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TWIST PRODUCT DERIVED FROM Γ-SEMIHYPERGROUP

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Abstract. The aim of this research work is to define a new class of hyperstructure that we call twist product. We first introduce the concept of left and right (Δ, G)-sets and by using this new idea we introduce the concept of flat Γ-semihypergroup, absolutely Γ-semihypergroup, twist product and extension property that product that play an important role in homology algebra.

1. Introduction

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [11], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [2–6].

Recently, the notion of Γ-hyperstructure introduced and studied by many researcher and represent an intensively studied field of research, for example, see [1, 7–10]. The concept of Γ-semihypergroups was introduced by Davvaz et al. [1, 10] and is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups.

In this paper, we denote the notion of left(right) (Δ, G)-set, (G_1, Δ, G_2)-biset, twist product, flat Γ-semihypergroup and absolutely flat Γ-semihypergroup. Also, we prove

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that twist product exists and unique up to isomorphism such that product that play an important role in homology algebra.

2. Introduction and Preliminaries

In this section, we present some basic notion of Γ-semihypergroup. These definitions and results are necessary for the next section.

Let$H$ be a non-empty set, then the map $\circ : H \times H \to P^*(H)$ is called *hyperoperation* or *join operation* on the set H, where $P^*(H)$ denotes the set of all non-empty subsets of H. A hypergroupoid is a set H together with a (binary) hyperoperation. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, we have $a \circ (b \circ c) = (a \circ b) \circ c$. A hypergroupoid $(H, \circ)$ is called *quasihypergroup* if for all $a \in H$, we have $a \circ H = H \circ a = H$. A hypergroupoid$(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

Definition 2.1. [10] Let G and Γ be nonempty set and $\alpha : G \times G \to P^*(G)$ be a hyperoperation, where $\alpha \in \Gamma$. Then, G is called Γ-*hypergroupoid*.

For any two nonempty subset G_1 and G_2 of G, we define

$$G_1 \alpha G_2 = \bigcup_{g_1 \in G_1, g_2 \in G_2} g_1 \alpha g_2, \quad G_1 \alpha \{x\} = G_1 \alpha x, \quad \{x\} \alpha G_2 = x \alpha G_2.$$

A Γ-hypergroupoid G is called Γ-*semihypergroup* if for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ we have

$$(x \alpha y) \beta z = x \alpha (y \beta z).$$

Example 2.1. Let $\Gamma \subseteq \mathbb{N}$ be a nonempty set. We define

$$x \alpha y = \{z \in \mathbb{N} : z \geq \max\{x, \alpha, y\}\},$$

where $\alpha \in \Gamma$ and $x, y \in \mathbb{N}$. Then, $\mathbb{N}$ is a Γ-semihypergroup.

Example 2.2. Let $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then, we define hyperoperations $x \alpha_k y = xyk\mathbb{Z}$. Hence $\mathbb{Z}$ is a Γ-semihypergroup.

Example 2.3. Let G be a nonempty set and Γ be a nonempty set of G. We define $x \alpha y = \{x, \alpha, y\}$. Then, G is a Γ-semihypergroup.

Example 2.4. Let $(\Gamma, \cdot)$ be a semigroup and $\{A_\alpha\}_{\alpha \in \Gamma}$ be a collection of nonempty disjoint sets and $G = \bigcup_{\alpha \in \Gamma} A_\alpha$. For every $g_1, g_2 \in G$ and $\alpha \in \Gamma$, we define $g_1 \alpha g_2 = A_{\alpha_1 \alpha \alpha_2}$, where $g_1 \in A_{\alpha_1}$ and $g_2 \in A_{\alpha_2}$.

Let G be a Γ-semihypergroup. Then, an element $e_\alpha \in G$ is called α-*identity* if for every $x \in G$, we have $x \in e_\alpha \alpha x \cap x \alpha e_\alpha$ and e_α is called *scaler* α-*identity* if $x = e_\alpha \alpha x = x \alpha e_\alpha$. We note that if for every $\alpha \in \Gamma$, e is a scaler α-identity, then $x \alpha y = x \beta y$, where $\alpha, \beta \in \Gamma$ and $x, y \in G$. Indeed,

$$x \alpha y = (x \beta e) \alpha y = x \beta (e \alpha y) = x \beta y.$$

Let G be a Γ-semihypergroup and for every $\alpha \in \Gamma$ has an α-identity. Then, G is called a Γ-semihypergroup with identity. In a same way, we can define Γ-semihypergroup with scaler identity.

A Γ-semihypergroup G is *commutative* when

$$x\alpha y = y\alpha x,$$

for every $x, y \in G$ and $\alpha \in \Gamma$.

