.75}} - \frac{1}{\Gamma(0.5)}(t+1)^{\overline{-0.5}}\\ &= \frac{\Gamma(0.25)}{\Gamma(0.75)}(t+1)^{\overline{-0.25}} - \frac{1}{\Gamma(0.5)}(t+1)^{\overline{-0.5}}\\ &\leq \frac{\Gamma(0.25)}{\Gamma(0.75)}.\end{aligned}$$

We know that

$$\lim_{t\to\infty}(t+1)^{\overline{-0.25}} = \lim_{t\to\infty}(t+1)^{\overline{-0.5}} = 0.$$

Thus, we have

$$\sup_{t\in\mathbb{N}_0}\left|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right|<\infty$$

and

$$\lim_{t\to\infty}\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right]=0,$$

implies, by Corollary 5.3, all solutions of (5.3) are bounded and asymptotically stable. A particular solution of (5.3) for $u(0) = 1$ is shown in Figure 2.

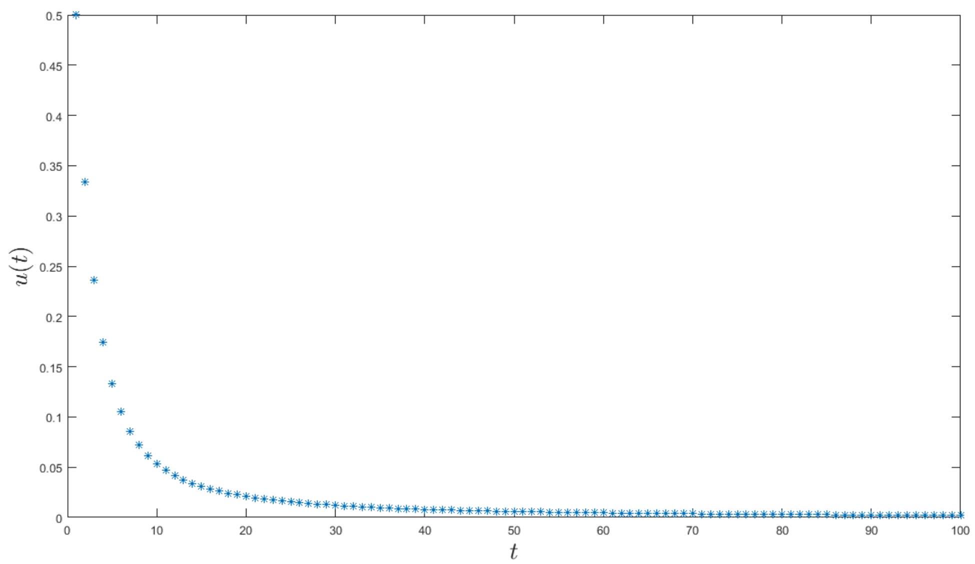


FIGURE 2.

Finally, consider (1.10). Clearly, $u(t)$ is a solution of (1.10) if and only if

$$u(t)=u(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}[\lambda u(s)+g(s)],\quad t\in\mathbb{N}_0. \tag{5.4}$$

We observe that (5.4) belongs to the class of (1.3)-(1.4) with

$$f(t)=\left[1-\lambda\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0)\right]+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s).$$

Consequently, we have the following result.

**Corollary 5.4.** *Assume that* $-1<\lambda\le 0$ *and*

$$\sup_{t\in\mathbb{N}_0}\left|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right|<\infty.$$

*Then, all solutions of* (1.10) *are bounded and stable. Further, if*

$$\lim_{t\to\infty}\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}g(s)\right]=0$$

*holds, then only the zero solution of* (1.10) *is asymptotically stable.*

## 6. Extension

In this section, we establish sufficient conditions on the qualitative behaviour of (1.1) and (1.2) using the results obtained in Sections 3 and 4. For this purpose, first we show that (1.1) and (1.2) can be expressed in the form of (1.3).

