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IDEALS OF IS-ALGEBRAS BASED ON $\mathcal{N}$-STRUCTURES

HASHEM BORDBAR[1*], MOHAMMAD MEHDI ZAHEDI[2], AND YOUNG BAE JUN[3]

Abstract. The notion of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal is introduced, and related properties are investigated. Characterizations of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal are considered. Translations of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal are studied. We show that the homomorphic image (preimage) of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal. The notion of retrenched left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideals is introduced, and their properties are investigated.

1. Introduction

Most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation because a (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [3] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. The $\mathcal{N}$-structures are applied to BE-algebras and subtraction algebras (see [1] and [5]).

In this paper, using the $\mathcal{N}$-structures, we introduce the notion of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal, and investigate related properties. We consider characterizations of a left

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(resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal, and study translations of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal. We show that the homomorphic image (preimage) of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal. We also introduction the notion of retrenched left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideals and investigate their properties.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = \theta$;
(ii) $(x * (x * y)) * y = \theta$;
(iii) $x * x = \theta$;
(iv) $x * y = y * x = \theta \ \Rightarrow \ x = y$;

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \ \Rightarrow \ x * y = \theta).$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * \theta = x), \tag{2.1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2.2}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$;
(I2) $(\forall x, y \in X)(x * y \in I,\ y \in I \ \Rightarrow x \in I)$.

We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

An **IS**-*algebra* (see [4]) is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant θ satisfying the conditions:

- $I(X) := (X, *, \theta)$ is a BCI-algebra;
- $S(X) := (X, \cdot)$ is a semigroup;
- the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is,

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \ \text{ and } \ (x * y) \cdot z = (x \cdot z) * (y \cdot z),$$

  for all $x, y, z \in X$.

In an **IS**-algebra X, the following hold:

$$(\forall x \in X)\ (\theta x = x\theta = \theta); \tag{2.3}$$

$$(\forall x, y, z \in X)\ (x \preceq y \ \Rightarrow \ xz \preceq yz,\ zx \preceq zy). \tag{2.4}$$

In what follows we use the notation xy instead of $x \cdot y$.

A nonempty subset A of an **IS**-algebra X is called a *left* (resp., *right*) $\mathcal{I}$-*ideal* of X (see [4]) if

(i) A is a left (resp., right) stable subset of $S(X)$, that is, $xa \in A$ (resp., $ax \in A$) whenever $x \in S(X)$ and $a \in A$;

(ii) $(\forall x, y \in I(X))\ (x * y \in A,\ y \in A\ \Rightarrow x \in A)$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\}, & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\}, & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\}, & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\}, & \text{otherwise.} \end{cases}$$

3. Ideals Based on $\mathcal{N}$-structures

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$-*function* on X). By an $\mathcal{N}$-*structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote an **IS**-algebra unless otherwise specified.

Definition 3.1. An $\mathcal{N}$-structure (X, f) is said to satisfy the *left* (resp., *right*) *condition* in $S(X)$ if $f(xy) \le f(y)$ (resp., $f(xy) \le f(x)$) for all x and y in $S(X)$.

Definition 3.2. An $\mathcal{N}$-structure (X, f) is called a *left* (resp., *right*) $\mathcal{N}_{\mathcal{I}}$-*ideal* of X if (X, f) satisfies the left (resp., right) condition in $S(X)$ and

$$(\forall x, y \in X)\ \Big(f(\theta) \le f(x) \le \bigvee\{f(x * y), f(y)\}\Big). \tag{3.1}$$

Example 3.1. Define two binary operations "$*$" and "$\cdot$" on a set $X = \{\theta, a, b, c\}$ as follows:

$*$	θ	a	b	c
θ	θ	θ	c	b
a	a	θ	c	b
b	b	b	θ	c
c	c	c	b	θ

,

$\cdot$	θ	a	b	c
θ	θ	θ	θ	θ
a	θ	θ	θ	θ
b	θ	θ	b	c
c	θ	θ	c	b

.

Then X is an **IS**-algebra (see [4]). Let (X, f) be an $\mathcal{N}$-structure in which f is given as follows:

$$f = \begin{pmatrix} \theta & a & b & c \\ -0.8 & -0.6 & -0.3 & -0.3 \end{pmatrix}.$$

It is routine to verify that (X, f) is both a left and a right $\mathcal{N}_{\mathcal{I}}$-ideal of X.