Definition 2.2. Let G be a Γ-semihypergroup and ρ be an equivalence relation on G. Then, ρ is called *right regular relation* if $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ there is $t_2 \in y\alpha g$ such that $t_1\rho t_2$ and for every $s_1 \in y\alpha g$ there is $s_1 \in x\alpha g$ such that $s_1\rho s_2$. In a same way, we can define *left regular relation.* An equivalence relation ρ is called *strong regular* when $x\rho y$ and g implies that for every $t_1 \in x\alpha g$ and $t_2 \in y\alpha g$ $t_1\rho t_2$.

Example 2.5. Let $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} A_n$ where $A_n = [n, n+1)$ and $x, y \in \mathbb{R}$ such that $x \in A_n$, $y \in A_m$ and $\alpha \in \mathbb{Z}$. Then, $\mathbb{R}$ is a $\mathbb{Z}$-semihypergroup such that we define $x\alpha y = A_{n\alpha m}$. Let

$$x\rho y \Leftrightarrow 2|n-m, x \in A_n, y \in A_m.$$

Then, the relation ρ is strong regular.

Proposition 2.1. *Let G be a Γ-semihypergroup and ρ be a regular relation on G. Then, $[G:\rho] = \{\rho(x) : x \in G\}$ is a $\widehat{\Gamma}$-semihypergroup with respect the following hyperoperation*

$$\rho(x)\widehat{\alpha}\rho(y) = \{\rho(z) : z \in \rho(x)\alpha\rho(y)\},$$

where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.

Proof. The proof is straightforward. $\square$

Corollary 2.1. *Let G be a Γ-semihypergroup and ρ be an equivalence relation G. Then, ρ is regular (strong regular) if and only if $[G:\rho]$ is $\widehat{\Gamma}$-semihypergroup ($\widehat{\Gamma}$-semigroup).*

Let X be a left (Δ, G)-set and n be a nonzero natural number. We say that

$$a\beta_n b \Leftrightarrow (\exists\delta_1, \delta_2, \dots, \delta_n \in \Delta, x \in X, g_1, g_2, \dots, g_n \in G)\ \{a, b\} \subseteq g_1\delta_1 g_2\delta_2, \dots, g_n\delta_n x.$$

Let $\beta = \bigcup_{n\geq 1} \beta_n$. Clearly, the relation SS is reflexive and symmetric. Denote by β^* the transitive closure.

Let G be a Γ-semihypergroup and $\alpha \in \Gamma$. We define $x \circ y = x\alpha y$ for every $x, y \in G$. Hence $(G, \circ)$ becomes a semihypergroup, we denote this semihypergroup by $G[\alpha]$.

Definition 2.3. Let G_1 and G_2 be Γ-semihypergroups with identity. Then, a map $\varphi : G_1 \to G_2$ is called *α-homomorphism* if $\varphi(x\alpha y) = \varphi(x)\alpha\varphi(y)$ and $\varphi(e_\alpha) = e_\alpha$ for every $x, y \in G_1$. If for every $\alpha \in \Gamma$, φ is an α-homomorphism, then φ is called *homomorphism.*

3. Twist Product

In this section we introduce a relation denoted by ρ^* which we shall use in order to define a new derived structure of Γ-semihypergroup that we called twist product.

Let G be a Γ-semihypergroup with identity and X, Δ be non-empty sets. We say that X is a *left* (Δ, G)*- set* if there is a scaler hyperaction $\delta : G \times X \to P^*(X)$ with the following properties:

$$(g_1\alpha g_2)\delta x = g_1\delta(g_2\delta x),$$
$$e_\alpha \delta x = x,$$

for every $g_1, g_2 \in G$, $\alpha \in \Gamma$, $\alpha \in \Gamma$, $x \in X$ and $\delta \in \Delta$.

In a same way, we can define a *right* (Δ, G)*-set*. Let G_1 and G_2 be Γ-semihypergroups and X be a non-empty set. Then, we say that X is a (G_1, Δ, G_2)*-bisets* if it is a left (Δ, G_1)-set, right (Δ, G_2)-set and

$$(g_1\delta_1 x)\delta_2 g_2 = g_1\delta_1(x\delta_2 g_2),$$

for every $\delta_1, \delta_2 \in \Delta$, $g_1 \in G_1$, $g_2 \in G_2$ and $x \in X$.

