

ALGEBRAIC PROPERTIES OF BI-PERIODIC DUAL FIBONACCI QUATERNIONS

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ABSTRACT. The purpose of the paper is to construct a new representation of dual quaternions called bi-periodic dual Fibonacci quaternions. These quaternions are originated as a generalization of the known quaternions in literature such as dual Fibonacci quaternions, dual Pell quaternions and dual k -Fibonacci quaternions. Moreover, some of them have not been introduced until now. Finally, we calculate the generating function, Binet formula and Catalan's identity of the bi-periodic dual Fibonacci quaternions.

1. INTRODUCTION

In linear algebra, the complex numbers extend the real numbers by adjoining one new element \mathbf{i} with the property $\mathbf{i}^2 = -1$ and the dual numbers extend the real numbers by adjoining one new nonzero element ε with the property $\varepsilon^2 = 0$. Every complex number has the form $z = a + \mathbf{i}b$ and the dual number has the form $d = a + \varepsilon b$.

The real quaternion is the number system in 4-dimensional vector space which is extended by the complex numbers. William Rowan Hamilton in 1843 first introduced the number system. Then, Clifford generalized the quaternions to the bi-quaternions in 1873. This number system is a useful tool for the analysis of complex number in literature. Also, they are important number system which use in kinematics, robotic technologies, spatial rotations, quantum physics and mechanics briefly both theoretical and applied mathematics or physics.

Generally, there has three types of quaternions called real, complex and dual quaternions. A real quaternion is defined as $q = q_0 + q_1i + q_2j + q_3k$ with four units $(1, i, j, k)$

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where (i, j, k) are orthogonal unit spatial vectors and q_m ($m = 0, 1, 2, 3$) are real numbers. A complex quaternion is an extension of complex numbers if the elements q_m ($m = 0, 1, 2, 3$) are the complex numbers and the other a dual quaternion is constructed by the dual numbers q_m ($m = 0, 1, 2, 3$). The family of dual quaternions is a Clifford algebra that can be used to represent spatial rigid body displacements in mathematics. Rigid motions in 3-dimensional space can be represented by dual quaternions of unit length this fact is used a theoretical kinematics and in applications to 3-dimensional computer graphics, robotics and computer visions ([13, 18, 19]).

Fibonacci sequence was the outcome of a mathematical problem about well-known rabbit breeding. In algebra, Fibonacci sequence is perhaps one of the most useful sequence and it has important applications to different disciplines such as fractal geometry, computer graphics, biology and CAD–CAM applications. Fibonacci numbers are characterized by the fact that every number after the first two numbers is the sum of the two proceeding numbers, that is, the numbers have the form

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

To find the n th terms of Fibonacci sequence, Binet's formula which was found by mathematician Jacques Philippe Marie Binet is used explicit formula. Then, there are well known two identities for Fibonacci sequence. In 1680, Jean-Dominique Cassini was discovered Cassini's identity for Fibonacci numbers. In 1879, Eugene Charles Catalan found the generalization of the Cassini's identity after him called Catalan's identity.

The Fibonacci quaternion is first expressed by Horadam [10] as $Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$, where F_n is the n th Fibonacci number. After the study, some results in this field have been investigated many authors ([1, 2, 4, 8, 9, 11, 12, 16, 20]). S. Halıcı [7] introduced the generating functions and Binet formulas for Fibonacci and Lucas quaternions and established some identities for these quaternions. C. B. Çimen and A. İpek [2] investigated the Pell and Pell-Lucas quaternions. In [17], they defined the bi-periodic Fibonacci quaternions. Furthermore, they gave the generalization of the Fibonacci quaternions, Pell quaternions, k -Fibonacci quaternions. Flaut and Shpakivskiy [5] investigated some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions in quaternion algebra. In [6], the authors investigated the geometric interpretation of the Fibonacci numbers and Fibonacci vectors.

In this study, we define the bi-periodic dual Fibonacci quaternions. We expressed the generating function, Binet formula, Catalan's identity and Cassini's identity for these quaternions. Hence, this work is a generalization of some quaternions known as dual Fibonacci quaternions, Pell quaternions and k -Fibonacci quaternions.

2. BASIC CONCEPTS AND ARGUMENTS

In the present days, many mathematicians studied several problems and results involving the Fibonacci sequences, and consequently we need the generalizations of this sequence.

The n th term of Fibonacci number is given by the formula

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

where $F_0 = 0$ and $F_1 = 1$.

