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ON SKEW LAPLACIAN SPECTRA AND SKEW LAPLACIAN ENERGY OF DIGRAPHS

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Abstract. Let $\mathscr{D}$ be a simple digraph with n vertices, m arcs having skew Laplacian eigenvalues $\nu_1, \nu_2, \dots, \nu_{n-1}, \nu_n = 0$. The skew Laplacian energy $SLE(\mathscr{D})$ of a digraph $\mathscr{D}$ is defined as $SLE(\mathscr{D}) = \sum_{i=1}^{n} |\nu_i|$. We obtain upper and lower bounds for $SLE(\mathscr{D})$, which improves some previously known bounds. We also show that every even positive integer is indeed the skew Laplacian energy of some digraph.

1. Introduction

Let $\mathscr{D}$ be a simple digraph with n vertices $v_1, v_2, \dots, v_n$ and m arcs. Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \dots, n$ be respectively, the out-degree, in-degree and degree of the vertices of the digraph $\mathscr{D}$. The out-adjacency matrix $A^+(\mathscr{D}) = (a_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix $A^-(\mathscr{D}) = (a_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. It is clear that $A^-(\mathscr{D}) = (A^+(\mathscr{D}))^t$.

The skew adjacency matrix $S(\mathscr{D}) = (s_{ij})$ of a digraph $\mathscr{D}$ is the $n \times n$ matrix, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $S(\mathscr{D})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. The energy of the matrix $S(\mathscr{D})$ was considered in [1], and is defined

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as

$$E_s(\mathscr{D}) = \sum_{i=1}^{n} |\xi_i|,$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of $S(\mathscr{D})$. This energy of a digraph $\mathscr{D}$ is called the skew energy by Adiga et al. [1]. For recent developments in the theory of skew spectrum and skew energy see the survey [10] and the references therein.

Let $D^+(G) = \operatorname{diag}(d_1^+, d_2^+, \ldots, d_n^+)$, $D^-(G) = \operatorname{diag}(d_1^-, d_2^-, \ldots, d_n^-)$ and $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrices of vertex out-degrees, vertex in-degrees and vertex degrees of $\mathscr{D}$, respectively. If $S(\mathscr{D})$ is the skew adjacency matrix of $\mathscr{D}$ and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph $\mathscr{D}$, then it is clear that $A(G) = A^+(\mathscr{D}) + A^-(\mathscr{D})$ and $S(\mathscr{D}) = A^+(\mathscr{D}) - A^-(\mathscr{D})$, where $A^+(\mathscr{D})$ and $A^-(\mathscr{D})$ are the out-adjacency and in-adjacency matrices of $\mathscr{D}$. Following the definition of Laplacian matrix of a graph, Cai et al. [2] called the matrix

$$\begin{aligned}\widetilde{SL}(\mathscr{D}) &= (D^+(\mathscr{D}) - D^-(\mathscr{D})) + (A^+(\mathscr{D}) - A^-(\mathscr{D}))\\ &= \widetilde{D}(\mathscr{D}) + S(\mathscr{D}),\end{aligned}$$

where $\widetilde{D}(\mathscr{D}) = D^+(\mathscr{D}) - D^-(\mathscr{D})$, as the skew Laplacian matrix of the digraph $\mathscr{D}$. It is clear that the matrix $\widetilde{SL}(\mathscr{D})$ is not symmetric, so its eigenvalues need not be real. However, the following observation was noted in [2].

Theorem 1.1. (i) *If $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\mathscr{D})$, then $\sum_{i=1}^{n} \nu_i = 0$.*

(ii) *0 is an eigenvalue of $\widetilde{SL}(\mathscr{D})$ with multiplicity at least p, where p is the number of components of $\mathscr{D}$, with all ones vector $(1, 1, \ldots, 1)$ as the corresponding eigenvector.*

The skew Laplacian spectrum of a digraph $\mathscr{D}$ is a new concept in the field of spectral theory of digraphs. A lot of literature can be found on spectral theory of skew matrix [10], but one can hardly find a paper on the spectral theory of skew Laplacian matrix. It will of interest in future to develop a spectral theory of digraphs for the skew Laplacian matrix.