If G is a commutative Γ-semihypergroup, then there is no distinction between a left and a right (Δ, G)-set. A *left* (Δ, G)*-subset* Y of X such that $Y\Delta X \subseteq Y$ is called *left* (Δ, G)*-subset* of X. A map $\varphi : X \to Y$ from a left (Δ, G)-set X into a left (Δ, G)-set Y is called *morphism* (G*-morphism*) if

$$\varphi(g\delta x) = g\delta\varphi(x),$$

for every $x \in X, \delta \in \Delta$ and $g \in G$.

Let X be a left (Δ, G)-set and ρ be an equivalence relation on X and A, B be nonempty subsets of X. Then,

$$A\overline{\rho}B \Leftrightarrow (\forall a \in A)(\exists b \in B)\ (a, b) \in \rho\ \wedge\ (\forall b \in B)(\exists a \in A)\ (a, b) \in \rho.$$

An equivalence relation ρ on left (Δ, G)-set X is called *regular*, if for every $x, y \in X$, $\delta \in \Delta$ and $g \in G$

$$x\rho y \Rightarrow (g\delta x)\overline{\rho}(g\delta y).$$

The quotient $[X : \rho]$ is a left $(\widehat{\Delta}, G)$-set by following operation:

$$g\widehat{\delta}(\rho(x)) = \{\rho(t) : t \in g\delta x\},$$

where $\widehat{\Delta} = \{\widehat{\delta} : \delta \in \Delta\}$. The map $\pi : X \to [X : \rho]$ defined by $\pi(x) = \rho(x)$, for every $x \in X$ is a morphism.

Example 3.1. Let G be a Γ-semihypergroup with scaler identity and G_1 be a Γ-subsemihypergroup of G. Then, G_1 is a (Γ, G_1)-biset in the obvious way.

Example 3.2. Let ρ be a left regular relation on Γ-semihypergroup G. Then, there is a well-defined action of G on $[G : \rho]$ given by

$$g\widehat{\alpha}(\rho(x)) = \{\rho(t) : t \in g\alpha x\},$$

where $\widehat{\alpha} \in \widehat{\Gamma}$ such that $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$. Hence, with this definition $[G : \rho]$ is a left $(\widehat{\Gamma}, G)$-system.

It is easy to see that the cartesian product $X \times Y$ of a left (Δ, G_1)-set X and a right (Δ, G_2)-set Y becomes $(G_1, \widehat{\Delta}, G_2)$-biset if we make the obvious definition

$$g_1\widehat{\delta}_1(x, y) = \{(t, y) : t \in g_1\delta_1 x\}, \quad (x, y)\widehat{\delta}_2 g_2 = \{(x, t) : t \in y\delta_2 g_2\},$$

where $\widehat{\delta}_1, \widehat{\delta}_2 \in \widehat{\Delta}$, $x \in X, y \in Y$ and $g_1 \in G_1, g_2 \in G_2$.

Let X and Y be (G_1, Δ, G_2)-and (G_2, Δ, G_3)-bisets, respectively and Z be a (G_1, Δ, G_3)-biset. Then, the cartesian product $X \times Y$ is (G_1, Δ, G_3)-biset.

A (G_1, Δ, G_3)-map $\varphi_\delta : X \times Y \to Z$ is called *δ-bimap* if

$$\varphi(x\delta g_2, y) = \varphi(x, g_2\delta y),$$

where $x \in X, y \in Y, g_2 \in G_2$ and $\delta \in \Delta$.

Theorem 3.1. *Let X be a left (Δ, G)-set. Then, β^* is the smallest strongly regular relation on X.*

Proof. Suppose that $a\beta^* b$ be an arbitrary element of X. It follows that there exist $x_0 = a, x_1, \ldots, x_n = b$ such that for all $i \in \{0, 1, 2, \ldots, n\}$ we have $x_i\beta x_{i+1}$. Let $u_1 \in g\delta a$ and $u_2 \in g\delta b$, where $g \in G$, $\delta \in \Delta$. From $x_i\beta x_{i+1}$ it follows that there exists a hyperproduct P_i, such that $\{x_i, x_{i+1}\} \subseteq P_i$ and so $g\delta x_i \subseteq g\delta P_i$ and $g\delta x_{i+1} \subseteq g\delta P_{i+1}$, which means that $g\delta x_i \overline{\overline{\beta}} g\delta x_{i+1}$. Hence for all $i \in \{0, 1, 2, \ldots, n-1\}$ and for all $s_i \in g\delta x_i$ we have $s_i\beta s_{i+1}$. We consider $s_0 = u_1$ and $s_n = u_2$ then we obtain $u_1\beta^* u_2$. Then β^* is strongly regular on a left.