We provide characterizations of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal.

Theorem 3.1. *An $\mathcal{N}$-structure (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X if and only if the following assertions are valid*

$$(\forall x, y \in X)\ \ (f(xy) \le f(y)), \tag{3.2}$$

$$(\forall x, y \in X)\ \ \Big(f(x) \le \bigvee\{f(x * y), f(y)\}\Big). \tag{3.3}$$

Proof. The necessity is clear. Assume that (X, f) satisfies two conditions (3.2) and (3.3). Using (2.3) and (3.2) induce $f(\theta) = f(\theta y) \leq f(y)$ for all $y \in X$. Hence (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X. □

Similarly we have the following theorem.

Theorem 3.2. *An $\mathcal{N}$-structure (X, f) is a right $\mathcal{N}_{\mathcal{I}}$-ideal of X if and only if f satisfies the condition* (3.3) *and*

$$(\forall x, y \in X)\ \ (f(xy) \leq f(x))\,. \tag{3.4}$$

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, the set

$$C(f; t) := \{x \in X \mid f(x) \leq t\}$$

is called a *closed t-support* of (X, f) (see [3]).

Theorem 3.3. *If an $\mathcal{N}$-structure (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X, then the closed t-support of (X, f) is a left $\mathcal{I}$-ideal of X for all $t \in [f(\theta), 0]$.*

Proof. Let $x \in S(X)$ and $a \in C(f; t)$ for $t \in [f(\theta), 0]$. Then $f(a) \leq t$, and so $f(xa) \leq f(a) \leq t$ which shows that $xa \in C(f; t)$. It follows from (2.3) that $\theta = \theta a \in C(f; t)$. Let $x, y \in X$ be such that $x * y \in C(f; t)$ and $y \in C(f; t)$ for $t \in [f(\theta), 0]$. Then $f(x * y) \leq t$ and $f(y) \leq t$. It follows from (3.3) that

$$f(x) \leq \bigvee\{f(x * y), f(y)\} \leq t$$

and so that $x \in C(f; t)$. Therefore $C(f; t)$ is an $\mathcal{I}$-ideal of X for all $t \in [f(\theta), 0]$. □

Theorem 3.4. *If an $\mathcal{N}$-structure (X, f) is a right $\mathcal{N}_{\mathcal{I}}$-ideal of X, then the closed t-support of (X, f) is a right $\mathcal{I}$-ideal of X for all $t \in [f(\theta), 0]$.*

Proof. It is similar to the proof of Theorem 3.3. □

Theorem 3.5. *Given an $\mathcal{N}$-structure (X, f), if the nonempty closed t-support of (X, f) is a left $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$, then (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Proof. Assume that $C(f; t)$ is a left $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$ with $C(f; t) \neq \emptyset$. If $f(ab) > f(b)$ for some $a, b \in X$, then there exists $t \in [-1, 0)$ such that $f(ab) > t \geq f(b)$. It follows that $b \in C(f; t)$ and $ab \notin C(f; t)$, which is a contradiction. Hence (3.2) is valid. Now suppose that (3.3) is false. Then there exists $a, b \in X$ such that

$$f(a) > \bigvee\{f(a * b), f(b)\}.$$

Taking $t := \frac{1}{2}\left(f(a) + \bigvee\{f(a * b), f(b)\}\right)$ implies that $a * b \in C(f; t)$, $b \in C(f; t)$ and $a \notin C(f; t)$. This is a contradiction, and so (3.3) is valid. Therefore (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X by Theorem 3.1. □

Similarly we have the following theorem.

Theorem 3.6. *Given an $\mathcal{N}$-structure (X, f), if the nonempty closed t-support of (X, f) is a right $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$, then (X, f) is a right $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Theorem 3.7. *For any left $\mathcal{I}$-ideal A of X and any fixed number t in an open interval $(-1,0)$, there exists a left $\mathcal{N}_{\mathcal{I}}$-ideal (X,f) of X on which A is the closed t-support of (X,f).*