Let $\mathscr{D}$ be a digraph of order n with m arcs and having skew Laplacian eigenvalues $\nu_1, \nu_2, \ldots, \nu_n$. The skew Laplacian energy of $\mathscr{D}$ is denoted by $SLE(\mathscr{D})$ and is defined as

$$SLE(\mathscr{D}) = \sum_{j=1}^{n} |\nu_j|.$$

This concept was introduced in 2013 by Cai et al. [2]. The idea of Cai et al. was to conceive a graph energy like quantity for a digraph, that instead of skew adjacency eigenvalues is defined in terms of skew Laplacian eigenvalues and that hopefully would preserve the main features of the original graph energy. The definition of $SLE(\mathscr{D})$ was therefore so chosen that all the properties possessed by graph energy should be preserved. The skew Laplacian energy is an extension of skew energy of a digraph just as Laplacian energy (see [4, 5, 11] and the references therein) is an extension of graph energy (see [6] and the references therein).

The rest of the paper is organized as follows. In Section 2, we obtain the skew Laplacian energy of star for any orientation and cycle for some orientations. In Section 3, we obtain bounds for $SLE(\mathscr{D})$ which are better than the already known bounds. We also leave some problems related to skew Laplacian spectrum and skew Laplacian energy. These problems will be of interest for the future research.

2. Skew Laplacian Energy of Some Digraphs

In this Section, we obtain the skew Laplacian energy of star for any orientation and cycle for some orientations.

A digraph $\mathscr{D}$ is said to be Eulerian if $d_i^+ = d_i^-$, for all $i = 1, 2, \dots, n$. Therefore, for an Eulerian digraph $\mathscr{D}$, we always have $\widetilde{D}(\mathscr{D}) = 0$, which gives $\widetilde{SL}(\mathscr{D}) = S(\mathscr{D})$. Using this, the following observation is immediate.

Theorem 2.1. *For an Eulerian digraph $\mathscr{D}$, we have $SLE(\mathscr{D}) = E_s(\mathscr{D})$, where $E_s(\mathscr{D})$ is the skew energy of $\mathscr{D}$.*

As an immediate consequence to Theorem 2.1 we have the following result.

Theorem 2.2. *For a directed cycle C_n, we have $SLE(C_n) = E_s(C_n)$, where $E_s(\mathscr{D})$ is the skew energy of $\mathscr{D}$.*

This shows that for a directed cycle C_n, the skew Laplacian energy is same as the corresponding skew energy. In [1], the skew energy of a cycle for any orientation, was completely determined. Like this, it will be interest to determine the skew Laplacian energy of a cycle for any orientation. We leave this as a problem for the future research at the end of the Section 3.

We obtain the skew Laplacian energy of a star for any orientation and as a consequence we show that every even positive integer is indeed the skew Laplacian energy of some digraph.

Theorem 2.3. *For the star $K_{1,n}$ of order $n+1$, we have $SLE(K_{1,n}) = 2(n-1)$, if all the edges are oriented towards or away from the center, and $SLE(K_{1,n}) = n - 2 + \sqrt{(n-2k)^2 - 4(n-1)}$, otherwise, where k, $1 \le k \le n-1$, is the number of edges oriented towards the center.*

Proof. Let $V(K_{1,n}) = \{v_1, v_2, \dots, v_{n+1}\}$ be the vertex set of $K_{1,n}$. If v_{n+1} is the center of $K_{1,n}$, orient all the edges toward v_{n+1}. Then

$$S(K_{1,n}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{D}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

Therefore,

$$\widetilde{SL}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ -1 & -1 & \cdots & -1 & -n \end{pmatrix}.$$

It is easy to see that the eigenvalues of this matrix are $\{-(n-1), 0, 1^{[n-1]}\}$, and so $SLE(K_{1,n}) = 2(n-1)$. On the other hand, if we orient the edges away from v_{n+1}, then it can be seen that

$$\widetilde{SL}(K_{1,n}) = \begin{pmatrix} -1 & 0 & \cdots & 0 & -1 \\ 0 & -1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix},$$

having eigenvalues $\{(n-1), 0, -1^{[n-1]}\}$, so $SLE(K_{1,n}) = 2(n-1)$. Thus, for a directed star $K_{1,n}$, we have $SLE(K_{1,n}) = 2(n-1)$.