Let ρ be a strongly regular relation on X. Then, we have

$$\beta_1 = \{(x, x) : x \in X\} \subseteq \rho,$$

since ρ is reflexive. Let $\beta_{n-1} \subseteq \rho$ and $a\beta_n b$. Then, there exist $g_1, g_2, \ldots, g_n \in G$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$ and $x \in X$ such that $\{a, b\} \subseteq \prod_{i=1}^n g_i\delta_i x = g_1\delta_1 \prod_{i=2}^n g_i\delta_i x$. This implies that there exits $u, v \in \prod_{i=2}^n g_i\delta_i x$ such that $a \in g_1\delta_1 u$ and $v \in g_1\delta_1 v$. We have $u\beta_{n-1} v$ and according to the hypothesis, we obtain $u\rho v$. Since ρ is regular it follows that $a\rho b$ and $\beta_n \subseteq \rho$. By induction, it follows that $\beta \subseteq \rho$. Therefore, $\beta^* \subseteq \rho$. □

Definition 3.1. A pair (P, ψ) consisting of (G_1, Δ, G_3)-biset P and a δ-bimap $\psi : X \times Y \to P$ will be called a *twist product* of X and Y over G_2 if for every (G_1, Δ, G_3)-biset Z and for every bimap $\omega : X \times Y \to Z$ there exists a unique bimap $\overline{\omega} : P \to Z$ such that $\overline{\omega} \circ \psi = \omega$.

Suppose that ρ is an equivalence relation on $X \times Y$ as follows:

$$\rho = \{(t_1, t_2) : t_1 \in x\delta g, t_2 \in g\delta y, x \in X, y \in Y, g \in G_2\}.$$

Let us define $X \ominus Y$ to be $[X \times Y : \rho^*]$, where ρ^* is a transitive closure of ρ. We denote a typical element $\rho^*(x, y)$ by $x \ominus y$. By definition of ρ^*, we have $x\delta g \ominus y = x \ominus g\delta y$, where $\delta \in \Delta$.

Proposition 3.1. *Let X and Y be (G_1, Δ, G_2)-and (G_2, Δ, G_3)-biset, respectively. Then, two element $x \ominus y$ and $x^{'} \ominus y^{'}$ are equal if and only if $(x, y) = (x^{'}, y^{'})$ or there exist $x_1, x_2, \ldots, x_{n-1}$ in X, $h_1, h_2, \ldots, h_{n-1} \in G_2$ and $\delta \in \Delta$ such that*

$$x \in x_1\delta g_1, x_1\delta h_1 = x_2\delta g_2, \ldots, x_i\delta g_i = x_{i+1}\delta g_{i+1}, x_{n-1}\delta h_{n-1} = x^{'}\delta g_n, g_n\delta y_{n-1} = y^{'},$$
$$g_1\delta y = h_1\delta y_1, g_2\delta y_1 = h_2\delta y_2, \ldots, g_{i+1}\delta y_i = h_{i+1}\delta y_{i+1} = g_n\delta y_{n-1} = y^{'}.$$

Proof. Suppose that we have the given sequence of equations. Then,

$$\begin{aligned} x \ominus y \in x_1\delta g_1 \ominus y = x_1 \ominus g_1\delta y = x_1 \ominus h_1\delta y_1 &= x_1\delta h_1 \ominus y_1 \\ &\vdots \\ &= x^{'} \ominus g_n\delta y_{n-1} \\ &= x^{'} \ominus y^{'}. \end{aligned}$$

Conversely, suppose that $x \ominus y = x^{'} \ominus y^{'}$. Then, there is a sequence

$$(x, y) = (t_1, s_1), (t_2, s_2), (t_3, s_3), \ldots, (t_n, s_n) = (x^{'}, y^{'}),$$

in which for each $1 \leq i \leq n$ such that $((t_i, s_i), (t_{i+1}, s_{i+1}) \in \rho$ or $((t_{i+1}, s_{i+1}), (t_i, s_i)) \in \rho$. This complete the proof. □

Theorem 3.2. *Let X and Y be (G_1, Δ, G_2)-and (G_2, Δ, G_3)-bisets.Then, $(X \ominus Y, \pi)$ is a twist product of X and Y over G_2.*

Proof. It is easy to see that $\pi : X \times Y \to X \ominus Y$ is a δ-bimap such that $\pi(x, y) = x \ominus y$. Let $\omega : X \times Y \to Z$, where Z is a (G_1, Δ, G_3)-biset and $\omega : X \times Y \to Z$ is a δ-bimap. We define $\overline{\omega} : X \ominus Y \to Z$ by

$$\overline{\omega}(x \ominus y) = \omega(x, y).$$

Let $x \ominus y = x^{'} \ominus y^{'}$. This implies that

$$\omega(x, y) = \omega(x_1\delta g_1, y) = \omega(x_1, g_1\delta y) = \omega(x_1, g_1\delta h_1) = \cdots = \omega(x^{'}, y^{'}).$$

