

ON WEIGHTED GENERALIZATION OF TRAPEZOID TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

H. BUDAK¹ AND M. Z. SARIKAYA¹

ABSTRACT. In this paper, we obtain generalized weighted trapezoid inequalities for functions of two independent variables with bounded variation. We also give some applications for qubature formulas.

1. INTRODUCTION

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [20].

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, we have the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality*.

In 2001, Dragomir [15] obtained following Ostrowski type inequality for functions of bounded variation.

Key words and phrases. Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral.

2010 *Mathematics Subject Classification.* Primary: 26D15, 26B30. Secondary: 26D10, 41A55.

Received: June 06, 2017.

Accepted: September 19, 2017.

Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then

$$\left| \int_a^b f(t)dt - (b-a)f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

In [21], Tseng et al. gave the following generalization of weighted trapezoidal inequality for functions of bounded variation.

Theorem 1.3. Let $w : [a, b] \rightarrow [0, \infty)$ nonnegative and continuous and $h : [a, b] \rightarrow \mathbb{R}$ be differentiable such that $h'(t) = w(t)$ on $[a, b]$. If $f : [a, b] \rightarrow \mathbb{R}$ be mapping of bounded variation on $[a, b]$, then

$$(1.2) \quad \left| \int_a^b w(t)f(t)dt - [(x-h(a))f(a) + (h(b)-x)f(b)] \right| \\ \leq \left[\frac{1}{2} \int_a^b w(t)dt + \left| x - \frac{h(a)+h(b)}{2} \right| \right] \bigvee_a^b(f),$$

for all $x \in [h(a), h(b)]$. The constant $\frac{1}{2}$ is the best possible.

2. PRELIMINARIES AND LEMMAS

In 1910, Fréchet [17] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x, y)$ and $g(x, y)$ are defined over the rectangle $Q = [a, b] \times [c, d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i, y = y_i$

$$a = x_0 < x_1 < \dots < x_n = b \text{ and } c = y_0 < y_1 < \dots < y_m = d,$$

let ξ_i, η_j be any numbers satisfying $\xi_i \in [x_{i-1}, x_i], \eta_j \in [y_{j-1}, y_j], i = 1, 2, \dots, n, j = 1, 2, \dots, m$ and for all i, j let

$$\Delta_{11}g(x_i, y_j) = g(x_{i-1}, y_{j-1}) - g(x_{i-1}, y_j) - g(x_i, y_{j-1}) + g(x_i, y_j).$$

Then if the sum

$$S = \sum_{i=1}^n \sum_{j=1}^m f(\xi_i, \eta_j) \Delta_{11}g(x_i, y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$(2.1) \quad \int_a^b \int_c^d f(x, y) d_y d_x g(x, y).$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i=1}^n \sum_{j=1}^m f(\xi_{ij}, \eta_{ij}) \Delta_{11}g(x_i, y_j),$$

where ξ_{ij}, η_{ij} are numbers satisfying $\xi_{ij} \in [x_{i-1}, x_i], \eta_{ij} \in [y_{j-1}, y_j]$ we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$(2.2) \quad (*) \int_a^b \int_c^d f(x, y) d_y d_x g(x, y).$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [13] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables.

2.1. Definitions. The function $f(x, y)$ is assumed to be defined in rectangle $R(a \leq x \leq b, c \leq y \leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i, \quad i = 0, 1, 2, \dots, m, \quad a = x_0 < x_1 < \dots < x_m = b, \\ y &= y_j, \quad j = 0, 1, 2, \dots, n, \quad c = y_0 < y_1 < \dots < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\begin{aligned} \Delta_{11}f(x_i, y_j) &= f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j), \\ \Delta f(x_i, y_j) &= f(x_{i+1}, y_{j+1}) - f(x_i, y_j). \end{aligned}$$

The total variation function, $\phi(\bar{x}) [\psi(\bar{y})]$, is defined as the total variation of $f(\bar{x}, y) [f(x, \bar{y})]$ considered as a function of $y [x]$ alone in interval $(c, d) [(a, b)]$, or as $+\infty$ if $f(\bar{x}, y) [f(x, \bar{y})]$ is of unbounded variation.