Edson and Yayenie [3] introduced the new generalization of the Fibonacci numbers called as bi-periodic Fibonacci numbers

$$(2.1) \quad F_n = \begin{cases} aF_{n-1} + F_{n-2}, & \text{if } n \text{ is even,} \\ bF_{n-1} + F_{n-2}, & \text{if } n \text{ is odd,} \end{cases}$$

with the initial values $F_0 = 0$ and $F_1 = 1$, where a and b are nonzero numbers. In (2.1), the classical Fibonacci sequence is obtained if $a = b = 1$, the Pell sequence is get if $a = b = 2$ and if we take $a = b = k$, we get the k -Fibonacci sequence. Also, the terms of the bi-periodic Fibonacci sequence satisfy the equation $F_n = (ab + 2)F_{n-2} - F_{n-4}$ ($n \geq 4$).

From (2.1), the generating function of the bi-periodic Fibonacci sequence is obtained following as:

$$F(x) = \frac{x + ax^2 - x^3}{1 - (ab + 2)x^2 + x^4}.$$

The Binet formula for the Fibonacci numbers theory is represented in [15]. Also, the Binet formula allows us to express the bi-periodic dual Fibonacci quaternion in function. The Binet formula of the bi-periodic Fibonacci numbers is denoted by

$$(2.2) \quad F_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{ab+(a^2b^2+4ab)^{\frac{1}{2}}}{2}$, $\beta = \frac{ab-(a^2b^2+4ab)^{\frac{1}{2}}}{2}$ that is, α and β are the roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2 \lfloor \frac{n}{2} \rfloor$. Namely, $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. It can be shown that $F_{-n} = (-1)^{n-1}F_n$.

The 19th century Irish mathematician and physicist William Rowan Hamilton was impressed by the role of complex numbers in two dimensional geometry. Hence, he discovered a 4-dimensional split algebra called the quaternions and the quaternions introduced by him

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$ and the vectors i, j and k are satisfied the following properties:

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

After him, the dual quaternions are defined a Clifford algebra that can be used to represent spatial rigid body displacements in mathematics and mechanics. A dual quaternion is constructed by eight real parameters as

$$Q = q + \varepsilon q^*,$$

where q, q^* are the real quaternions and ε is the dual unit such that $\varepsilon \neq 0$ and $\varepsilon^2 = 0$.

Horadam [10] put forward a new definition by taking the Fibonacci numbers replaced with the real numbers in the quaternions. Then, the authors [14] presented the n th dual Fibonacci quaternions $\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}$ using the n th dual Fibonacci numbers $\tilde{F}_n = F_n + \varepsilon F_{n+1}$.

Definition 2.1. The bi-periodic Fibonacci quaternions are

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

where F_n is the n th bi-periodic Fibonacci number (see for details in [17]).

3. BI-PERIODIC DUAL FIBONACCI QUATERNIONS

In this section, we give the new definitions and generalizations of the bi-periodic dual Fibonacci quaternions.

Definition 3.1. The bi-periodic dual Fibonacci numbers \tilde{F}_n is

$$\tilde{F}_n = F_n + \varepsilon F_{n+1},$$

where F_n, F_{n+1} are the n th and $(n+1)$ th bi-periodic Fibonacci numbers in (2.1), respectively.

Definition 3.2. The bi-periodic dual n th Fibonacci quaternion \tilde{Q}_n is defined by

$$(3.1) \quad \tilde{Q}_n = Q_n + \varepsilon Q_{n+1},$$

where i, j and k are the standard orthonormal bases in \mathbb{R}^3 .

Let $\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}$ and $\tilde{P}_n = P_n + \varepsilon P_{n+1}$ be two bi-periodic dual Fibonacci quaternions. Taking into account the above definitions, the addition is

$$\tilde{Q}_n + \tilde{P}_n = (Q_n + P_n) + \varepsilon(Q_{n+1} + P_{n+1}),$$

where $\tilde{Q}_n = \tilde{F}_n + \tilde{F}_{n+1}i + \tilde{F}_{n+2}j + \tilde{F}_{n+3}k$ and $\tilde{P}_n = \tilde{G}_n + \tilde{G}_{n+1}i + \tilde{G}_{n+2}j + \tilde{G}_{n+3}k$. Also, the multiplication of them is given by

$$\tilde{Q}_n \tilde{P}_n = Q_n P_n + \varepsilon(Q_{n+1} P_n + Q_n P_{n+1}).$$

Theorem 3.1 (Binet Formula). *The Binet formula for the bi-periodic dual Fibonacci quaternion \tilde{Q}_n is given as:*