Hence $\overline{\omega}(x \ominus y) = \overline{\omega}(x^{'} \ominus y^{'})$. It is easy to see that $\overline{\omega}$ is a δ-bimap, $\overline{\omega} \circ \pi = \omega$ and $\overline{\omega}$ is unique with respect. □

Let G be a Γ-semihypergroup. Then, we say that G_1 has the *extension property* in G if for every left (Δ, G)-set X and for every right (Δ, G)-set Y the map $X \ominus Y \to X \ominus G \ominus Y$ defined by $x \ominus y \to x \ominus e_\alpha \ominus y$ is one to one, for every $\alpha \in \Gamma$.

Theorem 3.3. *Let X and Y be (G_1, Δ, G_2)-and (G_2, Δ, G_3)-biset. Then, the twist product X and Y over G_2 is unique up to isomorphism.*

Proof. Suppose that (P, ψ) and $(P^{'}, \psi^{'})$ are twist product of X and Y over G_2. By definition 3.1, we find a unique $\overline{\psi^{'}} : P \to P^{'}$ and $\overline{\psi} : P^{'} \to P$ such that $\psi \circ \overline{\psi^{'}} = \psi^{'}$ and $\overline{\psi} : P^{'} \to P$ such $\psi^{'} \circ \overline{\psi} = \psi$. Since $\psi \circ \overline{\psi^{'}} \circ \overline{\psi} = \psi$, we have $\overline{\psi^{'}} \circ \overline{\psi} = Id$. By a similar argument, $\overline{\psi} \circ \overline{\psi^{'}} = Id$. □

We can generalize the notion of twist product three bisets. Let X, Y, Z and W be (G_1, Δ, G_2)-, (G_2, Δ, G_3)-, (G_3, Δ, G_4)- and (G_1, Δ, G_4)-biset. Then, a map $\varphi : X \times Y \times Z \to Z$ is called *δ-trimap* if for $x \in X, y \in Y$ and $z \in Z$ and $g_2 \in G_2$, $g_3 \in G_3$ and $\delta \in \Delta$,

$$\varphi(x\delta g_2, y, z) = \varphi(x, g_2\delta y, z), \quad \varphi(x, y\delta g_3, z) = \varphi(x, y, g_3\delta z).$$

A pair (P, ψ), where P is a (G_1, Δ, G_4)-biset and $\psi : X \times Y \times Z \to P$ is a δ-trimap is said to be twist product if for every (G_1, Δ, G_4)-biset W and every δ-trimap $\phi : X \times Y \times Z \to W$ there is a unique $\overline{\phi} : P \to W$ such that $\overline{\phi} \circ \psi = \phi$. By the similar proof of Theorem 3.2, shows that $X \ominus (Y \ominus Y)$, together with the obvious trimap $(x, y, z) \to x \ominus (y \ominus z))$ is also a twist product of X, Y and Z.

Proposition 3.2. *Let X, Y, Z be (G_1, Δ, G_2)-, (G_2, Δ, G_3)-, (G_3, Δ, G_4)-bisets, respectively. Then, $X \ominus (Y \ominus Z) \cong (X \ominus Y) \ominus Z$.*

Proof. By the similar proof of Theorem 3.2, $X \ominus (Y \ominus Z)$ and $(X \ominus Y) \ominus Z$ are twist product of X, Y and Z and by Theorem 3.3, we have the twist product of X, Y and Z are unique. Therefore, $X \ominus (Y \ominus Z) \cong (X \ominus Y) \ominus Z$. □

Definition 3.2. Let G_1 be a Γ-subsemihypergroup of G and $g \in G$. Then, we say that g, *α-fixed* by G_1 if for every Γ-semihypergroup H and α-homomorphism $\varphi_1, \varphi_2 : G[\alpha] \to H[\alpha]$, $\varphi_1(g_1) = \varphi_2(g_1)$, where $g_1 \in G_1$ implies that $\varphi_1(g) = \varphi_2(g)$. If for every $\alpha \in \Gamma$, g is an α-fixed element, then we say that g is a *fixed element.*

The set elements of G fixed by G_1 denoted by $\mathrm{Fix}_G(G_1)$.