Definition 2.1 (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x, y)$ is said to be of bounded variation if the sum

$$\sum_{i=0, j=0}^{m-1, n-1} |\Delta_{11}f(x_i, y_j)|$$

is bounded for all nets.

Definition 2.2 (Fréchet). The function $f(x, y)$ is said to be of bounded variation if the sum

$$\sum_{i=0, j=0}^{m-1, n-1} \epsilon_i \bar{\epsilon}_j |\Delta_{11}f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\bar{\epsilon}_j = \pm 1$.

Definition 2.3 (Hardy-Krause). The function $f(x, y)$ is said to be of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\bar{x}, y)$ is of bounded variation in y (i.e., $\phi(\bar{x})$ is finite) for at least one \bar{x} and $f(x, \bar{y})$ is of bounded variation in x (i.e., $\psi(\bar{y})$ is finite) for at least one \bar{y} .

Definition 2.4 (Arzelà). Let (x_i, y_i) , $i = 0, 1, 2, \dots, m$ be any set of points satisfying the conditions

$$\begin{aligned} a = x_0 < x_1 < \dots < x_m = b, \\ c = y_0 < y_1 < \dots < y_m = d. \end{aligned}$$

Then $f(x, y)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^m |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the concept of total variation of a function of variables, as follows.

Let f be of bounded variation on $Q = [a, b] \times [c, d]$ and let $\sum(P)$ denote the sum $\sum_{i=1}^n \sum_{j=1}^m |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q . The number

$$\bigvee_Q(f) := \bigvee_c^d \bigvee_a^b(f) := \sup \left\{ \sum(P) : P \in P(Q) \right\}$$

is called the total variation of f on Q .

The following lemmas will be used in our main result.

Lemma 2.1 (Integrating by parts). [19, Lemma 2] *If $f(t, s)$ is continuous on rectangle $Q = [a, b] \times [c, d]$ and $\alpha(t, s) \in BV_H(Q)$, then $\alpha(t, s)$ is integrable with respect to $f(t, s)$ over Q in the Riemann-Stieltjes sense and*

$$\begin{aligned} \int_a^b \int_c^d f(t, s) d_t d_s \alpha(t, s) &= \int_a^b \int_c^d \alpha(t, s) d_t d_s f(t, s) \\ &\quad - \int_a^b \alpha(t, d) d_t f(t, d) + \int_a^b \alpha(t, c) d_t f(t, c) \\ &\quad - \int_c^d \alpha(b, s) d_s f(b, s) + \int_c^d \alpha(a, s) d_s f(a, s) \\ &\quad + f(b, d) \alpha(b, d) - f(b, c) \alpha(b, c) - f(a, d) \alpha(a, d) \\ &\quad + f(a, c) \alpha(a, c). \end{aligned}$$

Lemma 2.2. [18] Assume that $\Omega \in RS(g)$ on Q and g is of bounded variation on Q , then

$$(2.3) \quad \left| \int_c^d \int_a^b \Omega(x, y) d_x d_y g(x, y) \right| \leq \sup_{(x, y) \in Q} |\Omega(x, y)| \bigvee_Q (g).$$

In [18], authors obtained the following Ostrowski type inequality for functions of two variables with bounded variation.

Theorem 2.1. If the function $f : Q = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is of bounded variation on Q , then for all $(x, y) \in Q$ we have the inequality

$$(2.4) \quad \left| f(x, y) - \frac{1}{b-a} \int_a^b f(t, y) dt - \frac{1}{d-c} \int_c^d f(x, s) ds + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ \leq \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \left[\frac{1}{2} + \frac{|y - \frac{c+d}{2}|}{d-c} \right] \bigvee_a^b \bigvee_c^d (f),$$

where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q .

Moreover, authors gave the following trapezoid inequality for mappings of two variables with bounded variation.

