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ON STRAIGHT-LINE EMBEDDING OF GRAPHS

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Abstract. Let G be a graph with n vertices, and P be a set of n points in the Euclidean space $\mathbb{R}^m$. A straight-line embedding of G onto P is an embedding of G onto P whose images of vertices are distinct points in P, and images of edges are straight line segments in $\mathbb{R}^m$. In this paper, we classify these kinds of sets.

1. Introduction

A well-known theorem of Fary [3] states that any (simple) planar graph can be embedded in the plane with straight edges. Much work has been done on straight-line embedding (see [2,5,6,10,11]). In this paper, we consider the more general case where the positions of vertices are extended to a subset of $\mathbb{R}^m$. To describe the problem more explicitly, we introduce the following notations. Throughout this paper, we let $G = (V(G), E(G))$ denote a graph of order n, and P denote a set of n points in the Euclidean space $\mathbb{R}^m$. A straight-line embedding Φ of G onto P in $\mathbb{R}^m$ is a one-to-one mapping of $V(G)$ onto P such that the images of edges are non-crossing line segments. The aim of this paper is to classify P.

2. Main Results

Definition 2.1. Let P be a set of points in $\mathbb{R}^m$. A straight-line embedding of G onto P in $\mathbb{R}^m$ is a bijection $\Phi : V(G) \to P$ which for any two distinct edges $uv, u'v' \in E(V(G))$, the open line segments $\Phi(u)\Phi(v)$ and $\Phi(u')\Phi(v')$ in $\mathbb{R}^m$ have no points in common.

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The points of P does not need to be in general position, necessarily. For example, there exists a straight-line embedding of P_n to arbitrary distinct points in $\mathbb{R}^m$ ($n \geq 3, m \geq 2$).

Definition 2.2. A graph G is line-drawable in $\mathbb{R}^m$ if there exists a set of points $P \subseteq \mathbb{R}^m$ such that G has a straight-line embedding of G onto P in $\mathbb{R}^m$.

Definition 2.3. A subset $A \subseteq \mathbb{R}^m$ is satisfiable if for any graph G, there exists a set of points $P \subseteq A$ such that G is line-drawable in $\mathbb{R}^m$.

Theorem 2.1. *A subset $A \subseteq \mathbb{R}^3$ is a satisfiable set if and only if A is not a subset of a finite union of planes.*

Proof. Clearly, A is an infinite set. For necessity, let $A \subseteq \mathbb{R}^3$ be satisfiable. Suppose to the contrary that there are planes P_i ($1 \leq i \leq n$) in $\mathbb{R}^3$ such that $A \subseteq \bigcup\limits_{i=1}^{n} P_i$. We claim that K_{4n+1} is not line-drawable in $\mathbb{R}^3$ with respect to A. By the pigeon hole principle, there exists P_j ($1 \leq j \leq n$) such that P_j contains at least 5 vertices of K_{4n+1}. With respect to classification of planar graphs with forbidden graphs by Kuratowski [7], the proof is complete.

For sufficiency, suppose that A is not a finite union of planes in $\mathbb{R}^3$. It is enough to prove it for K_n ($n \geq 1$). Then it will hold for any subgraph of K_n. Thus we prove it by induction on the number of vertices of the complete graph K_n. Clearly K_1 and K_2 are line-drawable. Suppose that the statement holds for K_n. Let $B \subseteq \mathbb{R}^3$ be the union of all planes containing any three vertices of K_n. In other words,

$$B = \{P | x, y, z \in P \cap V(K_n), P \text{ is a plane in } \mathbb{R}^3\} \ (n \geq 3).$$

Since A is not a subset of a finite union of planes, so that $A \backslash B \neq \phi$. We take $x_{n+1} \in A \backslash B$. x_n is not on a plane containing any three vertices of K_n ($n \geq 3$). Therefore, by connecting x_{n+1} to each x_i ($1 \leq i \leq n$) by a segment, K_{n+1} is line-drawable. ☐

Corollary 2.1. *A subset $A \subseteq \mathbb{R}^m$ where $m \geq 3$ is satisfiable if and only if A is not a subset of finite union of planes in $\mathbb{R}^m$ which each plane is topologically isometric with $\mathbb{R}^2$.*

Example 2.1. Plane $\mathbb{R}^2$ is not satisfiable. In other words, all graphs can not be drawn by straight lines in $\mathbb{R}^2$. The torus and double torus can be embedded on $\mathbb{R}^3$ [1] and since they are not subsets of finite union of planes, therefore they are satisfiable. On the other hand, the projective plane and Klein bottle cannot be embedded on $\mathbb{R}^3$ and with respect to strong Whitney theorem, there exist embeddings of them on $\mathbb{R}^4$ [1] and since similarly they are not subsets of finite union of planes, therefore they are satisfiable. Finally, each subset of $\mathbb{R}^m$ ($m \geq 3$) which has a non-zero Lebesgue measure is an example of a satisfiable set, as the measure of finite union of planes is zero in $\mathbb{R}^m$ ($m \geq 3$) [4].