Theorem 3.4. *Let G_1 be a Γ-subsemihypergroup of G and $g \in G$ such that $e_\alpha \ominus g = g \ominus e_\alpha$. Then, $g \in \mathrm{Fix}_G(G_1)$.*

Proof. Suppose that $e_\alpha \ominus g = g \ominus e_\alpha$ in the twist product $G \ominus G$. Let H be a Γ-semihypergroup and $\varphi_1, \varphi_2 : G[\alpha] \to H[\alpha]$ such that for every $x \in G_1$, $\varphi_1(x) = \varphi_2(x)$ Then, H is a (Γ, G_1)-biset if we define

$$y\widehat{\delta}h = \varphi_1(y)\delta h = \varphi_2(y)\delta h, \quad h\widehat{\delta}y = h\delta\varphi_1(y) = h\delta\varphi_2(y),$$

where $y \in G_1$, $\delta \in \Gamma$ and $h \in H$.

We define $\psi : G \times G \to H$ by the rule that

$$\psi(g, g^{'}) = \varphi_1(g)\delta\varphi_2(g^{'}),$$

where $g, g^{'} \in G$ and $\delta \in \Gamma$. Hence ψ is a δ-bimap. Indeed, for all $x_1, x_2 \in G$ and $g_1 \in G_1$, we have

$$\begin{aligned}\psi(x_1\delta g_1, x_2) &= \varphi_1(x_1)\delta\varphi_1(g_1)\delta\varphi_2(x_2) = \varphi_1(x_1)\delta\varphi_2(g_1)\delta\varphi_2(x_2)\\ &= \varphi_1(x_1)\delta\varphi_2((g_1\delta x_2)\\ &= \psi(x_1, g_1\delta x_2).\end{aligned}$$

This implies that there is a map $\overline{\psi} : G \ominus G \to H$ such that

$$\overline{\psi}(x_1 \ominus x_2) = \psi(x_1, x_2) = \varphi(x_1)\delta\varphi_2(x_2),$$

for every $(x_1, x_2) \in G \ominus G$. Since $g \ominus e_\delta = e_\delta \ominus g$,

$$\varphi_1(g) = \varphi_1(g)\delta\varphi_2(e_\alpha) = \overline{\psi}(g \ominus e_\alpha) = \overline{\psi}(e_\alpha \ominus g) = \varphi_1(e_\alpha)\delta\varphi_2(g) = \varphi_2(g).$$

Therefore, $g \in \mathrm{Fix}_G(G_1)$ and this completes the proof. □

Let X be a left (Δ, G)-set. We say that X is *flat* if for every monomorphism $\varphi : X_1 \to X_2$ of right (Δ, G)-sets the induced map $\varphi \ominus I : X_1 \ominus X \to X_2 \ominus X$ is one to one. A Γ-semihypergroup G is called *absolutely flat* if all left and right (Δ, G)-sets are flat. A Γ-semihypergroup G is called *absolutely extendable* if it has the extension property in every Γ-semihypergroup H containing it as Γ-subsemihypergroup G.

Proposition 3.3. *Let G be a absolutely flat Γ-semihypergroup. Then, G is a absolutely extendable.*

Proof. Suppose that G is a absolutely flat and G_1 is a Γ-semihypergroup such that $G \leq G_1$, X and Y are left (Δ, G)- and right (Δ, G)-sets. We have

$$X \cong X \ominus G \to X \ominus G_1.$$

Since the inclusion homomorphism from G into G_1 is one to one implies that the induced map $X \ominus G \to X \ominus G_1$ is one to one. By the flatness of Y the following map is one to one

$$X \ominus Y \to X \ominus G_1 \ominus Y.$$

This completes the proof. □

Theorem 3.5. *Let G be a Γ-semihypergroup such that G_1 be a Γ-subsemihypergroup of G and g be an α-fixed element. Then, $e_\alpha \ominus g = e_\alpha \ominus g$.*

Proof. Suppose that $X = G \ominus G$. Hence X is a (G, Γ, G)-biset by the following actions:

$$g\alpha(g_1 \ominus g_2) = \{t \ominus g_2 : t \in g\alpha g_1\}, \quad (g_1 \ominus g_2)\alpha g = \{g_1 \ominus t : t \in g_2\alpha g\},$$

where $g_1, g_2, g \in G$ and $\alpha \in \Gamma$. Let $F(X)$ be abelian group generated by X. We define a binary relation on $G \times F(X)$ as follows:

$$(g_1, x)\widehat{\alpha}(g_2, y) = \{(t_1, t_2) : t_1 \in g_1\alpha g_2, t_2 \in g_1\alpha y + x\alpha g_2\},$$