If all the edges of the star $K_{1,n}$ are oriented away from the center v_{n+1} except k, $1 \leq k \leq n-1$, edges which are oriented towards the center v_{n+1}, then it can be seen that the skew Laplacian matrix of $K_{1,n}$ is

$$\widetilde{SL}(K_{1,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & \cdots & 1 & -1 & \cdots & -1 & n-2k \end{pmatrix}.$$

By direct calculation, it can be seen that the skew Laplacian characteristic polynomial of this matrix is $x(x-1)^{k-1}(x+1)^{n-k-1}\left(x^2-(n-2k)x+n-1\right)$ and so its eigenvalues are $\left\{0, 1^{[k-1]}, -1^{[n-k-1]}, \frac{n-2k+\sqrt{(n-2k)^2-4(n-1)}}{2}, \frac{n-2k-\sqrt{(n-2k)^2-4(n-1)}}{2}\right\}$.

Therefore, $SLE(K_{1,n}) = n-2+\sqrt{(n-2k)^2-4(n-1)}$. Thus, we have $SLE(K_{1,n}) = 2(n-1)$, if all the edges are oriented towards or away from the center, and $SLE(K_{1,n}) = n-2+\sqrt{(n-2k)^2-4(n-1)}$, otherwise, where k, $1 \leq k \leq n-1$ is the number of edges oriented towards the center. ☐

As a consequence to Theorem 2.3, we have the following observation.

Corollary 2.1. *For a directed star of order n, skew Laplacian energy is always $2(n-2)$.*

The above Theorem gives a complete description of the skew Laplacian energy of orientations of $K_{1,n}$. It is clear that unlike the skew energy of any orientation of $K_{1,n}$, which is same as the corresponding energy, the skew Laplacian energy of orientations of $K_{1,n}$ is not same as the corresponding Laplacian energy. Moreover, it is also clear that any two orientations which contain edges directed from and directed towards the center of $K_{1,n}$ are non-isomorphic non-skew Laplacian cospectral digraphs. Further, it gives an infinite family of real non-symmetric matrices with real eigenvalues.

3. Bounds for Skew Laplacian Energy

In this Section, we obtain bounds for skew Laplacian energy $SLE(\mathscr{D})$, which gives its connection to various parameters associated to a digraph. We show these bounds for $SLE(\mathscr{D})$ are better than some of the previously known bounds.

We first mention some known bounds for $SLE(\mathscr{D})$. For a digraph with n vertices, m arcs having vertex out-degrees d_i^+ and vertex in-degrees d_i^-, $i = 1, 2, \ldots, n$, let $M = -m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i^+ - d_i^-\right)^2$ and $M_1 = M + 2m = m + \frac{1}{2}\sum_{i=1}^{n}\left(d_i^+ - d_i^-\right)^2$. Clearly, $M_1 \geq m$, with equality if and only if $\mathscr{D}$ is an Eulerian digraph.

The following bounds are obtained in the basic paper [2] for skew Laplacian energy $SLE(\mathscr{D})$ of a digraph $\mathscr{D}$, which are analogues to the corresponding bounds on Laplacian energy $LE(G)$.

Theorem 3.1. *Let $\mathscr{D}$ be a simple digraph possessing n vertices, m arcs and p components. Assume that d_i^+ and d_i^- respectively, are the out-degree and in-degree of the vertex v_i, $i = 1, 2, \ldots, n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are the skew Laplacian eigenvalues of $\mathscr{D}$. Then*

$$2\sqrt{|M|} \leq SLE(\mathscr{D}) \leq \sqrt{2M_1(n-p)}.$$

Equality occurs on the left if and only if for each pair of $\nu_{i_1}\nu_{j_1}$ and $\nu_{i_2}\nu_{j_2}$ ($i_1 \neq j_1$, $i_2 \neq j_2$), there exists a non-negative real number k such that $\nu_{i_1}\nu_{j_1} = k\nu_{i_2}\nu_{j_2}$; and for each pair of $\nu_{i_1}^2$ and $\nu_{i_2}^2$, there exists a non-negative real number l such that $\nu_{i_1}^2 = l\nu_{i_2}^2$. Equality occurs on the right if and only if $\mathscr{D}$ is 0-regular or $\mathscr{D}$ is an Eulerian digraph with skew Laplacian eigenvalues $0^{[p]}$, $(ai)^{[\frac{n-p}{2}]}$, $(-ai)^{[\frac{n-p}{2}]}$, $a > 0$, where $b^{[t]}$, means that the eigenvalues b is repeated t times in the spectrum.

