

A POINTFREE ANALOGUES OF LATTICE-VALUED BITOPOLOGICAL SPACES

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ABSTRACT. The concept of coupled semi-quantales is introduced. An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. The topological and the lattice-theoretic concepts of regularity and compactness are extended to both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

1. INTRODUCTION

In 1986 Mulvey [9], proposed the concept quantale as a non-commutative extension of frame (or pointfree topology) with aim to develop the concept of non-commutative topology [6] and provide a constructive foundations for both quantum mechanics and non-commutative logic [17]. Nowadays, the concepts of quantales and semi-quantales (as a generalization of quantales [14]) can boast many areas of applications, e.g., the area of non-commutative topology [5, 10, 11]. Further details about quantales can be found in [15].

In 2015 Höhle [7], established a non-commutative extension of the well known Papert-Papert-Isbell adjunction [8, 12] between the category of locales and the category of topological spaces to one between the category of quantales and the category of many valued topological spaces.

In [4], El-Saadly extended the Höhle's adjunction to a more general one between the category of semi-quantales and the category of lattice-valued quasi-topological spaces.

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In this paper we aim to introduce the concept of coupled semi-quantales as the pointfree analogues of lattice-valued bitopological spaces and extend the dual adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces to one between the category of coupled semi-quantales and the category of lattice valued biquasi-topological spaces. Also, the topological and the lattice-theoretic concepts of regularity and compactness are extended to lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively. Some relations among these axioms are established.

2. PRELIMINARIES

By a complete join-semilattice (or \vee -semilattice) we mean a partially ordered set (L, \leq) having arbitrary sups.

Definition 2.1. [14] A semi-quantale (L, \leq, \otimes) is a complete join-semilattice (L, \leq) equipped with a binary operation $\otimes : L \times L \rightarrow L$, with no additional assumptions, called a tensor product.

Definition 2.2. [14] Let L and M be semi-quantales. A function $h : L \rightarrow M$ is said to be:

- (1) a semi-quantale morphism if it preserves \otimes and arbitrary sups;
- (2) a strong semi-quantale morphism if it preserves \otimes , arbitrary sups and \top .

By **SQuant**(resp. **SSQuant**), we mean the category of all semi-quantales and semi-quantale morphisms (resp. strong semi-quantale morphism).

Definition 2.3. A semi-quantale (L, \leq, \otimes) is said to be:

- (1) a quantale [15] if whose multiplication \otimes is associative and distributes across \vee from both sides. **Quant** denotes the full subcategory of **SQuant** of all quantales.
- (2) a unital semi-quantale [14] if whose multiplication \otimes has an identity element $e \in L$ called the unit. **USQuant** denotes the category all unital semi-quantales together with all semi-quantales morphisms preserving the unit e .
- (3) a commutative semi-quantate [14] if whose multiplication \otimes satisfies that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. **CSQuant** denotes the full subcategory of **SQuant** of all commutative semi-quantales.
- (4) a distributive semi-quantate [16] if whose multiplication \otimes distributes across finite \vee from both sides. **DSQuant** is the category of distributive semi-quantales.

Definition 2.4. [4] Let $L \in |\mathbf{SQuant}|$, $M \subseteq L$, and $a, b \in M$. An element a is said to be well-inside of b (w.r.t. M), denoted $a \preceq b$, if

$$\text{exists } c \in M \text{ with } a \otimes c = \perp \text{ and } c \vee b = \top.$$

An $L \in |\mathbf{SQuant}|$ is said to be *regular* [4], if for each $a \in L$ there exists $D \subseteq I_a$, where $I_a = \{b \in L : b \preceq a\}$ such that $a = \vee D$.

Definition 2.5. [3] Let $L = (L, \leq, \otimes)$ be a semi-quantale. A subset $K \subseteq L$ is a subsemi-quantale of L if and only if the inclusion $K \hookrightarrow L$ is a semi-quantale morphism, i.e., K is closed under \otimes and arbitrary sups. A subsemi-quantale K of L is said to be strong if and only if \top belongs to K . If L is a unital semi-quantale with the identity e , then a subsemi-quantale K of L is called a unital subsemi-quantale of L if and only if e belongs to K .

Let $L = (L, \leq, \otimes)$ be a semi-quantale. For any non-empty set X , let L^X be the set of all L -valued maps $X \xrightarrow{f} L$. We can extend the algebraic and lattice-theoretic structure from L to L^X pointwisely, i.e., for all $x \in X, f, g \in L^X$ and $\{f_j : j \in J\} \subseteq L^X$, we have

$$\begin{aligned} f \leq g &\Leftrightarrow f(x) \leq g(x), \\ (f \otimes g)(x) &= f(x) \otimes g(x), \\ \left(\bigvee_{j \in J} f_j\right)(x) &= \bigvee_{j \in J} (f_j(x)). \end{aligned}$$

Then L^X is again a semi-quantale with respect to the multiplication \otimes . If L is a unital semi-quantale with unit e , then L^X becomes a unital semi-quantale with the unit \underline{e} (a mapping from X to L , defined by $\underline{e}(x) = e$ for all $x \in X$), where e is the unit of \otimes in L .