where $g_1, g_2 \in G$, $x, y \in F(X)$ and $\alpha \in \Gamma$. This binary relation is associative. Indeed,

$$\begin{aligned}
[(g_1, x_1)\widehat{\alpha}(g_2, x_2)]\widehat{\beta}(g_3, x_3) =& \{(t_1, t_2) : t_1 \in g_1\alpha g_2, t_2 \in g_1\alpha x_2 + x_1\alpha g_2\}\beta(g_3, x_3) \\
=& \{(t_1, t_2) : t_1 \in (g_1\alpha g_2)\beta g_3, t_2 \in (g_1\alpha g_2)\beta x_3 \\
& + (g_1\alpha x_3 + x_1\alpha g_2)\beta g_3\} \\
=& \{(t_1, t_2) : t_1 \in g_1\alpha(g_2\beta g_3), t_2 \in g_1\alpha(g_2\beta x_3 \\
& + x_3\beta g_3) + x_1\alpha(g_2\beta g_3)\} \\
=& (g_1, x_1)\widehat{\alpha}(g_2\alpha g_3, g_2\alpha x_3 + x_2\beta g_3) \\
=& (g_1, x_1)\widehat{\alpha}[(g_2, x_2)\widehat{\beta}(g_3, x_3)].
\end{aligned}$$

Let $g \in \mathrm{Fix}_G(G_1)$ and $\varphi_1 : G \to G \times F(X)$ and $\varphi_2 : G \to G \times F(X)$ as follows:

$$\varphi_1(g_1) = (g_1, 0), \quad \varphi_2(g_2) = (g_2, g_2 \ominus e_\alpha - e_\alpha \ominus g_2),$$

where $g_1, g_2 \in G$ and $\alpha \in \Gamma$ is a fixed element. One can see that φ_1 and φ_2 are α-homomorphism. Indeed,

$$\begin{aligned}\varphi(g_1)\alpha\varphi_2(g_2) &= (g_1, g_1 \ominus e_\alpha - e_\alpha \ominus g_1)\alpha(g_2, g_2 \ominus e_\alpha - e_\alpha \ominus g) \\ &= (g_1\alpha g_2, g_1\alpha(g_2 \ominus e_\alpha - e_\alpha \ominus g_2) + (g_1 \ominus e_\alpha - e_\alpha \ominus g_1)\alpha g_2) \\ &= (g_1\alpha g_2, ((g_1\alpha g_2) \ominus e_\alpha - e_\alpha \ominus (g_1\alpha g_2)) \\ &= \varphi_2(g_1\alpha g_2).\end{aligned}$$

If $g \in G_1$, then $g \ominus e_\alpha = e_\alpha \ominus g$. This implies that $\varphi_1(g) = \varphi_2(g)$ and hence for every $g \in Fix_G(G_1), \varphi_1(g) = \varphi_2(g)$ and so $g \ominus e_\alpha = e_\alpha \ominus g$. This completes the proof. □

Definition 3.3. Let X, Y, Y and P be (Δ, G)-sets such that $\beta : X \to Y$, $\gamma : X \to Z$, $\mu : Y \to P$ and $\nu : Z \to P$ be morphisms. Then, we say that P is a pushout (Δ, G)-system if there exist a left (Δ, G)-set, $\mu' : Y \to P'$ and $\nu' : Z \to P'$ such that $\mu' \circ \beta = \nu' \circ \gamma$, then there exists a unique morphism $\varphi : P \to P'$ such that $\varphi \circ \mu = \mu'$ and $\varphi \circ \nu = \nu'$.

Proposition 3.4. *Let X, Y and Z be (Δ, G)-sets such that $\beta : X \to Y$ and $\gamma : X \to Z$ be morphisms. Then, the pushout P exist.*

Proof. Suppose that ρ^* is a regular elation generated by ρ on disjoint union $X \cup Y \cup Z$

$$(a, b) \in \rho \Leftrightarrow a \in X, \beta(a) = b \text{ or } \gamma(a) = b.$$

We define $\varphi : Y \to [X \cup Y \cup Z : \rho^*]$ and $\psi : Z \to [X \cup Y \cup Z : \rho^*]$ by

$$\varphi(y) = \rho^*(y), \quad \psi(z) = \rho^*(z).$$

One can see that $[X \cup Y \cup Z : \rho^*]$ is a pushout system. □

Suppose that G is a Γ-semihypergroup, G_1 is a Γ-subsemihypergroup of G, X, Y are left (Δ, G)-set and $\varphi : X \to Y$ is a morphism. We define the relation ϱ on $T = Y \ominus_{G_1} G$ as follows:

$$\begin{aligned}(y_1 \ominus t_1, y_2 \ominus t_2) \in \varrho \Leftrightarrow & (\exists x_1, x_2 \in X, s_1, s_2 \in G, \delta \in \Delta)\ \varphi(x_1) = t_1, \varphi(x_2) = t_2, \\ & x_1\delta s_1 = x_2\delta s_2.\end{aligned}$$