As an immediate consequence to Theorem 3.1, we have the following result.

Corollary 3.1. *Let $\mathscr{D}$ be a simple digraph possessing p components $C_1, C_2, \ldots, C_p$. If $SLE(\mathscr{D}) = \sqrt{2M_1(n-p)}$, then each component C_i is Eulerian with odd number of vertices.*

Since $n - p \leq n$, we have the following consequence of Theorem 3.1.

Corollary 3.2. *For any simple digraph $\mathscr{D}$, we have $SLE(\mathscr{D}) \leq \sqrt{2M_1 n}$.*

If $\mathscr{D}$ has no isolated vertices, then $n \leq 2m$, and so $\sqrt{2M_1 n} \leq 2\sqrt{M_1 m} \leq 2M_1$. Thus, we have the following observation.

Corollary 3.3. *For any simple digraph $\mathscr{D}$, we have $SLE(\mathscr{D}) \leq 2M_1$.*

The following Theorem gives a Koolen type [9] upper bound for $SLE(\mathscr{D})$.

Theorem 3.2. *Let $\mathscr{D}$ be a simple digraph with n vertices, m arcs and p components. Assume that $t = |\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{n-p}| \geq 0$, where $\nu_1, \nu_2, \ldots, \nu_{n-p}, 0^{[p]}$ are the eigenvalues of $\widetilde{SL}(\mathscr{D})$. Then*

$$SLE(\mathscr{D}) \leq t + \sqrt{(n-p-1)(2M_1 - t^2)}. \tag{3.1}$$

Equality occurs if and only if $\mathscr{D}$ is 0-regular or $\mathscr{D}$ is an Eulerian digraph with skew Laplacian eigenvalues $0^{[p]}$, $(ai)^{\left[\frac{n-p}{2}\right]}$, $(-ai)^{\left[\frac{n-p}{2}\right]}$, $a > 0$.

Proof. Let $\widetilde{SL}(\mathscr{D}) = (l_{ij})$. By Schur's triangularization theorem [7], there exists a unitary matrix U such that $U^*\widetilde{SL}(\mathscr{D})U = T$, where $T = (t_{ij})$ is an upper triangular matrix with diagonal entries $t_{ii} = \nu_i,\ i = 1, 2, \ldots, n$. Therefore,

$$\sum_{i,j=1}^{n} |l_{ij}|^2 = \sum_{i,j=1}^{n} |t_{ij}|^2 \geq \sum_{i=1}^{n} |t_{ii}|^2 = \sum_{i=1}^{n} |\nu_i|^2, \tag{3.2}$$

that is,

$$\sum_{i=1}^{n} |\nu_i|^2 \leq \sum_{i,j=1}^{n} |l_{ij}|^2 = \sum_{i=1}^{n} (d_i^+ - d_i^-)^2 + 2m = 2M_1. \tag{3.3}$$

Now, applying Cauchy-Schwarz's inequality to vectors $(|\nu_2|, |\nu_3|, \ldots, |\nu_{n-p}|)$ and $(1, 1, \ldots, 1)$ and using (3.3), we have

$$\begin{aligned} SLE(\mathscr{D}) - |\nu_1| &= \sum_{i=2}^{n} |\nu_i| = \sum_{i=2}^{n-p} |\nu_i| \leq \sqrt{(n-p-1)\sum_{i=2}^{n-p} |\nu_i|^2} \\ &= \sqrt{(n-p-1)\sum_{i=2}^{n} |\nu_i|^2} \leq \sqrt{(n-p-1)(2M_1 - |\nu_1|^2)}. \end{aligned}$$

This gives

$$SLE(\mathscr{D}) \leq t + \sqrt{(n-p-1)(2M_1 - t^2)}.$$

Equality occurs in (3.1) if and only if equality occurs in (3.2) and equality occurs in Cauchy-Schwarz's inequality. Since equality occurs in (3.2) if and only if $T = (t_{ij})$ is a diagonal matrix. It follows that equality occurs in (3.1) if and only if $T = (t_{ij})$ is a diagonal matrix and $|\nu_2| = |\nu_3| = \cdots = |\nu_{n-p}|$.