Theorem 2.2. Let $f : Q \rightarrow \mathbb{R}$ be mapping of bounded variation on Q . Then for all $(x, y) \in Q$, we have inequality

$$(2.5) \quad \left| \frac{f(b, d) + f(b, c) + f(a, d) + f(a, c)}{4} - \frac{1}{2(d-c)} \left[\int_c^d f(a, s) ds + \int_c^d f(b, s) ds \right] \right. \\ \left. - \frac{1}{2(b-a)} \left[\int_a^b f(t, c) dt + \int_a^b f(t, d) dt \right] + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ \leq \frac{1}{4} \bigvee_a^b \bigvee_c^d (f).$$

The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see [6].

In [8], Budak and Sarikaya obtained the following generalized trapezoid inequality.

Theorem 2.3. Let $f : Q \rightarrow \mathbb{R}$ be a mapping of bounded variation on Q . Then for all $(x, y) \in Q$, we have inequality

$$(2.6) \quad |(b-x)(d-y)f(b, d) + (b-x)(y-c)f(b, c) \\ + (x-a)(d-y)f(a, d) + (x-a)(y-c)f(a, c)|$$

$$\begin{aligned}
& - (d - y) \int_a^b f(t, d) dt - (y - c) \int_a^b f(t, c) dt \\
& - (b - x) \int_c^d f(b, s) ds - (x - a) \int_c^d f(a, s) ds + \int_a^b \int_c^d f(t, s) ds dt \Big| \\
& \leq \left[\frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] \left[\frac{1}{2} (d - c) + \left| y - \frac{c + d}{2} \right| \right] \bigvee_a^b \bigvee_c^d (f),
\end{aligned}$$

where $\bigvee_a^b \bigvee_c^d (f)$ denotes the total variation of f on Q .

For more information and recent developments on inequalities for mappings of bounded variation, please refer to [1–11, 14–16, 18, 21–26].

The aim of this paper is to establish weighted generalization of trapezoid inequality for functions of two independent variables with bounded variation.

3. MAIN RESULTS

Let $w_1 : [a, b] \rightarrow [0, \infty)$ continuous and nonnegative on (a, b) and $h_1 : [a, b] \rightarrow \mathbb{R}$ be differentiable such that $h_1'(t) = w_1(t)$ on $[a, b]$. Similarly, let $w_2 : [c, d] \rightarrow [0, \infty)$ continuous and positive on (c, d) and $h_2 : [c, d] \rightarrow \mathbb{R}$ be differentiable such that $h_2'(t) = w_2(t)$ on $[c, d]$.

Theorem 3.1. *If $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a mapping of bounded variation on $[a, b] \times [c, d]$, then we have the following inequality for all $(x, y) \in [h_1(a), h_1(b)] \times [h_2(c), h_2(d)]$*

$$\begin{aligned}
(3.1) \quad & |(h_1(b) - x)(h_2(d) - y)f(b, d) + (h_1(b) - x)(y - h_2(c))f(b, c) \\
& + (x - h_1(a))(h_2(d) - y)f(a, d) + (x - h_1(a))(y - h_2(c))f(a, c) \\
& - (h_2(d) - y) \int_a^b w_1(t)f(t, d) dt - (y - h_2(c)) \int_a^b w_1(t)f(t, c) dt \\
& - (h_1(b) - x) \int_c^d w_2(s)f(b, s) ds - (x - h_1(a)) \int_c^d w_2(s)f(a, s) ds \\
& + \int_a^b \int_c^d w_1(t)w_2(s)f(t, s) ds dt \Big| \\
& \leq \left[\frac{1}{2} \int_a^b w_1(t) dt + \left| x - \frac{h_1(a) + h_1(b)}{2} \right| \right] \\
& \times \left[\frac{1}{2} \int_c^d w_2(t) dt + \left| y - \frac{h_2(c) + h_2(d)}{2} \right| \right] \bigvee_a^b \bigvee_c^d (f),
\end{aligned}$$

where $\bigvee_a^b \bigvee_c^d(f)$ denotes the total variation of f on interval $[a, b] \times [c, d]$.