Proof. Let (X,f) be an $\mathcal{N}$-structure on which f is given as follows:

$$f(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let $x, y \in X$. If $y \notin A$, then $f(y) = 0$ and thus

$$f(x) \leq 0 = \bigvee\{f(x * y), f(y)\}.$$

Assume that $y \in A$. If $x \in A$, then $x * y$ may or may not belong to A. In any case, we have

$$f(x) \leq \bigvee\{f(x * y), f(y)\}.$$

If $x \notin A$, then $x * y \notin A$ and hence

$$f(x) = 0 = \bigvee\{f(x * y), f(y)\}.$$

For any $x, y \in X$, if $y \in A$ then $xy \in A$. Hence $f(xy) = t = f(y)$. If $y \notin A$, then $f(y) = 0$ and so $f(xy) \leq 0 = f(y)$. It follows from Theorem 3.1 that (X,f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X. Obviously, $A = C(f;t)$. □

Similarly, we have the following theorem.

Theorem 3.8. *For any right $\mathcal{I}$-ideal A of X and any fixed number t in an open interval $(-1,0)$, there exists a right $\mathcal{N}_{\mathcal{I}}$-ideal (X,f) of X on which A is the closed t-support of (X,f).*

Theorem 3.9. *For any nonempty subset A of X and $t \in [-1,0)$, let (X,f) be an $\mathcal{N}$-structure on which f is given as follows:*

$$f(x) = \begin{cases} t, & \textit{if } x \in A, \\ 0, & \textit{if } x \notin A. \end{cases}$$

If A is a left (resp., right) $\mathcal{I}$-ideal of X, then (X,f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X.

Proof. Suppose that A is a left $\mathcal{I}$-ideal of X. Let $x, y \in X$. If $y \in A$, then $xy \in A$, and

(i) $x * y$ may or may not belong to A whenever $x \in A$;
(ii) $x * y \notin A$ whenever $x \notin A$.

Hence $f(xy) = t = f(y)$ and $f(x * y) \leq \bigvee\{f(x * y), f(y)\}$. If $y \notin A$, then $f(xy) \leq 0 = f(y)$ and $f(x * y) \leq 0 = \bigvee\{f(x * y), f(y)\}$. Therefore (X,f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X by Theorem 3.1. Similarly we can prove it for the right case. □

Corollary 3.1. *For any nonempty subset A of X and an $\mathcal{N}$-structure (X,f) with $\mathrm{Im}(f) = \{-1,0\}$, the following assertions are equivalent.*

(1) *A is a left (resp., right) $\mathcal{I}$-ideal of X.*
(2) *(X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Theorem 3.10. *If an $\mathcal{N}$-structure (X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X, then the set*

$$X_f := \{x \in X \mid f(x) = f(\theta)\}$$

is a left (resp., right) $\mathcal{I}$-ideal of X.

Proof. Assume that (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X and let $x, y \in X$. If $y \in X_f$, then $f(xy) \leq f(y) = f(\theta)$ and so $f(xy) = f(\theta)$, that is, $xy \in X_f$. Obviously, $\theta \in X_f$. Suppose that $x * y \in X_f$ and $y \in X_f$. Then

$$f(x) \leq \bigvee\{f(x * y), f(y)\} = f(\theta),$$

and so $f(x) = f(\theta)$, i.e., $x \in X_f$. Therefore X_f is a left $\mathcal{I}$-ideal of X. Similarly, we can prove it for the right case. □

Given an $\mathcal{N}$-structure (X, f), we denote

$$\perp := -1 - \bigwedge\{f(x) \mid x \in X\}.$$

For any $\alpha \in [\perp, 0]$, the *α-translation* of (X, f) is defined to be the new $\mathcal{N}$-structure (X, f_α) on which f_α is defined by $f_\alpha(x) = f(x) + \alpha$ for all $x \in X$.

Theorem 3.11. *For every $\alpha \in [\perp, 0]$, the α-translation of a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Proof. Let $\alpha \in [\perp, 0]$ and let (X, f) be a left $\mathcal{N}_{\mathcal{I}}$-ideal of X. For any $x, y \in X$, we have $f_\alpha(xy) = f(xy) + \alpha \leq f(y) + \alpha = f_\alpha(y)$ and

$$\begin{aligned} f_\alpha(x) = f(x) + \alpha &\leq \bigvee\{f(x * y), f(y)\} + \alpha \\ &= \bigvee\{f(x * y) + \alpha, f(y) + \alpha\} \\ &= \bigvee\{f_\alpha(x * y), f_\alpha(y)\}. \end{aligned}$$