From Schur's unitary triangularzation theorem [7], we know that $T = (t_{ij})$ is a diagonal matrix if and only if $\widetilde{SL}(\mathscr{D})$ is a normal matrix. That is,

$$\widetilde{SL}(\mathscr{D})\widetilde{SL}^*(\mathscr{D}) = \widetilde{SL}^*(\mathscr{D})\widetilde{SL}(\mathscr{D}). \tag{3.4}$$

Since $\widetilde{SL}(\mathscr{D})=\widetilde{D}(\mathscr{D})+S(\mathscr{D})$ and $\widetilde{SL}^*(\mathscr{D})=\widetilde{D}(\mathscr{D})+S(\mathscr{D})$, it follows from (3.4) that

$$\begin{aligned}&(\widetilde{D}(\mathscr{D})-S(\mathscr{D}))(\widetilde{D}(\mathscr{D})+S(\mathscr{D}))=(\widetilde{D}(\mathscr{D})-S(\mathscr{D}))(\widetilde{D}(\mathscr{D})-S(\mathscr{D}))\\ \Rightarrow &S(\mathscr{D})\widetilde{D}(\mathscr{D})=\widetilde{D}(\mathscr{D})S(\mathscr{D}).\end{aligned}$$

Comparing the element on the i^{th} row and the j^{th} column of the matrices on both sides, we arrive at

$$s_{ij}(d_j^+-d_j^-)=(d_i^+-d_i^-)s_{ij}. \tag{3.5}$$

If v_i and v_j are not adjacent, then $s_{ij}=0$ and so (3.5) always holds. Assume that v_i and v_j are adjacent, then $s_{ij}\neq 0$ and so (3.5) gives

$$d_i^+-d_i^-=d_j^+-d_j^-.$$

Let $C_1,C_2,\dots,C_p$ be the components of $\mathscr{D}$. As $C_k,\ 1\le k\le p$ is connected, there is a path between any two vertices, let $P:u=v_0,v_1,\dots,v_t=w$ be a path between u and w in C_k. Since for any two connected vertices in $\mathscr{D}$, the differences between the out-degree and in-degree is same, it follows that $d^+(u)-d^-(u)=d^+(w)-d^-(w)$, for all $u,w\in C_k$. Therefore, using the fact that $\sum_{v\in C_k}(d^+(v)-d^-(v))=0$, it follows that $d^+(v)-d^-(v)=0$, for all $v\in C_k$. That is, $d_i^+=d_i^-$, for all $v_i\in\mathscr{D}$, giving that $\widetilde{D}(\mathscr{D})=0$ and so $\widetilde{SL}(\mathscr{D})=-S(\mathscr{D})$. This shows that equality occurs in (3.1) if and only if the non-zero skew Laplacian eigenvalues $\nu_1,\nu_2,\dots,\nu_{n-p}$ of $\mathscr{D}$ are purely imaginary with $t=|\nu_1|$ and $|\nu_2|=\cdots=|\nu_{n-p}|$.

If $t=|\nu_1|$ and $|\nu_2|=\cdots=|\nu_{n-p}|=0$, then we must have $t=0$. For if $t>0$, then using the fact that the eigenvalues of $\widetilde{SL}(\mathscr{D})$ either zero or purely imaginary, it follows that the spectrum of $\widetilde{SL}(\mathscr{D})$ is $\{it,-it,0^{[n-1]}\}$, which is not possible as order of $\mathscr{D}$ is n. Therefore, we must have $t=0$ and so the spectrum of $\widetilde{SL}(\mathscr{D})$ contains 0 with multiplicity n. Since $d_i^+=d_i^-$, for all v_i it follows that $\mathscr{D}$ is a 0-regular digraph.

If $t=|\nu_1|$ and $|\nu_2|=\cdots=|\nu_{n-p}|=a,\ a>0$ then we must have $t=a$. For if $t>a>0$, then using the fact that the eigenvalues of $\widetilde{SL}(\mathscr{D})$ either zero or purely imaginary, it follows that the spectrum of $\widetilde{SL}(\mathscr{D})$ is $\{it,-it,\ (ia)^{\left[\frac{n-p-1}{2}\right]},\ (-ia)^{\left[\frac{n-p-1}{2}\right]},0^{[p]}\}$, which is not possible as order of $\mathscr{D}$ is n. Therefore, we must have $t=a$ and so the eigenvalues of $\widetilde{SL}(\mathscr{D})$ are $\{(ia)^{\left[\frac{n-p}{2}\right]},\ (-ia)^{\left[\frac{n-p}{2}\right]},0^{[p]}\}$. That completes the proof. □

The following arithmetic-geometric mean inequality can be found in [8].