For an ordinary mapping $f : X \rightarrow Y$, the forward and backward powerset operators [13, 14]:

$$f_L^{\rightarrow} : L^X \rightarrow L^Y \text{ and } f_L^{\leftarrow} : L^Y \rightarrow L^X,$$

defined by

$$f_L^{\rightarrow}(A)(y) = \bigvee \{A(x) : x \in X, f(x) = y\} \text{ and } f_L^{\leftarrow}(B) = B \circ f,$$

respectively.

Theorem 2.1. [14] *Let $L \in |\mathbf{SQquant}|$, X, Y be a nonempty ordinary sets and $f : X \rightarrow Y$ be an ordinary mapping, then we have:*

- (1) f_L^{\rightarrow} preserves arbitrary \bigvee ;
- (2) f_L^{\leftarrow} preserves arbitrary \bigvee, \otimes , and all constant maps;
- (3) f_L^{\leftarrow} preserves the unit if $L \in |\mathbf{USQuant}|$.

For a fixed $L \in |\mathbf{SQquant}|$ and a set X , an L -quasi-topology on X [14] is a subsemi-quantale τ of $L^X = (L^X, \leq, \otimes)$, i.e., satisfying the following conditions.

- (T_1) For all $A, B \in L^X$, if $A, B \in \tau$ then $A \otimes B \in \tau$.
- (T_2) For all $\{A_j : j \in J\} \subseteq L^X$, if $\{A_j : j \in J\} \subseteq \tau$ then $\bigvee_j A_j \in \tau$.

An L -quasi-topology τ is said to be strong [3] if and only if it is strong as a subsemi-quantale of L^X , i.e., τ satisfies the additional axiom:

- (T_3) $\perp \in \tau$.

If $L \in |\mathbf{USQuant}|$ with unit e , a unital subsemi-quantale τ of L^X is called an L -topology on X [14], i.e., τ satisfies (T_1) , (T_2) and the following:

$$(T_4) \quad \underline{e} \in \tau.$$

If $\tau \subseteq L^X$ is an L -quasi-topology (resp. L -topology), then the pair (X, τ) is said to be an L -quasi-topological (resp. L -topological) space. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be L -continuous (resp. L -open) [13] if $(f_L^{\leftarrow})|_{\rho} : \tau \leftarrow \sigma$ (resp. $(f_L^{\rightarrow})|_{\tau} : \tau \rightarrow \sigma$). An L -continuous bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is an L -homeomorphism [13] if f^{-1} is L -continuous.

It is clear that L -quasi-topological (resp. strong L -quasi-topological, L -topological) spaces and L -continuous maps form a category denoted by $L\text{-QTop}$ (resp. $L\text{-SQTop}$, $L\text{-Top}$).

One can easily prove that each of $L\text{-QTop}$, $L\text{-SQTop}$ and $L\text{-Top}$ is a topological category over the category \mathbf{Set} .

Definition 2.6. [4] An $(X, \tau) \in |L\text{-QTop}|$ is called

- (1) $L\text{-QT}_0$ if for every $x, y \in X$ with $x \neq y$ there exists $\mu \in \tau$ with $\mu(x) \neq \mu(y)$;
- (2) L -qsober if and only if $\eta_x : (X, \tau) \rightarrow (LPT(\tau), \Phi_L^{\rightarrow}(\tau))$ is bijective.

3. COUPLED SEMI-QUANTALES AND LATTICE-VALUED BIQUASI-TOPOLOGICAL SPACES

Before we go on, this section, we begin our study by the following.

Lemma 3.1. *If $\{A_j : j \in J\}$ is any collection of subsemi-quantales of a semi-quantale Q , then $\bigcap_j A_j$ is also a subsemi-quantale of Q , provided $\bigcap_j A_j \neq \phi$.*

Proof. Let $M = \bigcap_j A_j$ and $a, b \in M$. Then $a, b \in A_j \Rightarrow a \otimes b \in A_j$ for each subsemi-quantale $A_j \Rightarrow a \otimes b \in M \Rightarrow M$ is closed under \otimes . Also, one can easily prove that M is closed under sups. \square

For a fixed $Q \in |\mathbf{SQquant}|$, it follows, as a consequence of the above lemma, that the family of all subsemi-quantales of Q , ordered by inclusion, forms a complete lattice, with the meet $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ (the set-intersection), and the join $Q_1 \vee Q_2$ is the least subsemi-quantale of Q containing Q_1 and Q_2 (which is not their set-theoretical union). The supremum (joins) of a set $\{A_j : j \in J\}$ of subsemi-quantales of Q , is the intersection of subsemi-quantales of Q which contains the union $\bigcup_j A_j$. More generally there is for each subset $K \subseteq Q$ of a semi-quantale Q a smallest subsemi-quantale of Q (sometimes denoted by $[K]$) which contains K and is the subsemi-quantale generated by K .