Theorem 3.6. *Let G be a Γ-semihypergroup, G_1 be a Γ-subsemihypergroup of G, X and Y be left and right (Δ, G_1)-set, respectively, and $\phi : X \to Y$ be a morphism. Then, there exist a right (Δ, G)-set D and morphisms $\beta : X \to D$, $\alpha : Y \to D$ such that $\alpha \circ \phi = \beta$.*

Proof. Suppose that $T = Y \ominus_{G_1} G$ and $D = [T : \varrho^*]$. We define $\alpha : Y \to D$ by

$$\alpha(y) = \varrho^*(y \ominus e).$$

Then, for all $y \in Y$ and $g_1 \in G_1$

$$\alpha(y\delta g_1) = \varrho^*(y\delta g_1 \ominus e) = \sigma^*(y \ominus g_1) = \varrho^*(y \ominus e)\widehat{\delta} g_1 = \alpha(y)\widehat{\delta} g_1.$$

We define $\beta : X \to D$ by

$$\beta(x) = \varrho^*(\phi(x) \ominus e).$$

Let $g \in G$ and $x \in X$. Then,

$$\begin{aligned}\beta(x\delta g) &= \varrho^*(\phi(x\delta x) \ominus e)\\ &= \varrho^*(\phi(x) \ominus g)\\ &= \varrho^*(\phi(x) \ominus e)\widehat{\delta} g = \beta(x)\widehat{\delta} g.\end{aligned}$$

This implies that β is a morphism. Also,

$$\alpha \circ \phi(x) = \varrho^*(\phi(x) \ominus e) = \beta(x).$$

This completes the proof. □

Theorem 3.7. *Let G_1 be a Γ-subsemihypergroup of G, X and Y be left and right (Δ, G_1)-sets, respectively and $\phi : X \to Y$ be G_1- morphism. Then, $D = [Y \ominus_{G_1} G : \varrho^*]$ is a pushout where $\phi \ominus Id_G : X \ominus G \to Y \ominus G$, $\pi : Y \ominus G \to D$ is a natural morphism, $\beta : X \to D$ defined by $\beta(x) = \phi(x) \ominus e$ and $\gamma : X \ominus G \to X$ given by $\gamma(x \ominus g) = x\delta g$, where $g \in G$, $x \in X$ and $\delta \in \Delta$.*

Proof. It is clear that $\pi \circ (\phi \ominus Id_G) = \beta \circ \gamma$. Let D_1 be a another pushout such that $\xi : Y \ominus G \to D_1$, $\beta' : X \to D_1$, $\xi \circ (\phi \ominus Id_G) = \beta' \circ \gamma$. Let $(y_1 \ominus g_1, y_2 \ominus g_2) \in \varrho$ where $y_1, y_2 \in Y$ and $g_1, g_2 \in G$. Then, there exist $x_1, x_2 \in X$ and $\delta \in \Delta$ such that $\phi(x_1) = y_1, \phi(x_2) = y_2$ and $x_1\delta g_1 = x_2\delta g_2$. We have

$$\begin{aligned}\xi(y_1 \ominus g_1) &= \xi(\phi(x_1) \ominus g_1) = \xi(\phi \ominus Id_G)(x \ominus g) = \beta' \circ \gamma(x_1 \ominus g_1) = \beta'(x_1\delta g_1)\\ &= \beta'(x_2\delta y_2) = \xi(\phi(x_2) \ominus g_2) = \xi(y_2 \ominus g_2).\end{aligned}$$

This implies that $\varrho \subseteq \mathrm{Kerl}\xi$. Then, $\varrho^* \subseteq \mathrm{Kerl}\xi$. It follows that ξ induced a unique G-morphism $\omega : D \to D_1$ given by

$$\omega(\varrho^*(y \ominus g)) = \xi(y \ominus g),$$

where $g \in G$ and $y \in Y$. Finally,

$$\begin{aligned}\omega \circ \beta(x) &= \omega(\varrho^*(\phi(x) \ominus e)) = \xi(\phi(x) \ominus e)\\ &= \xi \circ (\phi \ominus Id)(x \ominus e) = \beta' \circ \gamma(x \ominus e)\\ &= \beta'(x).\end{aligned}$$

This completes the proof. □

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