It follows from Theorem 3.1 that (X, f_α) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X. For the right case, it is similar. □

Theorem 3.12. *For $\mathcal{N}$-structure (X, f), if there exists $\alpha \in [\perp, 0]$ such that every α-translation of (X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal, then (X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Proof. Assume that the α-translation (X, f_α) of (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X. For any $x, y \in X$, we have $f(xy) + \alpha = f_\alpha(xy) \leq f_\alpha(y) = f(y) + \alpha$ and

$$\begin{aligned} f(x) + \alpha = f_\alpha(x) &\leq \bigvee\{f_\alpha(x * y), f_\alpha(y)\} \\ &= \bigvee\{f(x * y) + \alpha, f(y) + \alpha\} \\ &= \bigvee\{f(x * y), f(y)\} + \alpha. \end{aligned}$$

It follows that $f(xy) \leq f(y)$ and $f(x) \leq \bigvee\{f(x * y), f(y)\}$. Therefore (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X by Theorem 3.1. $\square$

For any $\mathcal{N}$-structure (X, f), $\alpha \in [\perp, 0]$ and $t \in [-1, \alpha)$, the set

$$C_\alpha(f; t) := \{x \in X \mid f(x) \leq t - \alpha\}$$

is called the *α-translation of closed t-support* of (X, f)

Theorem 3.13. *Let (X, f) be an $\mathcal{N}$-structure and $\alpha \in [\perp, 0]$. If (X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X, then the α-translation of closed t-support of (X, f) is a left (resp., right) $\mathcal{I}$-ideal of X for all $t \in [-1, \alpha)$.*

Proof. Let $x, y \in X$. If $y \in C_\alpha(f; t)$, then $f(y) \leq t - \alpha$ and so

$$f(xy) \leq f(y) \leq t - \alpha. \tag{3.5}$$

Thus $xy \in C_\alpha(f; t)$. Suppose that $x * y \in C_\alpha(f; t)$ and $y \in C_\alpha(f; t)$. Then

$$f(\theta) \leq f(x) \leq \bigvee\{f(x * y), f(y)\} \leq t - \alpha$$

by (3.1). Thus $\theta \in C_\alpha(f; t)$ and $x \in C_\alpha(f; t)$. Consequently, $C_\alpha(f; t)$ is a left $\mathcal{I}$-ideal of X for all $t \in [-1, \alpha)$. Similarly we can prove it for the right case. $\square$

Theorem 3.14. *For any $\mathcal{N}$-structure (X, f) and $\alpha \in [\perp, 0]$, the following assertions are equivalent.*

(1) *The α-translation of closed t-support of (X, f) is a left (resp., right) $\mathcal{I}$-ideal of X for all $t \in [-1, \alpha)$.*
(2) *The α-translation of (X, f) is a left (resp., right) $\mathcal{N}_{\mathcal{I}}$-ideal of X.*

Proof. Suppose that (X, f_α) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X for $\alpha \in [\perp, 0]$ and let $t \in [-1, \alpha)$. For any $x, y \in X$, if $x * y \in C_\alpha(f; t)$ and $y \in C_\alpha(f; t)$, then

$$\begin{aligned} f(x) + \alpha = f_\alpha(x) &\leq \bigvee\{f_\alpha(x * y), f_\alpha(y)\} \\ &= \bigvee\{f(x * y) + \alpha, f(y) + \alpha\} \\ &= \bigvee\{f(x * y), f(y)\} + \alpha \\ &\leq t - \alpha + \alpha = t \end{aligned}$$

and so $f(x) \leq t - \alpha$. Thus $x \in C_\alpha(f; t)$. Since

$$f(\theta) + \alpha = f_\alpha(\theta) \leq f_\alpha(x) = f(x) + \alpha \leq t - \alpha + \alpha = t,$$

for any $x \in C_\alpha(f; t)$, we have $f(\theta) \leq t - \alpha$, i.e., $\theta \in C_\alpha(f; t)$. Now if $y \in C_\alpha(f; t)$, then $f(y) \leq t - \alpha$ which implies that

$$f(xy) + \alpha = f_\alpha(xy) \leq f_\alpha(y) = f(y) + \alpha \leq t,$$

that is, $f(xy) \leq t - \alpha$ for all $x \in X$. Hence $xy \in C_\alpha(f; t)$, and therefore $C_\alpha(f; t)$ is a left $\mathcal{I}$-ideal of X.