Definition 3.1. (The category of coupled semi-quantales)

- (1) A coupled semi-quantale is a triple $Q = (Q_0, Q_1, Q_2)$ in which Q_0 is a semi-quantale, Q_1 and Q_2 are subsemi-quantales of Q_0 such that $Q_1 \cup Q_2$ generates Q_0 .

- (2) A map $h : Q \rightarrow P$ between coupled semi-quantales is a semi-quantale morphism $Q_0 \rightarrow P_0$ for which the restrictions $h|_{Q_i} : Q_i \rightarrow P_i$ are semi-quantale morphisms i.e., $h(Q_i) \subseteq P_i$ for $i = 1, 2$.
- (3) The resulting category will be denoted by **CSQuant**.

We refer to Q_0 as the total part of Q , and Q_1, Q_2 as its first and second parts, respectively.

Definition 3.2. A coupled semi-quantale $Q = (Q_0, Q_1, Q_2)$ is said to be:

- (1) *unital* if and only if Q_0 is unital and e belongs to both Q_1 and Q_2 .
UnCSQuant is the full subcategory of **CSQuant** of all unital coupled semi-quantales.
- (2) *coupled quantal* [1] if Q_0 is a quantale and both Q_1 and Q_2 are subquantales.
CQuant is the full subcategory of **CSQuant** of all coupled quantales.
- (3) *strong coupled quantal* if both Q_1 and Q_2 are strong subquantales of Q_0 .
- (4) *symmetric* if and only if $Q_0 = Q_1 = Q_2$.
- (5) *right-sided* (resp. *left-sided*) if and only if $a \otimes \top \leq a$ (resp. $\top \otimes a \leq a$) for all $a \in Q_0$.
- (6) *idempotent* if and only if the total part Q_0 is idempotent, i.e., $a \otimes a = a$ for all $a \in Q_0$.
- (7) *commutative* if the operation \otimes is commutative, i.e., $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1 \in Q_i$ and $q_2 \in Q_k$. **ComCSQuant** is the full subcategory of **CSQuant** of all commutative coupled semi-quantales.

Example 3.1. For a fixed $L \in |\mathbf{SQuant}|$ and a non-empty set X . For $i = 1, 2$, let $\tau_i \subset L^X$ be a subsemi-quantale of L^X , i.e., L -quasi-topologies on X . The triple $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a coupled semi-quantale where $\tau_1 \vee \tau_2$ is the coarsest L -quasi-topology finer than both τ_1 and τ_2 .

Example 3.2. Let $Q = \{\perp, a, b, \top\}$ be the four Boolean lattice and let $\otimes : Q \times Q \rightarrow Q$ defined by

\otimes	\perp	a	b	\top
\perp	\perp	\perp	\perp	\perp
a	\perp	a	\perp	a
b	\perp	\perp	b	b
\top	\perp	a	b	\top

It is clear that Q is a coupled quantales with $Q_0 = \{\perp, a, b, \top\}$ as the total part, $Q_1 = \{\perp, a, \top\}$ as the first part and $Q_2 = \{\perp, b, \top\}$ as the second part.

Example 3.3. Any biframe $A = (A_0, A_1, A_2)$ [2] is a commutative coupled quantale provided that $\otimes = \wedge$ and any element of $a \in A_0$ can be expressed as $a = \vee\{a_1 \otimes a_2 : a_1 \in A_1, a_2 \in A_2\}$.

Definition 3.3. (The category of L -biquasi-topological spaces)

- (1) An L -biquasi-topological space is a triple (X, τ_1, τ_2) consisting of a non-empty set X and two L -quasi-topologies τ_1 and τ_2 on X .
- (2) A morphism $f : X \rightarrow Y$ between L -biquasi-topological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) is a function between their underlying sets for which

$$f : (X, \tau_1) \rightarrow (Y, \sigma_1) \text{ and } f : (X, \tau_2) \rightarrow (Y, \sigma_2)$$

are L -continuous.

- (3) The category of L -biquasi-topological spaces and their morphisms will be denoted by $L\text{-BiQTop}$.

Between the category $L\text{-QTop}$ and $L\text{-BiQTop}$ there is a faithful functor

$$E_S : L\text{-BiQTop} \rightarrow L\text{-QTop} ,$$

which we describe as follows. If $X = (X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$, then $E_S(X) = (X, \tau_1 \vee \tau_2)$, where $\tau_1 \vee \tau_2$ is the coarsest L -quasi topology finer than both τ_1 and τ_2 , $E_S(f) = f$.

The left adjoint of S is the functor

$$E_d : L\text{-QTop} \rightarrow L\text{-BiQTop},$$

by the following correspondences:

$$E_d(X, \tau) = (X, \tau, \tau), E_d(f) = f.$$

One notes that since E_S embeds $L\text{-QTop}$ in $L\text{-BiQTop}$, then we will regard the constructions in $L\text{-BiQTop}$ as extensions of the constructions in the category $L\text{-QTop}$.