$$(3.2) \quad \tilde{Q}_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} \right) \\ + \varepsilon \left(\frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^{n+1} - \beta^{**} \beta^{n+1}}{\alpha - \beta} \right) \right), & \text{if } n \text{ is even,} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^n - \beta^{**} \beta^n}{\alpha - \beta} \right) \\ + \varepsilon \left(\frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha^* \alpha^{n+1} - \beta^* \beta^{n+1}}{\alpha - \beta} \right) \right), & \text{if } n \text{ is odd,} \end{cases}$$

where $\alpha^* := a + \alpha i + \frac{a}{ab} \alpha^2 j + \frac{1}{ab} \alpha^3 k$, $\beta^* := a + \beta i + \frac{a}{ab} \beta^2 j + \frac{1}{ab} \beta^3 k$ and $\alpha^{**} := 1 + \frac{a}{ab} \alpha i + \frac{1}{ab} \alpha^2 j + \frac{1}{(ab)^2} \alpha^3 k$, $\beta^{**} := 1 + \frac{a}{ab} \beta i + \frac{1}{ab} \beta^2 j + \frac{1}{(ab)^2} \beta^3 k$.

Proof. Induction method can be used to prove that the Binet formula in (3.2). First, we show that the statement holds for $n = 0$, i.e.,

$$\tilde{Q}_0 = (0, 1, a, ab + 1) + \varepsilon(1, a, ab + 1, a(ab + 2))$$

and from the Binet formula, we have

$$\begin{aligned} \tilde{Q}_0 &= \frac{\alpha^* - \beta^*}{\alpha - \beta} + \varepsilon \left(\frac{\alpha^{**} \alpha - \beta^{**} \beta}{\alpha - \beta} \right) \\ &= \frac{1}{\alpha - \beta} (0, \alpha - \beta, \frac{a}{ab} (\alpha^2 - \beta^2), \frac{1}{ab} (\alpha^3 - \beta^3)) \\ &\quad + \varepsilon \frac{1}{\alpha - \beta} (\alpha - \beta, \frac{a}{ab} (\alpha^2 - \beta^2), \frac{1}{ab} (\alpha^3 - \beta^3), \frac{a}{(ab)^2} (\alpha^4 - \beta^4)). \end{aligned}$$

By simplifying the above equation using the equalities $\alpha - \beta = -ab$, $\alpha + \beta = ab$, $\alpha^2 + \beta^2 = ab(ab + 2)$, we get

$$\tilde{Q}_0 = Q_0 + \varepsilon Q_1.$$

Hence, the Binet formula in (3.2) is satisfied for $n = 0$.

Assume that the \tilde{Q}_n holds. It must then be shown that \tilde{Q}_{n+1} holds. In here, there are two situations, n is even or odd.

If we take n is even, it is satisfied the following equation

$$\tilde{Q}_n = \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} \right) + \varepsilon \left(\frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^{n+1} - \beta^{**} \beta^{n+1}}{\alpha - \beta} \right) \right).$$

From the Definition 3.2, the bi-periodic dual $(n + 1)$ th Fibonacci quaternion \tilde{Q}_{n+1} is written as follows:

$$\tilde{Q}_{n+1} = Q_{n+2} + \varepsilon Q_{n+3}.$$

By using Theorem 2 in [17], we can write

$$Q_{n+2} = \frac{1}{(ab)^{\lfloor \frac{n+2}{2} \rfloor}} \left(\frac{\alpha^* \alpha^{n+2} - \beta^* \beta^{n+2}}{(\alpha - \beta)} \right)$$

and

$$Q_{n+3} = \frac{1}{(ab)^{\lfloor \frac{n+3}{2} \rfloor}} \left(\frac{\alpha^{**} \alpha^{n+3} - \beta^{**} \beta^{n+3}}{(\alpha - \beta)} \right).$$

So, the Binet formula is obtained for all even indices. Also, the Binet formula can be proven using the similar method for all odd indices. \square

Remark 3.1. If we take $a = b = 1$ in (3.2), then we obtain the Binet formula of the dual Fibonacci quaternions in [14].