Lemma 3.1. *If $a_1,a_2,\dots,a_n$ are non-negative numbers, then*

$$\begin{aligned}n\left[\frac{1}{n}\sum_{j=1}^n a_j-\left(\prod_{j=1}^n a_j\right)^{\frac{1}{n}}\right]&\le n\sum_{j=1}^n a_j-\left(\sum_{j=1}^n\sqrt{a_j}\right)^2\\ &\le n(n-1)\left[\frac{1}{n}\sum_{j=1}^n a_j-\left(\prod_{j=1}^n a_j\right)^{\frac{1}{n}}\right].\end{aligned}$$

Moreover, equality occurs if and only if $a_1=a_2=\cdots=a_n$.

The following inequality was obtained by Furuichi [3].

Lemma 3.2. *For* $a_1, a_2, \dots, a_n \geq 0$ *and* $p_1, p_2, \dots, p_n \geq 0$ *such that* $\sum_{j=1}^{n} p_i = 1$, *then*

$$\sum_{j=1}^{n} a_j p_j - \prod_{j=1}^{n} a_j^{p_j} \geq n\lambda \left(\frac{1}{n} \sum_{j=1}^{n} a_j - \prod_{j=1}^{n} a_j^{\frac{1}{n}} \right),$$

where $\lambda = \min\{p_1, p_2, \dots, p_n\}$. *Moreover, equality occurs if and only if* $a_1 = a_2 = \cdots = a_n$.

For a connected digraph $\mathscr{D}$ with absolute values of the skew Laplacian eigenvalues $|\nu_1| \geq |\nu_2| \geq |\nu_{n-1}| \geq 0$, let $K = \prod_{j=1}^{n-1} |\nu_j|$.

We obtain a lower bound for $SLE(\mathscr{D})$, in terms of the number of vertices n and the number K.

Theorem 3.3. *Let* $\mathscr{D}$ *be a simple connected digraph with* n *vertices and* m *arcs having skew Laplacian eigenvalues* $\nu_1, \nu_2, \dots, \nu_{n-1}, 0$ *with* $t = |\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{n-1}| \geq 0$. *Then*

$$SLE(\mathscr{D}) \geq t + (n-2) K^{\frac{1}{n-1}} \left(\frac{K^{\frac{1}{2(n-1)(n-2)}}}{t^{\frac{1}{2n-4}}} - 1 \right), \tag{3.6}$$

with equality if and only if $t = |\nu_1| = |\nu_2| = \cdots = |\nu_{n-1}|$.

Proof. Setting $n := n-1$, $a_j = |\nu_j|$, for $j = 1, 2, \dots, n-1$, $p_1 = \frac{1}{2(n-1)}$, $p_j = \frac{2n-3}{2(n-1)(n-2)}$, for $j = 2, 3, \dots, n-1$ in Lemma 3.2, we have

$$\begin{aligned}
&\frac{|\nu_1|}{2(n-1)} + \frac{2n-3}{2(n-1)(n-2)} \sum_{j=2}^{n-1} |\nu_j| - |\nu_1|^{\frac{1}{2(n-1)}} \prod_{j=2}^{n-1} |\nu_j|^{\frac{2n-3}{2(n-1)(n-2)}} \\
\geq & \frac{1}{2(n-1)} \sum_{j=1}^{n-1} |\nu_j| - \frac{1}{2} \prod_{j=1}^{n-1} |\nu_j|^{\frac{1}{n-1}},
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{|\nu_1|}{2(n-1)} + \frac{2n-3}{2(n-1)(n-2)} (SLE(\mathscr{D}) - |\nu_1|) - |\nu_1|^{\frac{-1}{2(n-2)}} K^{\frac{2n-3}{2(n-1)(n-2)}} \\
\geq & \frac{1}{2(n-1)} SLE(\mathscr{D}) - \frac{1}{2} K^{\frac{1}{n-1}},
\end{aligned}$$

this gives

$$SLE(\mathscr{D}) \geq 2(n-2) \left(\frac{|\nu_1|}{2(n-2)} + \frac{K^{\frac{2n-3}{2(n-1)(n-2)}}}{|\nu_1|^{\frac{1}{2(n-2)}}} - \frac{1}{2} K^{\frac{1}{n-1}} \right).$$

From previous the result follows.