For $L \in |\mathbf{SQuant}|$ and $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$. The functor

$$\mathcal{O}_L : L\text{-BiQTop} \rightarrow \mathbf{CSQuant}^{op}$$

is defined as follows

$$\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2).$$

For the L -biquasi-topological space (X, τ_1, τ_2) , the triple $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a coupled semi-quantale where $\tau_1 \vee \tau_2$ is the coarsest L -quasi-topology finer than both τ_1 and τ_2 , and

$$\mathcal{O}_L(f : (X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)) = [(f_L^{\leftarrow})|_{\theta_i}]^{op} : \tau_i \rightarrow \theta_i, \quad i = 1, 2.$$

Now, we will introduce some ideas needed to define a functor in the opposite direction. For a coupled semi-quantale $Q = (Q_0, Q_1, Q_2)$, let

$$LPT(Q_0) = \{p : Q_0 \rightarrow L : p \in |\mathbf{SQuant}|\}.$$

Also, we define a coupled semi-quantale map

$$\Phi_L : (Q_0, Q_1, Q_2) \rightarrow (L^{LPT(Q_0)}, L^{LPT(Q_0)}, L^{LPT(Q_0)})$$

such that

- (1) $\Phi_L : Q_0 \rightarrow L^{LPT(Q_0)}$ is a semi-quantale map, where $\Phi_L(a)(p) = p(a)$;
- (2) $\Phi_L^{\rightarrow}(Q_1) \subseteq L^{LPT(Q_0)}$;
- (3) $\Phi_L^{\rightarrow}(Q_2) \subseteq L^{LPT(Q_0)}$.

As given in [4] the function Φ_L preserves \otimes and arbitrary \vee , where these are inherited by the codomain of Φ_L from L . Also, for $i = 1, 2$, we have $\Phi_L^\rightarrow(Q_i)$ is closed under these operations and hence is an L -quasi topology on $LPT(Q_0)$. Thus we have

$$LPT : L\text{-BiQTop} \leftarrow \mathbf{CSQuant}^{op},$$

defined by

$$(Q_0, Q_1, Q_2) \rightarrow (LPT(Q_0), \Phi_L^\rightarrow(Q_1), \Phi_L^\rightarrow(Q_2)),$$

where $LPT(f : A \rightarrow B) = [f]^{op}$, that is, $LPT(f)(p) = p \circ f^{op}$, $f^{op} : B \rightarrow A$, is a concrete map in $\mathbf{CSQuant}$. It is clear that $\{\Phi_L(a_i) : a_i \in Q_i, i = 1, 2\}$ is an L -quasi-topology on $LPT(Q_0)$ and, therefore, we have $(LPT(Q_0), \Phi_L^\rightarrow(Q_1), \Phi_L^\rightarrow(Q_2)) \in |L\text{-BiQTop}|$.

Proposition 3.1. *For a fixed $L \in |\mathbf{SQuant}|$ and $Q, P \in |\mathbf{CSQuant}|$, the mapping*

$$LPT(f) : (LPT(Q_0), \Phi_L^\rightarrow(Q_1), \Phi_L^\rightarrow(Q_2)) \rightarrow (LPT(P_0), \Phi_L^\rightarrow(P_1), \Phi_L^\rightarrow(P_2))$$

is L -bicontinuous.

Proof. We need to check the L -continuity of both the functions

- (1) $LPT(f) : (LPT(Q_0), \Phi_L^\rightarrow(Q_1)) \rightarrow (LPT(P_0), \Phi_L^\rightarrow(P_1))$ and
- (2) $LPT(f) : (LPT(Q_0), \Phi_L^\rightarrow(Q_2)) \rightarrow (LPT(P_0), \Phi_L^\rightarrow(P_2))$.

The first function is L -continuous since for all $q_2 \in P_0, p \in LPT(Q_0)$, we have

$$\begin{aligned} LPT(f)^\leftarrow(\Phi_L(q_2)(p)) &= \Phi_L(q_2)(LPT(f)(p)) \\ &= \Phi_L(q_2)(p \circ f^{op}) \\ &= \Phi_L(f^{op}(q_2))(p). \end{aligned}$$

Similarly, we can check the L -continuity of the second function and this completes the proof. □

Then we have the spectrum or point functor

$$LPT : \mathbf{CSQuant}^{op} \rightarrow L\text{-BiQTop}.$$

To study the adjunction between the functors

$$LPT : \mathbf{CSQuant}^{op} \rightarrow L\text{-BiQTop}$$

and

$$\mathcal{O}_L : L\text{-BiQTop} \rightarrow \mathbf{CSQuant}^{op}.$$

we give the following definitions.

For fixed $L \in |\mathbf{SQuant}|$, $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ and $Q \in |\mathbf{CSQuant}|$ define the maps:

- (1) $\eta_X : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2))$, by setting, for all $x \in X$ and $\mu \in \mathcal{O}_L(X)$, $\eta_X(x)(\mu) = \mu(x)$;
- (2) $\varepsilon_Q^{op} : Q \rightarrow \mathcal{O}_L(LPT(Q))$, by setting $\varepsilon_{Q_0}^{op} = \Phi_L|_{\Phi_L^\rightarrow(Q_0)}$.