Proof. For $(x, y) \in [h_1(a), h_1(b)] \times [h_2(c), h_2(d)]$, using integration by parts, we have

$$\begin{aligned}
 (3.2) \quad & \int_a^b \int_c^d (x - h_1(t))(y - h_2(s)) d_t d_s f(t, s) \\
 &= (h_1(b) - x)(h_2(d) - y) f(b, d) + (h_1(b) - x)(y - h_2(c)) f(b, c) \\
 &\quad + (x - h_1(a))(h_2(d) - y) f(a, d) + (x - h_1(a))(y - h_2(c)) f(a, c) \\
 &\quad - (h_2(d) - y) \int_a^b w_1(t) f(t, d) dt - (y - h_2(c)) \int_a^b w_1(t) f(t, c) dt \\
 &\quad - (h_1(b) - x) \int_c^d w_2(s) f(b, s) ds - (x - h_1(a)) \int_c^d w_2(s) f(a, s) ds \\
 &\quad + \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt.
 \end{aligned}$$

Taking modulus (3.2) and using Lemma 2.2, we get

$$\begin{aligned}
 & |(h_1(b) - x)(h_2(d) - y) f(b, d) + (h_1(b) - x)(y - h_2(c)) f(b, c) \\
 & + (x - h_1(a))(h_2(d) - y) f(a, d) + (x - h_1(a))(y - h_2(c)) f(a, c) \\
 & - (h_2(d) - y) \int_a^b w_1(t) f(t, d) dt - (y - h_2(c)) \int_a^b w_1(t) f(t, c) dt \\
 & - (h_1(b) - x) \int_c^d w_2(s) f(b, s) ds - (x - h_1(a)) \int_c^d w_2(s) f(a, s) ds \\
 & + \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt| \\
 &= \left| \int_a^b \int_c^d (x - h_1(t))(y - h_2(s)) d_t d_s f(t, s) \right| \\
 &\leq \sup_{t \in [a, b]} |x - h_1(t)| \sup_{s \in [c, d]} |y - h_2(s)| \bigvee_a^b \bigvee_c^d(f).
 \end{aligned}$$

Since $x - h_1(t)$ is decreasing on $[a, b]$, $h_1(a) \leq x \leq h_1(b)$, and $h_1'(t) = w_1(t)$ on $[a, b]$, we have

$$\begin{aligned}
 (3.3) \quad & \sup_{t \in [a, b]} |x - h_1(t)| = \max \{x - h_1(a), h_1(b) - x\} \\
 &= \frac{h_1(a) + h_1(b)}{2} + \left| x - \frac{h_1(a) + h_1(b)}{2} \right|
 \end{aligned}$$

$$= \frac{1}{2} \int_a^b w_1(t) dt + \left| x - \frac{h_1(a) + h_1(b)}{2} \right|.$$

Similarly, we have

$$\sup_{s \in [c, d]} |y - h_2(s)| = \frac{1}{2} \int_c^d w_2(t) dt + \left| y - \frac{h_2(c) + h_2(d)}{2} \right|.$$

This completes the proof. \square

Remark 3.1. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c, d]$ in Theorem 3.1, then the inequality (3.1) reduces the inequality (2.6).

Corollary 3.1 (Weighted trapezoid). *Under the assumption of Theorem 3.1 with $x = \frac{h_1(a) + h_1(b)}{2}$ and $y = \frac{h_2(c) + h_2(d)}{2}$, then we have the following weighted trapezoid inequality*

$$(3.4) \quad \left| \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) \frac{f(b, d) + f(b, c) + f(a, d) + f(a, c)}{4} \right. \\ \left. - \left(\int_c^d w_2(t) dt \right) \left[\int_a^b w_1(t) f(t, d) dt + \int_a^b w_1(t) f(t, c) dt \right] \right. \\ \left. - \left(\int_a^b w_1(t) dt \right) \left[\int_c^d w_2(s) f(b, s) ds + \int_c^d w_2(s) f(a, s) ds \right] \right. \\ \left. + \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt \right| \\ \leq \frac{1}{4} \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) \bigvee_a^b \bigvee_c^d (f).$$

The constant $\frac{1}{4}$ is the best possible.