Equality occurs in (3.6) if and only if equality occurs in Lemma 3.2, that is, if and only if $t = |\nu_1| = |\nu_2| = \cdots = |\nu_{n-1}|$. That completes the proof. □

We now obtain the bounds for $SLE(\mathscr{D})$, in terms of the number of vertices n, the numbers K, M and M_1 associated with the digraph $\mathscr{D}$.

Theorem 3.4. *Let $\mathscr{D}$ be a simple connected digraph with n vertices and m arcs having skew Laplacian eigenvalues $\nu_1, \nu_2, \ldots, \nu_{n-1}, 0$ with $|\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{n-1}| \geq 0$. Then*

$$\sqrt{2|M| + (n-1)(n-2)K^{\frac{2}{n-1}}} \leq SLE(\mathscr{D}) \leq \sqrt{2M_1(n-2) + (n-1)K^{\frac{2}{n-1}}}, \tag{3.7}$$

with equality on the left if and only if $\nu_1^2 = \nu_2^2 = \cdots = \nu_{n-1}^2$ and the equality on right occurs if and only if $\mathscr{D}$ is 0-regular or $\mathscr{D}$ is an Eulerian digraph with skew Laplacian eigenvalues $0^{[p]}$, $(ai)^{[\frac{n-p}{2}]}$, $(-ai)^{[\frac{n-p}{2}]}$, $a > 0$.

Proof. Setting $n := n-1$ and $a_j = |\nu_j|^2$, for $j = 1, 2, \ldots, n-1$ in Lemma 3.1, we have

$$\alpha \leq (n-1)\sum_{j=1}^{n-1} |\nu_j|^2 - \left(\sum_{j=1}^{n-1} |\nu_j|\right)^2 \leq (n-2)\alpha,$$

that is,

$$\alpha \leq (n-1)\sum_{j=1}^{n-1} |\nu_j|^2 - (SLE(\mathscr{D}))^2 \leq (n-2)\alpha, \tag{3.8}$$

where

$$\begin{aligned}\alpha &= (n-1)\left[\frac{1}{n-1}\sum_{j=1}^{n-1} |\nu_j|^2 - \left(\prod_{j=1}^{n-1} |\nu_j|^2\right)^{\frac{1}{n-1}}\right]\\ &= \sum_{j=1}^{n-1} |\nu_j|^2 - (n-1)\left(\prod_{j=1}^{n-1} |\nu_j|\right)^{\frac{2}{n-1}}\\ &= \sum_{j=1}^{n-1} |\nu_j|^2 - (n-1)K^{\frac{2}{n-1}}.\end{aligned}$$

Using (3.3) and the value of α, we have from the left inequality of (3.8)

$$(SLE(\mathscr{D}))^2 \leq (n-2)\sum_{j=1}^{n-1} |\nu_j|^2 + (n-1)K^{\frac{2}{n-1}},$$

that is,

$$SLE(\mathscr{D}) \leq \sqrt{2M_1(n-2) + (n-1)K^{\frac{2}{n-1}}},$$

which proves the right inequality.

Now, using inequality (7) from [2] and the value of α, we have from the right inequality of (3.8)

$$(SLE(\mathscr{D}))^2 \geq \sum_{j=1}^{n-1} |\nu_j|^2 + (n-1)(n-2)K^{\frac{2}{n-1}},$$

that is,

$$SLE(\mathscr{D}) \geq \sqrt{2|M| + (n-1)(n-2)K^{\frac{2}{n-1}}},$$

which proves the left inequality.

Equality occurs on the left of (3.7) if and only if equality occurs in Lemma 3.1 and equality occurs in (6) and (7) in [2]. Since equality occurs in Lemma 3.1 if and only if all a_i^s are equal and equality occurs in (6) and(7) of [2] if and only if for each pair of $\nu_{i_1}\nu_{j_1}$ and $\nu_{i_2}\nu_{j_2}$, $i_1 \neq j_1, i_2 \neq j_2$, there exists a non-negative real number k such that $\nu_{i_1}\nu_{j_1} = k\nu_{i_2}\nu_{j_2}$; and for each pair $\nu_{i_1}^2$ and $\nu_{i_2}^2$, there exists a non-negative real number l such that $\nu_{i_1}^2 = l\nu_{i_2}^2$. It follows that equality occurs on the left of (3.7) if and only if $\nu_1^2 = \nu_2^2 = \cdots = \nu_{n-1}^2$.