It is clear that by definition ε_Q^{op} always surjective.

Lemma 3.2. *Let $L \in |\mathbf{SQuant}|$, $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ and $Q \in |\mathbf{CSQuant}|$. Then*

- (1) *the map $\eta_X : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2))$, is L -bicontinuous, and pairwise L -open w.r.t. its range in $(LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2))$ and*
- (2) *the map $\varepsilon_Q^{op} : Q \rightarrow \mathcal{O}_L(LPT(Q))$ is a coupled semi-quantale morphism.*

Proof. (1) To prove that the mapping η_X is L -bicontinuous and pairwise L -open, it suffices to prove that both the mappings $\eta_X : (X, \tau_1) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1))$ and $\eta_X : (X, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_2))$ are L -continuous and L -open with respect to their respective ranges.

- (i) L -continuity: for $i \in \{1, 2\}$, for all $\mu \in \Phi_L^\rightarrow(\tau_i)$, and for all $x \in X$, there exists $\rho \in \tau_i$ such that $\Phi_L(\rho) = \mu$, $(\eta_X)_L^\leftarrow(\mu)(x) = (\eta_X)_L^\leftarrow(\Phi_L(\rho))(x) = \rho(x)$, that is, $(\eta_X)_L^\leftarrow(\mu) \in \tau_i$. Hence η_X is L -bicontinuous.
- (ii) Openness: in fact, for $\nu \in \tau_i$, $i \in \{1, 2\}$, and $p \in LPT(\tau_1 \vee \tau_2)$:

$$\begin{aligned} (\eta_X)_L^\rightarrow(\nu)(p) &= \bigvee_{x \in X} \{\nu(x) : \eta_X(x) = p\} \\ &= \bigvee_{x \in X} \{\eta_X(x)(\nu) : \eta_X(x) = p\} \\ &= p(\nu) = \Phi_L^\rightarrow(\nu)(p). \end{aligned}$$

Now, $\Phi_L(\nu) \in \Phi_L^\rightarrow(\tau_i)$, the L -quasi-topology on $LPT(\tau_1 \vee \tau_2)$, and it follows that $(\eta_X)_L^\rightarrow(\nu) = \Phi_L(\nu)$, that is, $(\eta_X)_L^\rightarrow(\nu)|_{(\eta_X)_L^\rightarrow(X)} = \Phi_L(\nu)|_{(\eta_X)_L^\rightarrow(X)}$. Thus $(\eta_X)_L^\rightarrow(\nu)$ is open w.r.t. the subspace topology of $(\eta_X)_L^\rightarrow(X)$ induced from $LPT(\tau_1 \vee \tau_2)$, that is, η_X is a pairwise L -open map.

- (2) As given in [4], we note that the mapping $\varepsilon_{Q_0}^{op} : Q_0 \rightarrow \mathcal{O}_L(LPT(Q_0))$ is a semi-quantale homomorphism and so the mappings $\varepsilon_Q^{op}|_{Q_i} : Q_0 \rightarrow \mathcal{O}_L(LPT(Q_0))$, for $i = 1, 2$. Thus we have that the mapping $\varepsilon_Q^{op} : Q \rightarrow \mathcal{O}_L(LPT(Q))$ is a coupled semi-quantale morphism. \square

Theorem 3.1. *The functor*

$$LPT : L\text{-BiQTop} \leftarrow \mathbf{CSQuant}^{op}$$

is a right adjoint of the functor

$$\mathcal{O}_L : L\text{-BiQTop} \rightarrow \mathbf{CSQuant}^{op}$$

with unit $\eta_X : X \rightarrow LPT^\rightarrow(\mathcal{O}_L(X, \tau_1, \tau_2))$ and counit $\varepsilon_Q : Q \leftarrow \mathcal{O}_L(LPT(Q))$.

Proof. It will be enough to show that for every $Q \in |\mathbf{CSQuant}|$ and an $L\text{-BiQTop}$ -morphism $(X, \tau_1, \tau_2) \xrightarrow{f} LPT(Q)$, there exists uniquely a $\mathbf{CSQuant}$ -morphism $Q \xrightarrow{f^*} \mathcal{O}_L(X, \tau_1, \tau_2)$ such that the left diagram of the following diagram in Figure 1 is commutative, where by τ_0 we mean the coarsest L -quasi-topology $\tau_1 \vee \tau_2$.