Definition 2.4. A subset $A \subseteq \mathbb{R}^m$ is n-satisfiable if for any graph with at most n vertices, there exists $P \subseteq A$ such that G is line-drawable in $\mathbb{R}^m$.

Example 2.2. Let A be a set of n points. Then A is 4-satisfiable, if and only if A can form a polygon with at least one point inside it.

Theorem 2.2. *If a subset $A \subseteq \mathbb{R}^3$ is not a subset of $\binom{n-1}{3}$ planes, where $n \geq 4$, then A is n-satisfiable.*

Proof. We prove it by induction on the number of vertices of the complete graph K_n. Clearly, it is true for $n = 4$. Suppose that the statement holds for K_n ($n \geq 4$). We want to show that K_{n+1} is line-drawable in $\mathbb{R}^3$ with vertices chosen in A which A is not a subset of $\binom{n}{3}$ planes. Since $\binom{n-1}{3} < \binom{n}{3}$, therefore $x_1, x_2, \ldots, x_n \in A$ exist such that K_n with $x_1, \ldots, x_n$ is line-drawable in $\mathbb{R}^3$ with at most $\binom{n}{3}$ planes passing through (each plane pass through 3 vertices). Since A is not a subset of $\binom{n}{3}$ planes, so we can choose $x_{n+1} \in A$ which is not in common with any of these $\binom{n}{3}$ planes. Clearly, by connecting x_{n+1} to each x_i ($1 \leq i \leq n$) by a segment, K_{n+1} is line-drawable. □

Definition 2.5. A subset $A \subseteq \mathbb{R}^m$ is strongly satisfiable if every graph is line-drawable with arbitrary distinct vertices chosen from A.

It is clear that if a subset $A \subseteq \mathbb{R}^m$ is strongly satisfiable, then it consists of points, not lines and planes. Therefore, we denote a strongly satisfiable set by P.

Theorem 2.3. *A subset $P \subseteq \mathbb{R}^3$ is strongly satisfiable if and only if the following conditions hold.*

(a) *P is infinite.*
(b) *Each line in $\mathbb{R}^3$ intersects P in at most 2 points.*
(c) *Each plane in $\mathbb{R}^3$ intersects P in at most 4 points.*
(d) *If a plane in $\mathbb{R}^3$ intersect P in 4 points, then those 4 points are a 4-satisfiable set.*

Proof. The proof is clear with respect to a straight-forward argument. □

In the following examples, we will show some strongly satisfiable sets in $\mathbb{R}^3$.

Example 2.3. Let $P = \{(t, \sin(t), \cos(t)) \mid t \in [0, \frac{\pi}{2}]\}$.

The intersections of a plane in general position and a helix reduces to the intersections of the cosine function and a line in general position [9]. Therefore, in the above interval every plane in $\mathbb{R}^3$ intersects P in at most 3 points. Moreover, every line in $\mathbb{R}^3$ intersects P in at most 2 points. Therefore, P satisfies all conditions of the latter theorem and P is a strongly satisfiable set.

It is clear that $\mathbb{R}^3$ is not a strongly satisfiable set. However, in the next example we show a satisfiable set which is dense in $\mathbb{R}^3$. We construct such P by induction.

Example 2.4. Since $\mathbb{R}^3$ is separable, so $\mathbb{R}^3$ has a countable base [8]. Let $\{B_i\}_{i=1}^{\infty}$ be a countable base for $\mathbb{R}^3$ which each B_i is an open ball. Choose $x_1 \in B_1$ arbitrary in B_1 and then choose $x_2 \in B_2$ $(x_2 \neq x_1)$ arbitrary in B_2. Suppose we have chosen $x_1, \ldots, x_n$ which $x_i \in B_i$ for $i = 1, \ldots, n$.

Consider all lines that pass through x_i and x_j $(1 \leq i < j \leq n)$ and all planes that pass through x_i, x_j and x_k $(1 \leq i < j < k \leq n)$. Since B_{n+1} is an open ball, we choose $x_{n+1} \in B_{n+1}$ in such a way that does not lie on the above lines and planes.

$P = \{x_i\}_{i=1}^{\infty}$ is the desired set.

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