Equality occurs on the right of (3.7) if and only if equality occurs in Lemma 3.1 and equality occurs in (3.3). Since equality occurs in Lemma 3.1 if and only if all a_i^s are equal and equality occurs in (3.3) if and only if the matrix $T = (t_{ij})$ in Theorem 3.2, is a diagonal matrix. That is, equality occurs on the right of (3.7) if and only if $T = (t_{ij})$ is a diagonal matrix and $|\nu_1|^2 = |\nu_2|^2 = \cdots = |\nu_{n-1}|^2$. Now, proceeding similarly as in Theorem 3.2, it can be seen that equality occurs on the right of (3.7) if and only if $\mathscr{D}$ is 0-regular or $\mathscr{D}$ is an Eulerian digraph with the eigenvalues of $\widetilde{SL}(\mathscr{D})$ as $0^{[p]}$, $(ai)^{\left[\frac{n-p}{2}\right]}$, $(-ai)^{\left[\frac{n-p}{2}\right]}$, $a > 0$. □

Remark 3.1. The upper bound given by Theorem 3.4 is better than the upper bound given by Theorem 3.1 for all connected digraphs $\mathscr{D}$. As by arithmetic-geometric mean inequality, we have

$$2M_1 \geq \sum_{j=1}^{n-1} |\nu_j|^2 \geq (n-1)\left(\prod_{j=1}^{n-1} |\nu_j|\right)^{\frac{2}{n-1}} = (n-1)K^{\frac{2}{n-1}},$$

adding $2M_1(n-2)$ on both sides, we obtain

$$2M_1(n-1) \geq 2M_1(n-2) + (n-1)K^{\frac{2}{n-1}},$$

from which the result follows.

Remark 3.2. The lower bound given by Theorem 3.4, is better than the lower bound given by Theorem 3.1 for all connected digraphs $\mathscr{D}$, with $2|M| \leq (n-1)(n-2)K^{\frac{2}{n-1}}$.

We conclude this paper with the following problems which will be of interest for the future research.

For a simple digraph $\mathscr{D}$, the relation between the coefficients of the characteristic polynomial of $S(\mathscr{D})$ and the structure of $\mathscr{D}$ is known [10]. Like wise, the following problem will be of interest.

Problem 3.1. For a simple non-Eulerian digraph $\mathscr{D}$, interpret if possible the coefficients of the characteristic polynomial of $\widetilde{SL}(\mathscr{D})$, in terms of structure of $\mathscr{D}$.

For a simple digraph $\mathscr{D}$, the largest among the absolute values of skew eigenvalues is called skew spectral radius [10]. Like wise, we call the largest among the absolute values of skew Laplacian eigenvalues as the skew Laplacian spectral radius of the digraph $\mathscr{D}$. The following problem will be of interest.

Problem 3.2. Establish possible relations between the skew spectral radius and the skew Laplacian spectral radius of a digraph $\mathscr{D}$. Also, establish possible relations between the skew Laplacian spectral radius with the parameters associated with the structure of the digraph.

Various relations between the skew spectrum of a simple digraph $\mathscr{D}$ and the adjacency spectrum of the underlying graph are established see [10]. The problem can be of interest.

Problem 3.3. Establish possible relations between the skew Laplacian spectrum of a digraph $\mathscr{D}$ and the Laplacian spectrum of the corresponding underlying graph.

A complete description of the skew energy of a cycle is given in [1]. Like wise, the following problem will be of interest.

Problem 3.4. For any orientation, give the complete description for the skew Laplacian energy of the cycle C_n.

Problem 3.5. Characterise all the non-Eulerian digraphs $\mathscr{D}$ for which $SLE(\mathscr{D}) = E_s(\mathscr{D})$.

Problem 3.6. If possible, interpret skew Laplacian energy in chemistry and other disciplines.

For a simple digraph $\mathscr{D}$, the rank of the matrix $S(\mathscr{D})$ is called skew rank of the underlying graph G. Like wise, if we call the rank of the matrix $\widetilde{SL}(\mathscr{D})$ as the skew Laplacian rank of the corresponding underlying graph G, the following problem will be of interest.

Problem 3.7. For a simple digraph $\mathscr{D}$, establish possible relations between the skew Laplacian rank and the rank of underlying graph.

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