To prove the existence, let $f^* = \mathcal{O}_L(f) \circ \varepsilon_Q$. From the definitions of $\mathcal{O}_L(f)$ and ε_Q one can easily prove that $f^* : Q \rightarrow \mathcal{O}_L(X, \tau_1, \tau_2)$ is a $\mathbf{CSQuant}$ -morphism. For commutativity of the above-mentioned left diagram notice that for $x \in X$ and $a \in Q_0$, we have

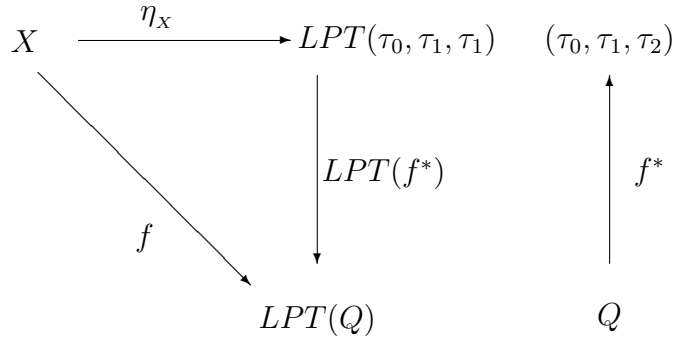


FIGURE 1.

$$\begin{aligned}
 pt(f^*) \circ \eta_X(x)(a) &= \eta_X(x)(f^*(a)) \\
 &= \eta_X(x)(\mathcal{O}_L(f) \circ \varepsilon_Q(a)) \\
 &= (\mathcal{O}_L(f)(\Phi_L(a)))(x) \\
 &= (f_L^{\leftarrow}(\Phi_L(a)))(x) \\
 &= (\Phi_L(a) \circ f)(x) \\
 &= f(x)(a).
 \end{aligned}$$

Uniqueness of the function f^* follows from the observation that given another **CSQuant**-morphism $Q \xrightarrow{g} \Omega(X, \tau_1, \tau_2)$ with the same property: for all $x \in X$, and for all $a \in Q_0$, we have

$$\begin{aligned}
 f(x)(a) &= \eta_X(x)(g(a)) \\
 &= \eta_X(x)(\mathcal{O}_L(g) \circ \varepsilon_L(a)) \\
 &= (g_L^{\leftarrow} \Phi_L(a))(x) \\
 &= (\Phi_L(a) \circ g)(x) \\
 &= g(a)(x).
 \end{aligned}$$

Hence for all $x \in X$ and for all $a \in Q_0$, we have $f^*(a) = g(a)$, i.e., $f^* = g$. □

Definition 3.4. An $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ is said to be pairwise $L\text{-QT}_0$ (i.e., fulfills the T_0 -axiom) if and only if for every pair $(x, y) \in X \times X$ with $x \neq y$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$.

By $L\text{-T}_0\text{BiQTop}$, we mean a full subcategory of $L\text{-BiQTop}$ consisting of those $L\text{-BiQTop}$ objects, which are pairwise $L\text{-QT}_0$.

As a consequence of Definition 2.6, we have the following easily established proposition.

Proposition 3.2. $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{T_0BiQTop}|$ if and only if $S(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ is $L\text{-}QT_0$.

Proposition 3.3. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is pairwise $L\text{-}QT_0$ if and only if the mapping

$$\eta_x : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2))$$

is pairwise L -embedding.

Proof. First, suppose that $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is pairwise $L\text{-}QT_0$, then for $x \neq y \in X$, there exists $\mu \in \tau_1 \vee \tau_2$ such that $\mu(x) \neq \mu(y)$. Therefore, $\eta_x(x)(\mu) = \mu(x) \neq \mu(y) = \eta_x(y)(\mu)$, that is, the mapping η_x is injective. Also, since the mapping η_x is pairwise L -continuous and L -open (see Lemma 3.2), then η_x is L -embedding. \square

Now, we will introduce the concept of sobriety of objects in the category $L\text{-}\mathbf{BiQTop}$.

Definition 3.5. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is L -sober if and only if the mapping

$$\eta_x : X \rightarrow LPT_{\rightarrow}(\mathcal{O}_L(X, \tau_1, \tau_2))$$

is bijective.

By $L\text{-}\mathbf{SobBiQTop}$, we mean the full subcategory of $L\text{-}\mathbf{BiQTop}$ of all sober objects.

Lemma 3.3. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is L -sober if and only if the mapping

$$\eta_x : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2))$$

is a pairwise homomorphism.

Proof. L -sobriety of an $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is equivalent to the fact of bijectivity of the mapping

$$\eta_x : (X, \tau_1, \tau_2) \rightarrow (LPT(\tau_1 \vee \tau_2), \Phi_L^\rightarrow(\tau_1), \Phi_L^\rightarrow(\tau_2)).$$

Also, the mapping η_x is pairwise L -continuous and L -open (see Lemma 3.2), and this is equivalent to the fact that η_x is pairwise L -homomorphism. \square

By the above and Definition 2.6, one have the following easily established result.

Proposition 3.4. An $(X, \tau_1, \tau_2) \in |L\text{-}\mathbf{BiQTop}|$ is L -sober if and only if $(X, \tau_1 \vee \tau_2)$ is L -qsober.

Definition 3.6. The coupled semi-quantales $Q = (Q_0, Q_1, Q_2)$ is spatial if and only if the total part Q_0 is spatial. Equivalently the map

$$\varepsilon_Q^{\text{op}} : Q_0 \rightarrow \mathcal{O}_L(LPT(Q_0))$$

is a semi-quantale isomorphism [4].