Conversely, assume that the α-translation of closed t-support of (X, f) is a left $\mathcal{I}$-ideal of X for all $t \in [-1, \alpha)$. Suppose that there exist $a, b \in X$ and $t_0 \in [-1, \alpha)$

such that $f_\alpha(ab) > t_0 \geq f_\alpha(b)$. Then $f(ab)+\alpha > t_0$ and $f(b)+\alpha \leq t_0$, which imply that $b \in C_\alpha(f;t_0)$ and $ab \notin C_\alpha(f;t_0)$. This is a contradiction, and thus $f_\alpha(xy) \leq f_\alpha(y)$ for all $x, y \in X$. If

$$f_\alpha(a) > \bigvee\{f_\alpha(a*b), f_\alpha(b)\},$$

for some $a, b \in X$, then there exists $t_1 \in [-1, \alpha)$ such that

$$f_\alpha(a) > t_1 \geq \bigvee\{f_\alpha(a*b), f_\alpha(b)\},$$

which implies that $f(a) > t_1 - \alpha$, $f(a*b) \leq t_1 - \alpha$ and $f(b) \leq t_1 - \alpha$. Hence $a*b \in C_\alpha(f;t_1)$ and $b \in C_\alpha(f;t_1)$, but $a \notin C_\alpha(f;t_1)$, which is a contradiction. Hence $f_\alpha(x) \leq \bigvee\{f_\alpha(x*y), f_\alpha(y)\}$ for all $x, y \in X$. Therefore (X, f_α) is a left $\mathcal{N}_\mathcal{J}$-ideal of X by Theorem 3.1. $\square$

Given two $\mathcal{N}$-structures (X,f) and (X,g), we say that (X,f) is a *retrenchment* of (X,g) if $f \subseteq g$, that is, $f(x) \leq g(x)$ for all $x \in X$.

Definition 3.3. Given two $\mathcal{N}$-structures (X,f) and (X,g), we say that (X,f) is a *retrenched left* (*resp., right*) $\mathcal{N}_\mathcal{J}$*-ideal* of (X,g), denoted by

$$(X,f) \,\tilde{\subseteq}_l\, (X,g) \text{ (resp., } (X,f) \,\tilde{\subseteq}_r\, (X,g)),$$

if (X,f) is a retrenchment of (X,g), and (X,f) is a left (resp., right) $\mathcal{N}_\mathcal{J}$-ideal of X whenever (X,g) is a left (resp., right) $\mathcal{N}_\mathcal{J}$-ideal of X.

Theorem 3.15. *Let (X,g) be a left (resp., right) $\mathcal{N}_\mathcal{J}$-ideal of X. For every $\alpha \in [\perp, 0]$, the α-translation (X, g_α) of (X,g) is a retrenched left (resp., right) $\mathcal{N}_\mathcal{J}$-ideal of X.*

Proof. For any $x \in X$, we have $g_\alpha(x) = g(x)+\alpha \leq g(x)$. Thus (X, g_α) is a retrenchment of (X,g). If (X,g) is a left $\mathcal{N}_\mathcal{J}$-ideal of X, then Theorem 3.11 shows that (X, g_α) is a left $\mathcal{N}_\mathcal{J}$-ideal of X. Therefore (X, g_α) is a retrenched left $\mathcal{N}_\mathcal{J}$-ideal of X. Similarly, we can prove it for the right case. $\square$

Theorem 3.16. *Let (X,g) be a left (resp., right) $\mathcal{N}_\mathcal{J}$-ideal of X. If (X,f_1) and (X,f_2) are retrenched left (resp., right) $\mathcal{N}_\mathcal{J}$-ideals of (X,g), then so is $(X, f_1 \cup f_2)$, where $(f_1 \cup f_2)(x) = \bigvee\{f_1(x), f_2(x)\}$ for all $x \in X$.*