**Lemma 6.1.** *$u$ is a solution of* (1.1) *if and only if $u$ is a solution of*

$$u(t)=F_1(t)+\sum_{s=0}^{t}K(t-s;\mu)u(s), \tag{6.1}$$

*where*

$$F_1(t)=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0)\left[1-\mu-b(0)\right]$$

*and*

$$K(t;\mu)=\mu\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}+\left(\nabla_0^{-\alpha}b\right)(t).$$

*Proof.* Consider (1.1). Applying the operator $\nabla_1^{-\alpha}$ on both sides of (1.1), we get

$$\nabla_1^{-\alpha}\nabla_0^{\alpha}u(t)=\mu\left(\nabla_1^{-\alpha}u\right)(t)+\nabla_1^{-\alpha}\left[\sum_{s=0}^{t}b(t-s)u(s)\right],$$

which can be written in the form

$$\nabla_1^{-\alpha}\nabla\nabla_0^{-(1-\alpha)}u(t)=\mu\left(\nabla_1^{-\alpha}u\right)(t)+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-s+1)^{\overline{\alpha-1}}\left[\sum_{r=0}^{s}b(s-r)u(r)\right].$$

Applying Theorem 2.3 on left hand side and changing the order of summation on right hand side, we obtain

$$\begin{aligned}&\nabla\nabla_0^{-\alpha}\nabla_0^{-(1-\alpha)}u(t)-\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\left[\nabla_0^{-(1-\alpha)}u(t)\right]_{t=0}\\&=-\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0)\left[\mu+b(0)\right]+\mu\left(\nabla_0^{-\alpha}u\right)(t)+\sum_{r=0}^{t}u(r)\left[\frac{1}{\Gamma(\alpha)}\sum_{s=r}^{t}(t-s+1)^{\overline{\alpha-1}}b(s-r)\right].\end{aligned}$$

Hence we have

$$\begin{aligned}&\nabla\nabla_0^{-\alpha}\nabla_0^{-(1-\alpha)}u(t)\\&=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u(0)\left[1-\mu-b(0)\right]+\sum_{r=0}^{t}\left[\mu\frac{(t-r+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}+\left(\nabla_0^{-\alpha}b\right)(t-r)\right]u(r).\end{aligned}$$

It follows that

$$u(t) = F_1(t) + \sum_{r=0}^{t} K(t-r;\mu)u(r).$$

Hence the proof. □

**Lemma 6.2.** *u is a solution of* (1.2) *if and only if u is a solution of*

$$u(t) = F_2(t) + \sum_{s=0}^{t} K(t-s;\mu)u(s), \tag{6.2}$$

*where*

$$F_2(t) = u(0) - \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(0)\Big[\mu + b(0)\Big].$$

A necessary and sufficient condition for the existence and unicity of the solutions of (1.1) and (1.2) is that $\mu + b(0) \neq 1$. Next, we obtain the resolvent kernel corresponding to (6.1) and (6.2). The resolvent kernel $R(t)$ for the kernel $K(t)$ in (6.1) is given by

$$N_1\left[R(t-1)\right] = \frac{N_1\left[K(t-1)\right]}{N_1\left[K(t-1)\right]-1} \text{ or } N_0\left[R(t)\right] = \frac{N_0\left[K(t)\right]}{(1-z)N_0\left[K(t)\right]-1}$$

or

$$N_0\left[R(t)\right] = (1-z)^{-1}\left(\frac{\mu + (1-z)N_0\left[b(t)\right]}{\mu + (1-z)N_0\left[b(t)\right] - z^{\alpha}}\right). \tag{6.3}$$

Thus, the unique solution of (1.1) is given by

$$u(t) = F_1(t) - \sum_{s=0}^{t} R(t-s)F_1(s), \quad t \in \mathbb{N}_0. \tag{6.4}$$

Similarly, the unique solution of (1.2) is given by

$$u(t) = F_2(t) - \sum_{s=0}^{t} R(t-s)F_2(s), \quad t \in \mathbb{N}_0. \tag{6.5}$$

Finally, we conclude this article with a discussion on boundedness and stability of (1.1) and (1.2) in $l^{\infty}$. We have

$$\sup_{t\in\mathbb{N}_0} |F_1(t)| = |u(0)|\Big|1-\mu-b(0)\Big|, \quad \lim_{t\to\infty} F_1(t) = 0$$

and

$$\sup_{t\in\mathbb{N}_0} |F_2(t)| = |u(0)|, \quad \lim_{t\to\infty} F_2(t) = u(0).$$

So, $F_1$, $F_2 \in l^{\infty}$. From (6.4) and (6.5), we have

$$|u(t)| \leq |F_1(t)| + \sum_{s=0}^{t} |R(t-s)||F_1(s)| \leq \|F_1\|_{\infty} + \|F_1\|_{\infty}\sum_{s=0}^{t}|R(s)|$$

and

$$|u(t)| \leq |F_2(t)| + \sum_{s=0}^{t} |R(t-s)||F_1(s)| \leq \|F_2\|_{\infty} + \|F_2\|_{\infty}\sum_{s=0}^{t}|R(s)|,$$

respectively. Then, we have the following observations.