By $\mathbf{SpatCSQuant}$, we mean the full subcategory of the spatial coupled semi-quantales in $\mathbf{CSQuant}$.

Lemma 3.4. For all $Q = (Q_0, Q_1, Q_2) \in |\mathbf{CSQuant}|$, $Q = (Q_0, Q_1, Q_2)$ is spatial if and only if the mapping

$$\varepsilon_Q^{op} : (Q_0, Q_1, Q_2) \rightarrow \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism.

Proof. Let $Q = (Q_0, Q_1, Q_2)$ be a spatial coupled semi-quantale. Then, by the definition, the total part Q_0 is spatial, and this is equivalent to the fact that the map

$$\varepsilon_Q^{op} : Q_0 \rightarrow \mathcal{O}_L(LPT(Q_0))$$

is a semi-quantale isomorphism, and this implies that the map

$$\varepsilon_Q^{op} : (Q_0, Q_1, Q_2) \rightarrow \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism. □

Lemma 3.5. *For all $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ and for all $Q \in |\text{CSQuant}|$, then*

- (i) $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is spatial;
- (ii) $LPT(Q_0, Q_1, Q_2) = (LPT(Q_0), \Phi_L^\rightarrow(Q_1), \Phi_L^\rightarrow(Q_2))$ is L -sober.

Proof. As to (i), clearly, the map

$$\varepsilon_{\tau_1 \vee \tau_2}^{op} : (\tau_1 \vee \tau_2) \rightarrow \mathcal{O}_L(LPT(\tau_1 \vee \tau_2)) = \Phi_L^\rightarrow(\tau_1 \vee \tau_2)$$

is a semi-quantale isomorphism, which implies that $\tau_1 \vee \tau_2$ is a spatial semi-quantale and, therefore, the coupled semi-quantale $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is spatial.

As to (ii), by definition, it suffices to prove that the mapping

$$\eta_X : LPT(Q) \rightarrow LPT(\mathcal{O}_L(LPT(Q))) = LPT((\Phi_L^\rightarrow(Q_1) \vee \Phi_L^\rightarrow(Q_2)), \Phi_L^\rightarrow(Q_1), \Phi_L^\rightarrow(Q_2))$$

is bijective. Now, we have the following.

- (a) η_X is one-to-one. For all $p_1, p_2 \in LPT(Q_0)$ with $p_1 \neq p_2$, there exist some $a \in Q_0$ with $p_1(a) \neq p_2(a)$, and this implies that

$$\eta_X(p_1)(\Phi_L^\rightarrow(a)) = \Phi_L^\rightarrow(a)(p_1) = p_1(a) \neq p_2(a) = \eta_X(p_2)(\Phi_L^\rightarrow(a)).$$

Hence η_X is one-to-one.

- (b) η_X is onto. For all $q \in LPT(\Phi_L^\rightarrow(Q_1 \vee Q_2))$, let $p = q \circ \Phi_L^\rightarrow : Q_0 \rightarrow \Phi_L^\rightarrow(Q_0) \rightarrow L$, then $p \in LPT(Q_0)$ and $a \in Q_0$. We have $\eta_X(p)(\Phi_L^\rightarrow(a)) = \Phi_L^\rightarrow(a)(p) = p(a) = q(\Phi_L^\rightarrow(a))$. Hence $\eta_X(p) = q$, that is, η_X is onto. From (a) and (b), it follows that η_X is bijective, and this completes the proof. □

Proposition 3.5. *The following functors are valid:*

- (i) $\mathcal{O}_L : L\text{-BiQTop} \rightarrow \text{SpatCSQuant}^{op}$;
- (ii) $LPT : L\text{-SobBiQTop} \leftarrow \text{CSQuant}^{op}$.

The equivalence between the categories $L\text{-SobBiQTop}$ and SpatCSQuant is proven as follows.

Theorem 3.2. *For all $L \in |\text{SQuant}|$, $L\text{-SobBiQTop} \approx \text{SpatCSQuant}^{op}$.*

Proof. The categorical equivalence $L\text{-SobBiQTop} \approx \text{SpatCSQuant}^{op}$ follows directly from the adjunction $\mathcal{O}_L \dashv LPT$ and the fact that both the unit and counit are isomorphisms in the categories $L\text{-SobBiQTop}$ and SpatCSQuant^{op} , respectively. \square

4. REGULARITY AND PAIRWISE COMPACTNESS

Now, we will define the regularity and compactness for a certain $L\text{-BiQTop}$ and CSQuant objects.

Definition 4.1. Let $Q = (Q_0, Q_1, Q_2) \in |\text{CSQuant}|$ and $a, b \in Q_i$, $i = 1, 2$. An element a is said to be well inside of b (w.r.t. Q_i) and denoted by $a \preceq_i b$, if and only if exists $c \in Q_k$, $k \neq i$, such that $a \otimes c = \perp$ and $c \vee b = \top$.