Proof. Assume that (X,f_1) and (X,f_2) are retrenched left $\mathcal{N}_\mathcal{J}$-ideals of a left $\mathcal{N}_\mathcal{J}$-ideal (X,g) of X. Then $f_1(x) \leq g(x)$ and $f_2(x) \leq g(x)$, for all $x \in X$. Thus $(f_1 \cup f_2)(x) = \bigvee\{f_1(x), f_2(x)\} \leq g(x)$ for all $x \in X$, and so $(X, f_1 \cup f_2)$ is a retrenchment of (X,g). For any $x, y \in X$, we have

$$\begin{aligned}(f_1 \cup f_2)(xy) &= \bigvee\{f_1(xy), f_2(xy)\}\\ &\leq \bigvee\{f_1(y), f_2(y)\}\\ &= (f_1 \cup f_2)(y)\end{aligned}$$

and

$$\begin{aligned}(f_1 \cup f_2)(x) &= \bigvee\{f_1(x), f_2(x)\} \\ &\leq \bigvee\left\{\bigvee\{f_1(x*y), f_1(y)\}, \bigvee\{f_2(x*y), f_2(y)\}\right\} \\ &= \bigvee\left\{\bigvee\{f_1(x*y), f_2(x*y)\}, \bigvee\{f_1(y), f_2(y)\}\right\} \\ &= \bigvee\{(f_1 \cup f_2)(x*y), (f_1 \cup f_2)(y)\}.\end{aligned}$$

It follows from Theorem 3.1 that $(X, f_1 \cup f_2)$ is a left $\mathcal{N}_{\mathcal{J}}$-ideal of X. Therefore $(X, f_1 \cup f_2)$ is a retrenched left $\mathcal{N}_{\mathcal{J}}$-ideal of (X, g). The proof is similar for the right case. □

Theorem 3.17. *Let (X, g) be a left $\mathcal{N}_{\mathcal{J}}$-ideal of X and let $\alpha, \beta \in [\perp, 0]$. If $\alpha \leq \beta$, then the α-translation (X, g_α) of (X, g) is a retrenched left $\mathcal{N}_{\mathcal{J}}$-ideal of the β-translation (X, g_β) of (X, g).*

Proof. Note that the α-translation (X, g_α) and the β-translation (X, g_β) of (X, g) are left $\mathcal{N}_{\mathcal{J}}$-ideal of X by Theorem 3.11. If $\alpha \leq \beta$, then

$$g_\alpha(x) = g(x) + \alpha \leq g(x) + \beta = g_\beta(x),$$

for all $x \in X$. Hence (X, g_α) is a retrenchment of (X, g_β). Therefore (X, g_α) is a retrenched left $\mathcal{N}_{\mathcal{J}}$-ideal of (X, g_β). □

Similarly we have the following theorem for the right case.

Theorem 3.18. *Let (X, g) be a right $\mathcal{N}_{\mathcal{J}}$-ideal of X and let $\alpha, \beta \in [\perp, 0]$. If $\alpha \leq \beta$, then the α-translation (X, g_α) of (X, g) is a retrenched right $\mathcal{N}_{\mathcal{J}}$-ideal of the β-translation (X, g_β) of (X, g).*

Theorem 3.19. *Let (X, g) be a left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal of X and let $\beta \in [\perp, 0]$. For every retrenched left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal (X, f) of the β-translation (X, g_β) of (X, g), there exists $\alpha \in [\perp, 0]$ such that $\alpha \leq \beta$ and (X, f) is a retrenched left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal of the α-translation (X, g_α) of (X, g).*

Proof. It is straightforward. □

A mapping $\varphi : X \to Y$ is called a *homomorphism* of **IS**-algebras if $\varphi(x * y) = \varphi(x) * \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in X$.

Let $\varphi : X \to Y$ be an onto mapping. Given an $\mathcal{N}$-structure (Y, g), the $\mathcal{N}$-structure (X, f), where $f = g \circ \varphi$, is called the *preimage* of (Y, g) under φ. Given an $\mathcal{N}$-structure (X, f), the *image* of (X, f) under φ is defined to be the $\mathcal{N}$-structure (Y, g) on which g is denoted by $\varphi(f)$ and is given by

$$g(y) = \bigwedge_{x \in \varphi^{-1}(y)} f(x),$$

for all $y \in Y$.