**Theorem 6.1.** *All solutions of* (1.1) *and* (1.2) *are bounded and stable in* $l^\infty$ *if*

$$\sup_{t\in\mathbb{N}_0}\sum_{s=0}^{t}|R(s)| = M < \infty. \tag{6.6}$$

**Theorem 6.2.** *All solutions of* (1.1) *are asymptotically stable in* $l_0^\infty$ *if* (6.6) *holds and*

$$\lim_{t\to\infty} R(t) = 0. \tag{6.7}$$

*Proof.* We know that $F_1 \in l_0^\infty$. Then, for any $\epsilon > 0$, there exists $N_1 = N_1(\epsilon)$ such that

$$|F_1(t)| < \varepsilon, \quad t \in \mathbb{N}_{N_1}.$$

It follows from (6.7) that there exists $N_2 = N_2(\epsilon)$ such that

$$\sum_{s=0}^{N_1-1}|R(t-s)| < \varepsilon, \quad t \in \mathbb{N}_{N_2}.$$

Then, for $t > \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|u(t)| &= \left|F_1(t) - \sum_{s=0}^{t} R(t-s)F_1(s)\right| \\
&= \left|F_1(t) - \sum_{s=0}^{N_1-1} R(t-s)F_1(s) - \sum_{s=N_1}^{t} R(t-s)F_1(s)\right| \\
&\le |F_1(t)| + \|F_1\|_\infty \sum_{s=0}^{N_1-1}|R(t-s)| + \varepsilon\sum_{s=N_1}^{t}|R(t-s)| \\
&\le |F_1(t)| + \varepsilon\|F_1\|_\infty + M\varepsilon.
\end{aligned}$$

Consequently, we get

$$\lim_{t\to\infty} u(t) = 0.$$

Hence the proof. ☐

**Theorem 6.3.** *The null solution corresponding to* (1.2) *is asymptotically stable in* $l_0^\infty$ *if* (6.6) *and* (6.7) *hold.*

*Example* 6.1. Consider the fractional nabla difference equation of Volterra type

$$\left(\nabla_0^{0.5}u\right)(t) = \sum_{s=0}^{t}\frac{1}{2^{t-s+1}}u(s), \quad t \in \mathbb{N}_1. \tag{6.8}$$

Here $\alpha = 0.5 \in (0,1)$, $\mu = 0$, $b(t) = \frac{1}{2^{t+1}}$ such that $\mu + b(0) \neq 1$. We have

$$(1-z)N_0\left[b(t)\right] = \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{1-z}{2}\right)^j = \frac{1}{1+z}.$$

Then, from (6.3) we have

$$N_0\left[R(t)\right] = -(1-z)^{-1}\sum_{n=0}^{\infty}\left[\frac{1}{z^{0.5}(1+z)}\right]^{n+1}.$$

Applying inverse $N_0$ on both sides, we get

$$\begin{aligned}
R(t) &= -\sum_{n=0}^{\infty} N_0^{-1}\left(N_1\left[\frac{t^{\overline{n}}}{2^{t+n}\Gamma(n+1)}\right] N_0\left[\frac{(t+1)^{\overline{0.5(n-1)}}}{\Gamma((0.5)(n+1))}\right]\right)\\
&= -\sum_{n=0}^{\infty}\left(\frac{t^{\overline{n}}}{2^{t+n}\Gamma(n+1)} *_0 \frac{(t+1)^{\overline{0.5(n-1)}}}{\Gamma((0.5)(n+1))}\right)\\
&= -\sum_{n=0}^{\infty}\sum_{s=0}^{t}\frac{(t-s+1)^{\overline{0.5(n-1)}}}{\Gamma((0.5)(n+1))}\frac{(s+1)^{\overline{n}}}{2^{s+n}\Gamma(n+1)}\\
&= -\sum_{n=0}^{\infty}\frac{1}{2^n\Gamma(n+1)}\nabla_0^{-(0.5)(n+1)}\left[\frac{(t+1)^{\overline{n}}}{2^t}\right].
\end{aligned}$$

Since $\frac{(t+1)^{\overline{n}}}{2^t} \le (t+1)^{\overline{n}}$ for $t \in \mathbb{N}_0$, we have

$$\nabla_0^{-(0.5)(n+1)}\left[\frac{(t+1)^{\overline{n}}}{2^t}\right] \le \nabla_0^{-(0.5)(n+1)}(t+1)^{\overline{n}} = \frac{\Gamma(n+1)}{\Gamma((1.5)(n+1))}(t+1)^{\overline{(1.5)n+0.5}}.$$