Remark 3.2. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c, d]$ in Corollary 3.1, then the inequality (3.4) reduces the inequality (2.5).

Corollary 3.2 (Weighted left rectangle inequality). *Under the assumption of Theorem 3.1 with $x = h_1(b)$ and $y = h_2(d)$, then we have the following weighted left rectangle inequality*

$$(3.5) \quad \left| \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) f(a, c) - \left(\int_c^d w_2(t) dt \right) \int_a^b w_1(t) f(t, c) dt \right. \\ \left. - \left(\int_a^b w_1(t) dt \right) \int_c^d w_2(s) f(a, s) ds + \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt \right|$$

$$\leq \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) \bigvee_a^b \bigvee_c^d (f).$$

Remark 3.3. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c, d]$ in Corollary 3.2, then Corollary 3.2 reduces the Remark 1.a in [5].

Corollary 3.3 (Weighted right rectangle inequality). *Under the assumption of Theorem 3.1 with $x = h_1(a)$ and $y = h_2(c)$, then we have the following weighted right rectangle inequality*

$$\begin{aligned} & \left| \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) f(b, d) - \left(\int_c^d w_2(t) dt \right) \int_a^b w_1(t) f(t, d) dt \right. \\ & \quad \left. - \left(\int_a^b w_1(t) dt \right) \int_c^d w_2(s) f(b, s) ds + \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt \right| \\ & \leq \left(\int_a^b w_1(t) dt \right) \left(\int_c^d w_2(t) dt \right) \bigvee_a^b \bigvee_c^d (f). \end{aligned}$$

Remark 3.4. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c, d]$ in Corollary 3.3, then Corollary 3.3 reduces the Remark 1.b in [5].

4. APPLICATION TO A CUBATURE RULE

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \dots < x_n = b$, and $J_m : c = y_0 < y_1 < \dots < y_m = d$ with $l_1^i := x_{i+1} - x_i$ and $l_2^j := y_{j+1} - y_j$, and let

$$\begin{aligned} v(l_1) &:= \max \{ l_1^i \mid i = 0, \dots, n-1 \}, \\ v(l_2) &:= \max \{ l_2^j \mid j = 0, \dots, m-1 \}, \end{aligned}$$

$$v(W_1) := \max \{ W_1^i \mid i = 0, \dots, n-1 \}, \quad W_1^i := \int_{x_i}^{x_{i+1}} w_1(u) du = h_1(x_{i+1}) - h_1(x_i)$$

and

$$v(W_2) := \max \{ W_2^j \mid j = 0, \dots, m-1 \}, \quad W_2^j := \int_{y_j}^{y_{j+1}} w_2(u) du = h_2(y_{j+1}) - h_2(y_j).$$

Let us have w_1, h_1, w_2 and h_2 defined as in Theorem 3.1 and let $\xi_i \in [h_1(x_i), h_1(x_{i+1})]$, $i = 0, \dots, n-1$ and $\eta_j \in [h_2(y_j), h_2(y_{j+1})]$, $j = 0, \dots, m-1$.