Theorem 3.20. *Every preimage of a left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal under onto homomorphism is a left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal.*

Proof. Let $\varphi : X \to Y$ be an onto homomorphism of **IS**-algebras and let an $\mathcal{N}$-structure (X, f) is the preimage of a left $\mathcal{N}_{\mathcal{J}}$-ideal (Y, g) of Y. For any $x, y \in X$, we have

$$\begin{aligned} f(xy) &= (g \circ \varphi)(xy) = g(\varphi(xy)) \\ &= g(\varphi(x)\varphi(y)) \leq g(\varphi(y)) \\ &= (g \circ \varphi)(y) = f(y) \end{aligned}$$

and

$$\begin{aligned} f(x) &= (g \circ \varphi)(x) = g(\varphi(x)) \\ &\leq \bigvee\{g(\varphi(x) * y'), g(y')\} \text{ for all } y' \in Y \\ &= \bigvee\{g(\varphi(x) * \varphi(y)), g(\varphi(y))\} \\ &= \bigvee\{g(\varphi(x * y)), g(\varphi(y))\} \\ &= \bigvee\{(g \circ \varphi)(x * y), (g \circ \varphi)(y)\} \\ &= \bigvee\{f(x * y), f(y)\}. \end{aligned}$$

It follows from Theorem 3.1 that (X, f) is a left $\mathcal{N}_{\mathcal{J}}$-ideal of X. Similarly we can verify it for the right case. □

Lemma 3.1. *Let $\varphi : X \to Y$ be an onto mapping. Given an $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, we have*

$$C(\varphi(f); t) = \bigcap_{t<s<0} \varphi(C(f; t - s)).$$

Proof. For any $y = f(x) \in Y$, if $y \in C(\varphi(f); t)$, then

$$\bigwedge_{z \in \varphi^{-1}(\varphi(x))} f(z) = \varphi(f)(\varphi(x)) = \varphi(f)(y) \leq t.$$

Hence, for every $s \in (t, 0)$, there exists $x_0 \in \varphi^{-1}(y)$ such that $f(x_0) \leq t - s$. Thus $y = \varphi(x_0) \in \varphi(C(f; t - s))$, and so $y \in \bigcap_{t<s<0} \varphi(C(f; t - s))$.

Conversely, let $y \in \bigcap_{t<s<0} \varphi(C(f; t - s))$. Then $y \in \varphi(C(f; t - s))$ for every $s \in (t, 0)$, and hence there exists $x_0 \in C(f; t-s)$ such that $y = \varphi(x_0)$. It follows that $f(x_0) \leq t-s$ and $x_0 \in \varphi^{-1}(y)$. Therefore

$$\varphi(f)(y) = \bigwedge_{x \in \varphi^{-1}(y)} f(x) \leq \bigwedge_{t<s<0} \{t - s\} = t,$$

and thus $y \in C(\varphi(f); t)$. □

Theorem 3.21. *Every image of a left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal under onto homomorphism is a left (resp., right) $\mathcal{N}_{\mathcal{J}}$-ideal.*

Proof. Let $\varphi : X \to Y$ be an onto homomorphism of **IS**-algebras and let an $\mathcal{N}$-structure (Y, g) is the image of a left $\mathcal{N}_{\mathcal{I}}$-ideal (X, f) of X. Let $t \in [-1, 0)$ be such that $C(\varphi(f); t) \neq \emptyset$. Then

$$C(\varphi(f); t) = \bigcap_{t<s<0} \varphi(C(f; t-s)),$$

by Lemma 3.1, and so $\varphi(C(f; t-s))$ is nonempty for all $s \in (t, 0)$. Since (X, f) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of X, $C(f; t-s)$ is a left $\mathcal{I}$-ideal of X and so the onto homomorphic image $\varphi(C(f; t-s))$ of $C(f; t-s)$ under φ is a left $\mathcal{I}$-ideal of Y. Hence $C(\varphi(f); t)$ is a left $\mathcal{I}$-ideal of Y. It follows from Theorem 3.5 that (Y, g) is a left $\mathcal{N}_{\mathcal{I}}$-ideal of Y. □

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[1]Department of Mathematics,
Shahid Beheshti University, Tehran, Iran
Email address: Bordbar.amirh@gmail.com

[2]Department of Mathematics,
Graduate University of Advanced Technology, Mahan-Kerman, Iran
Email address: zahedi_mm@kgut.ac.ir

[3]Department of Mathematics Education,
Gyeongsang National University, Jinju 52828, Korea
Email address: skywine@gmail.com

*Corresponding Author