Theorem 3.2 (Generating function). *The generating function for the bi-periodic dual Fibonacci quaternions is*

$$(3.3) \quad G(t) = \frac{Q_0 + (Q_1 - bQ_0)t + (a - b)R(t)}{1 - bt - t^2} + \varepsilon \left(\frac{Q_1 + (Q_2 - bQ_1)t + (a - b)S(t)}{1 - bt - t^2} \right),$$

where

$$\begin{aligned} R(t) &= tf(t) + (f(t) - t)i + \left(\frac{f(t)}{t} - 1 \right) j + \left(\frac{f(t)}{t^2} - \frac{1}{t} - (ab + 1)t \right) k, \\ S(t) &= (f(t) - t) + \left(\frac{f(t)}{t} - 1 \right) i + \left(\frac{f(t)}{t^2} - \frac{1}{t} - (ab + 1)t \right) j \\ &\quad + \left(\frac{f(t)}{t^3} - \frac{1}{t^2} - (ab + 1) \right) k, \end{aligned}$$

with $f(t) := \sum_{k=1}^{\infty} F_{2k-1} t^{2k-1}$.

Proof. The ordinary generating function of the bi-periodic dual Fibonacci quaternions is $G(t) = \sum_{n=0}^{\infty} \tilde{Q}_n t^n$ and using the equations $btG(t)$ and $t^2G(t)$,

$$\begin{aligned} G(t) &= \tilde{Q}_0 + \tilde{Q}_1 t + \tilde{Q}_2 t^2 + \cdots + \tilde{Q}_n t^n + \cdots, \\ btG(t) &= b\tilde{Q}_0 t + b\tilde{Q}_1 t^2 + b\tilde{Q}_2 t^3 + \cdots + b\tilde{Q}_n t^{n+1} + \cdots, \\ t^2G(t) &= \tilde{Q}_0 t^2 + \tilde{Q}_1 t^3 + \tilde{Q}_2 t^4 + \cdots + \tilde{Q}_n t^{n+2} + \cdots \end{aligned}$$

and take into account the equation $\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}$ in the above equations, we get

$$\begin{aligned} G(t) - btG(t) - t^2G(t) &= Q_0 + \varepsilon Q_1 + [(Q_1 - bQ_0) + \varepsilon(Q_2 - bQ_1)]t \\ &\quad + [(Q_2 - bQ_1 - Q_0) + \varepsilon(Q_3 - bQ_2 - Q_1)]t^2 \\ &\quad + [(Q_3 - bQ_2 - Q_1) + \varepsilon(Q_4 - bQ_3 - Q_2)]t^3 + \cdots \\ &\quad + [(Q_n - bQ_{n-1} - Q_{n-2}) + \varepsilon(Q_{n+1} - bQ_n - Q_{n-1})]t^n + \cdots \end{aligned}$$

$$\begin{aligned}
 &= Q_0 + (Q_1 - bQ_0)t + \sum_{n=2}^{\infty} (Q_n - bQ_{n-1} - Q_{n-2})t^n \\
 &\quad + \varepsilon \left\{ Q_1 + (Q_2 - bQ_1)t + \sum_{n=2}^{\infty} (Q_{n+1} - bQ_n - Q_{n-1})t^n \right\}.
 \end{aligned}$$

And using (2.1), we can calculate

$$\begin{aligned}
 \sum_{n=2}^{\infty} (Q_n - bQ_{n-1} - Q_{n-2})t^n &= (a - b)R(t), \\
 \sum_{n=2}^{\infty} (Q_{n+1} - bQ_n - Q_{n-1})t^n &= (a - b)S(t).
 \end{aligned}$$

So, the generating function becomes in (3.3) for the bi-periodic dual Fibonacci quaternions. \square

Remark 3.2. The generating function of the dual Fibonacci quaternions for $a = b = 1$ is obtained as follows

$$G(t) = \frac{Q_0 + (Q_1 - Q_0)t}{1 - t - t^2} + \varepsilon \left(\frac{Q_1 + (Q_2 - Q_1)t}{1 - t - t^2} \right).$$

Now, we give the important identities of the bi-periodic dual Fibonacci quaternions known as Catalan and Cassini’s identities.

Theorem 3.3 (Catalan’s identity). *Let r be nonnegative even integer number. The Catalan’s identity for the bi-periodic dual Fibonacci quaternions is given by*