Define the sum

$$\begin{aligned} (4.1) \quad & A(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta) \\ & := \sum_{j=0}^{m-1} (h_2(y_{j+1}) - \eta_j) \int_a^b w_1(t) f(t, y_{j+1}) dt + \sum_{j=0}^{m-1} (\eta_j - h_2(y_j)) \int_a^b w_1(t) f(t, y_j) dt \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{n-1} (h_1(x_{i+1}) - \xi_i) \int_c^d w_2(s) f(x_{i+1}, s) ds + \sum_{i=0}^{n-1} (\xi_i - h_1(x_i)) \int_c^d w_2(s) f(x_i, s) ds \\
& - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (h_1(x_{i+1}) - \xi_i) (h_2(y_{j+1}) - \eta_j) f(x_{i+1}, y_{j+1}) \\
& - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (h_1(x_{i+1}) - \xi_i) (\eta_j - h_2(y_j)) f(x_{i+1}, y_j) \\
& - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (\xi_i - h_1(x_i)) (h_2(y_{j+1}) - \eta_j) f(x_i, y_{j+1}) \\
& - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (\xi_i - h_1(x_i)) (\eta_j - h_2(y_j)) f(x_i, y_j).
\end{aligned}$$

Theorem 4.1. *Let f defined as in Theorem 3.1 and let*

$$\begin{aligned}
& \int_a^b \int_c^d w_1(t) w_2(s) f(t, s) ds dt \\
& = A(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta) + R(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta),
\end{aligned}$$

where the remainder term $R(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta)$ satisfies

$$\begin{aligned}
(4.2) \quad & |R(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta)| \\
& \leq \left[\frac{1}{2} v(W_1) + \max_{0 \leq i \leq n} \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
& \quad \times \left[\frac{1}{2} v(W_2) + \max_{0 \leq j \leq m} \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \bigvee_a^b \bigvee_c^d (f) \\
& \leq v(W_1) v(W_2) \bigvee_a^b \bigvee_c^d (f).
\end{aligned}$$

Proof. Applying Theorem 3.1 to the bidimensional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we have

$$\begin{aligned}
(4.3) \quad & |(h_1(x_{i+1}) - \xi_i) (h_2(y_{j+1}) - \eta_j) f(x_{i+1}, y_{j+1}) \\
& + (h_1(x_{i+1}) - \xi_i) (\eta_j - h_2(y_j)) f(x_{i+1}, y_j) \\
& + (\xi_i - h_1(x_i)) (h_2(y_{j+1}) - \eta_j) f(x_i, y_{j+1}) \\
& + (\xi_i - h_1(x_i)) (\eta_j - h_2(y_j)) f(x_i, y_j) \\
& - (h_2(y_{j+1}) - \eta_j) \int_{x_i}^{x_{i+1}} w_1(t) f(t, y_{j+1}) dt - (\eta_j - h_2(y_j)) \int_{x_i}^{x_{i+1}} w_1(t) f(t, y_{j+1}) dt
\end{aligned}$$

$$\begin{aligned}
 & - (h_1(x_{i+1}) - \xi_i) \int_{y_j}^{y_{j+1}} w_2(s) f(x_{i+1}, s) ds - (\xi_i - h_1(x_i)) \int_{y_j}^{y_{j+1}} w_2(s) f(x_i, s) ds \\
 & + \left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t, s) ds dt \right| \\
 \leq & \left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
 & \times \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f),
 \end{aligned}$$

for any $\xi_i \in [h_1(x_i), h_1(x_{i+1})]$, $i = 0, \dots, n - 1$ and $\eta_j \in [h_2(y_j), h_2(y_{j+1})]$, $j = 0, \dots, m - 1$.

Summing the inequality (4.3) over i from 0 to $n - 1$ and j from 0 to $m - 1$ and using the generalized triangle inequality, we get

$$\begin{aligned}
 (4.4) \quad & |R(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta)| \\
 & \leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
 & \quad \times \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
 & \leq \max_{0 \leq i \leq n} \left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
 & \quad \times \max_{0 \leq j \leq m} \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f),
 \end{aligned}$$

which finishes the proof of the first inequality in (4.2).

In the last inequality in (4.4), we have

$$\left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \leq \frac{1}{2} W_1^i, \quad i = 0, 1, \dots, n - 1,$$

and so,

$$\max_{i=0, \dots, n-1} \left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \leq \frac{1}{2} v(W_1).$$

Similarly, we get

$$\max_{j=0, \dots, m-1} \left| h_2(\eta_j) - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \leq \frac{1}{2} v(W_2).$$

The proof of the theorem is completely completed. □

Remark 4.1. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) \equiv 1$, $h_2(s) = s$ on $[c, d]$ in Theorem 4.1, then the inequalities (4.2) reduce to the inequalities (4.2) in [8].