Then,

$$|R(t)| \le \sum_{n=0}^{\infty}\frac{1}{2^n\Gamma((1.5)(n+1))}(t+1)^{\overline{(1.5)n+0.5}}.$$

Now, consider

$$\begin{aligned}
\sum_{s=0}^{t}|R(s)| &\le \sum_{s=0}^{t}\sum_{n=0}^{\infty}\frac{1}{2^n\Gamma((1.5)(n+1))}(t+1)^{\overline{(1.5)n+0.5}}\\
&= \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{1}{\Gamma((1.5)(n+1))}\sum_{s=0}^{t}(t-s+1)^{\overline{(1.5)(n+1)-1}}\right]\\
&= \sum_{n=0}^{\infty}\frac{1}{2^n}\nabla_0^{-(1.5)(n+1)}(t+1)^{\overline{0}}\\
&= \sum_{n=0}^{\infty}\frac{1}{2^n\Gamma((1.5)n+2.5)}(t+1)^{\overline{(1.5)(n+1)}}.
\end{aligned}$$

Let

$$a_n = \frac{1}{2^n\Gamma((1.5)n+2.5)}(t+1)^{\overline{(1.5)(n+1)}} = \frac{1}{2^n}\frac{\Gamma(t+(1.5)n+2.5)}{\Gamma(t+1)\Gamma((1.5)n+2.5)}.$$

Consider

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{2}\lim_{n\to\infty}\frac{\Gamma((1.5)n+1)}{\Gamma((1.5)n+2.5)}\lim_{n\to\infty}\frac{\Gamma(t+(1.5)n+2.5)}{\Gamma(t+(1.5)n+1)} = \frac{1}{2} < 1.$$

Then, by Ratio test,

$$\sum_{n=0}^{\infty} a_n$$

converges implies

$$\sup_{t\in\mathbb{N}_0} \sum_{s=0}^{t} |R(s)| < \infty.$$

Thus, by Theorem 6.1, all solutions of (6.8)) are bounded and stable in $l^\infty$.

*Example* 6.2. Consider the fractional nabla difference equation of Volterra type

$$\left(\nabla_0^{0.5} u\right)(t) = -(0.75) \sum_{s=0}^{t} \frac{(t-s+1)^{\overline{-0.75}}}{\Gamma(0.25)} u(s), \quad t \in \mathbb{N}_1. \tag{6.9}$$

Here $\alpha = 0.5 \in (0,1)$, $\mu = 0$, $b(t) = -(0.75)\frac{(t+1)^{\overline{-0.75}}}{\Gamma(0.25)}$ such that $\mu + b(0) \neq 1$. We have

$$(1-z)N_0\left[b(t)\right] = -(0.75)z^{-0.25}.$$

Then, from (6.6) we have

$$N_0\left[R(t)\right] = (1-z)^{-1} \frac{0.75}{0.75 + z^{0.75}}.$$

Applying inverse $N_0$ on both sides, we get

$$R(t) = \frac{0.75}{1.75} \hat{e}_{0.75,0.75}\left(-0.75, t^{\overline{0.75}}\right).$$

Now, consider

$$\sum_{s=0}^{t} |R(s)| = \frac{0.75}{1.75} \sum_{s=0}^{t} \hat{e}_{0.75,0.75}\left(-0.75, t^{\overline{0.75}}\right) = 1 - F_{0.75}\left(-0.75, (t+1)^{\overline{0.75}}\right) \leq 1.$$

Also,

$$\lim_{t\to\infty} R(t) = 0.$$

Thus, by Theorem 6.2, all solutions of (6.9) are asymptotically stable in $l_0^\infty$.

## Conclusion

In this article, we analysed a particular class of convolution type Volterra fractional nabla difference equations (1.1) and (1.2). For this purpose, we expressed (1.1) and (1.2) in the form of a linear implicit difference equation of Volterra type (1.3) and established sufficient conditions on the existence, uniqueness, boundedness and stability of the solutions of (1.1) and (1.2) using resolvent kernel and $N$-transform techniques. This work can be extended to study a more general class of Volterra type fractional nabla difference equations using Lyapunov functions and fixed point theorems.

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Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078, Telangana, India
*Email address*: j.jaganmohan@hotmail.com