$$\begin{aligned}
 &\tilde{Q}_{n-r} \tilde{Q}_{n+r} - \tilde{Q}_n^2 \\
 &= \left\{ \begin{aligned} &\frac{\alpha^* \beta^* ((ab)^r - \beta^{2r}) + \beta^* \alpha^* ((ab)^r - \alpha^{2r})}{(ab)^r (\alpha - \beta)^2} \\ &+ \varepsilon \left[\frac{(\alpha^{**} \beta^* \alpha + \alpha^* \beta^{**} \beta) ((ab)^r - \beta^{2r}) + (\beta^* \alpha^{**} \alpha + \beta^{**} \alpha^* \beta) ((ab)^r - \alpha^{2r})}{(ab)^r (\alpha - \beta)^2} \right], \quad \text{if } n \text{ is even,} \\ &-\frac{\alpha^{**} \beta^{**} ((ab)^r - \beta^{2r}) + \beta^{**} \alpha^{**} ((ab)^r - \alpha^{2r})}{(ab)^{r-1} (\alpha - \beta)^2} \\ &+ \varepsilon \left\{ -\frac{(\alpha^* \beta^{**} \alpha + \alpha^{**} \beta^* \beta) ((ab)^r - \beta^{2r}) + (\beta^* \alpha^{**} \beta + \beta^{**} \alpha^* \alpha) ((ab)^r - \alpha^{2r})}{(ab)^r (\alpha - \beta)^2} \right\}, \quad \text{if } n \text{ is odd,} \end{aligned} \right.
 \end{aligned}$$

where n and r are the nonnegative integer numbers with $n \geq r$.

Proof. From (3.1), we can write the following equality

$$\begin{aligned}
 \tilde{Q}_{n-r} \tilde{Q}_{n+r} - \tilde{Q}_n^2 &= (Q_{n-r} + \varepsilon Q_{n-r+1})(Q_{n+r} + \varepsilon Q_{n+r+1}) - (Q_n + \varepsilon Q_{n+1})^2 \\
 &= Q_{n-r} Q_{n+r} - Q_n^2 + \varepsilon (Q_{n-r+1} Q_{n+r} + Q_{n-r} Q_{n+r+1} - Q_n Q_{n+1} - Q_{n+1} Q_n)
 \end{aligned}$$

and use the Binet formula for the bi-periodic dual Fibonacci quaternions in (3.2), we can find the desired result. \square

Theorem 3.4 (Cassini's identity). (i) *Cassini's formula of the bi-periodic dual Fibonacci quaternions for odd indices is*

$$\begin{aligned} \tilde{Q}_{2m-1} \tilde{Q}_{2m+3} - \tilde{Q}_{2m+1}^2 = & - \left\{ \frac{((ab)^2 - \beta^4)\alpha^{**}\beta^{**} + ((ab)^2 - \alpha^4)\beta^{**}\alpha^{**}}{(ab)(\alpha - \beta)^2} \right\} \\ & - \varepsilon \left\{ \frac{\alpha [((ab)^2 - \beta^4)\alpha^*\beta^{**} + ((ab)^2 - \alpha^4)\beta^{**}\alpha^*]}{(ab)^2(\alpha - \beta)^2} \right. \\ & \left. + \frac{\beta [((ab)^2 - \alpha^4)\beta^*\alpha^{**} + ((ab)^2 - \beta^4)\alpha^{**}\beta^*]}{(ab)^2(\alpha - \beta)^2} \right\}. \end{aligned}$$

(ii) *For even indices Cassini's formula is*

$$\begin{aligned} \tilde{Q}_{2m-2} \tilde{Q}_{2m+2} - \tilde{Q}_{2m}^2 = & \frac{((ab)^2 - \beta^4)\alpha^*\beta^* + ((ab)^2 - \alpha^4)\beta^*\alpha^*}{(ab)^2(\alpha - \beta)^2} \\ & + \varepsilon \left\{ \frac{(\alpha^{**}\beta^*\alpha + \alpha^*\beta^{**}\beta)((ab)^2 - \beta^4)}{(ab)^2(\alpha - \beta)^2} \right. \\ & \left. + \frac{(\beta^*\alpha^{**}\alpha + \beta^{**}\alpha^*\beta)((ab)^2 - \alpha^4)}{(ab)^2(\alpha - \beta)^2} \right\}. \end{aligned}$$

Proof. (i) If we take $r = 2$ and $n = 2m + 1$ in the Catalan's formula then we get the Cassini's formula.

(ii) We put the $r = 2$ and $n = 2m$ in the Catalan's formula, we obtain the Cassini's formula for even indices. \square

4. CONCLUSIONS

We want to emphasized that the characterizations some of the dual Fibonacci quaternions can be introduced with the convenient parameter a and b . For example, the following results are stated:

- (i) for dual Fibonacci quaternions with the parameters $a = b = 1$;
- (ii) for dual Pell quaternions with the parameters $a = b = 2$;
- (iii) for dual k -Fibonacci quaternions with the parameters $a = b = k$.

Although, the option (i) is introduced by Nurkan and Güven in [14], the other options (ii) and (iii) are unnecessary to publish in literature.

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