Corollary 4.1. *Under the assumption of Theorem 4.1 with $\xi_i = h_1(x_{i+1})$ and $\eta_j = h_2(y_{j+1})$, we get*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t, s)dsdt = A_L(f, w_1, h_1, w_2, h_2, I_n, J_m) + R_L(f, w_1, h_1, w_2, h_2, I_n, J_m),$$

where $A_L(f, w_1, h_1, w_2, h_2, I_n, J_m)$ is built from the weighted left rectangle rule

$$\begin{aligned} & A_L(f, w_1, h_1, w_2, h_2, I_n, J_m) \\ &= \sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t)f(t, y_j)dt + \sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s)f(x_i, s)ds \\ & \quad - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_i, y_j)W_1^i W_2^j \end{aligned}$$

and remainder term $R_L(f, w_1, h_1, w_2, h_2, I_n, J_m)$ satisfies

$$|R_L(f, w_1, h_1, w_2, h_2, I_n, J_m)| \leq v(W_1)v(W_2) \bigvee_a^b \bigvee_c^d (f).$$

Corollary 4.2. *Under the assumption of Theorem 4.1 with $\xi_i = h_1(x_i)$ and $\eta_j = h_2(y_j)$, we have*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t, s)dsdt = A_R(f, w_1, h_1, w_2, h_2, I_n, J_m) + R_R(f, w_1, h_1, w_2, h_2, I_n, J_m),$$

where $A_R(f, w_1, h_1, w_2, h_2, I_n, J_m)$ is constructed from the weighted right rectangle rule

$$\begin{aligned} & A_R(f, w_1, h_1, w_2, h_2, I_n, J_m) \\ &= \sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t)f(t, y_{j+1})dt + \sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s)f(x_{i+1}, s)ds \\ & \quad - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_{i+1}, y_{j+1})W_1^i W_2^j \end{aligned}$$

and remainder term satisfies

$$|R_L(f, w_1, h_1, w_2, h_2, I_n, J_m)| \leq v(W_1)v(W_2) \bigvee_a^b \bigvee_c^d (f).$$

Corollary 4.3. *Under the assumption of Theorem 4.1 with $\xi_i = \frac{h_1(x_i) + h_1(x_{i+1})}{2}$ and $\eta_j = \frac{h_2(y_j) + h_2(y_{j+1})}{2}$ then*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t, s)dsdt = A_T(f, w_1, h_1, w_2, h_2, I_n, J_m) + R_T(f, w_1, h_1, w_2, h_2, I_n, J_m),$$

where $A_T(f, w_1, h_1, w_2, h_2, I_n, J_m)$ is constructed from the weighted trapezoid rule

$$\begin{aligned} & A_T(f, w_1, h_1, w_2, h_2, I_n, J_m) \\ &= \frac{1}{2} \sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t) [f(t, y_j) + f(t, y_{j+1})] dt \\ & \quad + \frac{1}{2} \sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s) [f(x_i, s) + f(x_{i+1}, s)] ds \\ & \quad - \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} [f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)] W_1^i W_2^j \end{aligned}$$

and remainder term satisfies

$$|R_T(f, w_1, h_1, w_2, h_2, I_n, J_m)| \leq \frac{1}{4} v(W_1) v(W_2) \bigvee_a^b \bigvee_c^d (f).$$

The constant $\frac{1}{4}$ is the best possible.

REFERENCES

- [1] M. W. Alomari, *A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation*, RGMIA Research Report Collection, **14** (2011), Article ID 87, 11 pages.
- [2] M. W. Alomari and M. A. Latif, *Weighted companion for the Ostrowski and the generalized trapezoid inequalities for mappings of bounded variation*, RGMIA Research Report Collection, **14** (2011), Article ID92, 10 pages.
- [3] M. W. Alomari, *A Generalization of Dragomir's generalization of Ostrowski integral inequality and applications in numerical integration*, Ukrainian Math. J. **64**(4) (2012).
- [4] N. S. Barnett, S. S. Dragomir and I. Gomm, *A companion for the Ostrowski and the generalized trapezoid inequalities*, Math. Comput. Modelling **50** (2009), 179–187.
- [5] H. Budak and M. Z. Sarikaya, *On Ostrowski type inequalities for functions of two variables with bounded variation*, Int. J. Anal. Appl. **12**(2) (2016), 142–156.
- [6] H. Budak and M. Z. Sarikaya, *On generalization Ostrowski type inequalities for functions of two variables with bounded variation*, Palest. J. Math. **5**(1) (2016), 86–97.
- [7] H. Budak and M. Z. Sarikaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, **17** (2014), Article ID155, 10 pages.
- [8] H. Budak and M. Z. Sarikaya, *On generalization trapezoid inequality for functions of two variables with bounded variation and applications*, Int. J. Nonlinear Anal. Appl. **7**(1) (2016), 77–85.
- [9] H. Budak and M. Z. Sarikaya, *A companion of Ostrowski type inequalities for functions of two variables with bounded variation*, J. Adv. Math. Stud. **8**(2) (2015), 170–184.
- [10] H. Budak and M. Z. Sarikaya, *A companion of generalization of Ostrowski type inequalities for functions of two variables with bounded variation*, Appl. Comput. Math. **15**(3) (2016), 297–312.
- [11] P. Cerone, S. S. Dragomir and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turkish J. Math. **24** (2000), 147–163.
- [12] J. A. Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*, Bull. Amer. Math. Soc. **35** (1933), 824–854.
- [13] J. A. Clarkson, *On double Riemann-Stieltjes integrals*, Bull. Amer. Math. Soc. **39** (1933), 929–936.

- [14] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull. Aust. Math. Soc. **60**(1) (1999), 495–508.
- [15] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. **4**(1) (2001), 59–66.
- [16] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. **5**(1) (2014), 89–97.
- [17] M. Fréchet, *Extension au cas des intégrals multiples d'une définition de l'intégrale due á Stieltjes*, Nouvelles Annales de Math Ématiques **10** (1910), 241–256.
- [18] Y. Jawarneh and M. S. M Noorani, *Inequalities of Ostrowski and Simpson type for mappings of two variables with bounded variation and Applications*, Transylvanian Journal of Mathematics and Mechanics **3**(2) (2011), 81–94.
- [19] F. Moricz, *Order of magnitude of double Fourier coefficients of functions of bounded variation*, Analysis (Munich), **22**(4) (2002), 335–345.
- [20] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.
- [21] K-L. Tseng, G-S. Yang and S. S. Dragomir, *Generalizations of weighted trapezoidal inequality for mappings of bounded variation and their applications*, Math. Comput. Modelling **40** (2004), 77–84.
- [22] K-L. Tseng, *Improvements of some inequalities of Ostrowski type and their applications*, Taiwanese J. Math. **12**(9) (2008), 2427–2441.
- [23] K-L. Tseng, S-R. Hwang and S. S. Dragomir, *Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and applications*, Comput. Math. Appl. **55** (2008), 1785–1793.
- [24] K-L. Tseng, S-R. Hwang, G-S. Yang and Y-M. Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Appl. Math. Comput. **217** (2010), 2348–2355.
- [25] K-L. Tseng, S-R. Hwang, G-S. Yang and Y-M. Chou, *Weighted Ostrowski integral inequality for mappings of bounded variation*, Taiwanese J. Math. **15**(2) (2011), 573–585.
- [26] K-L. Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Appl. Math. Comput. **218** (2012), 5841–5847.

¹ DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE AND ARTS,
DÜZCE UNIVERSITY
DÜZCE-TURKEY
Email address: hsyn.budak@gmail.com

Email address: sarikayamz@gmail.com