Lemma 4.1. For any strong CSQuant -morphism $h : Q \rightarrow P$

$$a \preceq_i b \Rightarrow h(a) \preceq_i h(b).$$

Proof. Let $a, b \in Q_i$ with $a \preceq_i b$, then exists $c \in Q_k$, $k \neq i$, with $c \otimes a = \perp$, $c \vee b = \top$. Since $h : Q \rightarrow P$ is a strong semi-quantale homomorphism, then $h(c \otimes a) = h(c) \otimes h(a) = \perp$ and $h(c \vee b) = h(c) \vee h(b) = h(\top) = \top$. So exists $h(c) \in P_k$, $k \neq i$, such that $h(c) \otimes h(a) = \perp$ and $h(c) \vee h(b) = \top$ which means that $h(a) \preceq_i h(b)$. \square

Definition 4.2. An $Q = (Q_0, Q_1, Q_2) \in |\text{CSQuant}|$ is said to be regular if and only if both Q_1 and Q_2 are regular subsemi-quantales. Or equivalently

$$\text{for all } a \in Q_i, \text{ exists } D \subseteq \{b \in Q_i : b \preceq_i a\} \text{ such that } a = \bigvee D, i = 1, 2.$$

By RegCSQuant , we mean the full subcategory of CSQuant of regular objects.

A coupled semi-quantale map $h : Q \rightarrow P$ is said to be *surjective* if and only if $h|_{Q_i} : Q_i \rightarrow P_i$ is surjective for $i = 1, 2$.

Lemma 4.2. If $h : Q \rightarrow P$ is a surjective strong coupled semi-quantale homomorphism and $Q \in |\text{RegCSQuant}|$, then $P \in |\text{RegCSQuant}|$.

Proof. For $i = 1, 2$, let $x \in P_i$. Then $x = h(a)$ for some $a \in Q_i$. Regularity of Q means that exists $D \subseteq \{b \in Q_i : b \preceq_i a\}$, $a = \bigvee D$, $i = 1, 2$. Therefore there exists $E \subseteq \{h(b) \in P_i : b \preceq_i a\}$ such that $E = h(D)$. Since $a \preceq_i b$ implies $x = h(a) \preceq_i h(b) = y$. Hence $E \subseteq \{y \in P_i : y \preceq_i x\}$ and $x = \bigvee E$. Thus $P \in |\text{RegCSQuant}|$. \square

Definition 4.3. Let $L \in |\text{SQuant}|$. An (X, τ_1, τ_2) is regular if and only if $\mathcal{O}_L(X, \tau_1, \tau_2) \in |\text{RegCSQuant}|$.

By $L\text{-RegBiQTop}$, we mean the full subcategory of $L\text{-BiQTop}$ of regular objects.

Proposition 4.1. For $Q = (Q_0, Q_1, Q_2) \in |\text{DCSQuant}|$ and $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$.

- (1) An $Q = (Q_0, Q_1, Q_2)$ is regular if and only if

$$a = \bigvee \{b \in Q_i : b \preceq_i a\} \text{ for all } a \in Q_i.$$

(2) For $L \in |\mathbf{DSQuant}|$. An (X, τ_1, τ_2) is regular if and only if

$$\mu = \bigvee \{\nu \in \tau_i : \nu \preceq_i \mu\} \text{ for all } \mu \in \tau_i.$$

Proof. (1) Let $Q = (Q_0, Q_1, Q_2) \in |\mathbf{DCSQuant}|$. Distributivity and $b \preceq_i a$ imply $a \leq b$. Let $D \subseteq \{b \in Q_i : b \preceq_i a\}$ such that $a = \bigvee D$. Then,

$$\bigvee D \leq \bigvee \{b \in Q_i : b \preceq_i a\} \leq \bigvee \{b \in Q_i : b \leq a\} = a = \bigvee D.$$

This shows $a = \bigvee D = \bigvee \{b \in Q_i : b \preceq_i a\}$ and from this follows the claims.

(2) Follows from (1). □

As the preceding proposition offers the preserving of the regular axiom under the functor

$$LPT : L\text{-BiQTop} \leftarrow \mathbf{CSQuant}^{op}$$

and with the aid of Definition 4.3, we have the following easily established proposition.

Proposition 4.2. *The following functors holds:*

$$\begin{aligned} \mathcal{O}_L : L\text{-RegBiQTop} &\rightarrow \mathbf{RegCSQuant}^{op}, \\ LPT : L\text{-RegBiQTop} &\leftarrow \mathbf{RegCSQuant}^{op}. \end{aligned}$$

Definition 4.4. An $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$ is said to be pairwise compact if $E_s(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2)$ is compact.

Theorem 4.1. *Let $L \in |\mathbf{SQuant}|$, $Q \in |\mathbf{CSQuant}|$ and $(X, \tau_1, \tau_2) \in |L\text{-BiQTop}|$. Then*

- (1) (X, τ_1, τ_2) is pairwise compact if and only if $\mathcal{O}_L(X, \tau_1, \tau_2) = (\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is compact;
- (2) if Q is spatial, then Q is compact if and only if $LPT(Q_0, Q_1, Q_2)$ is pairwise compact.

Proof. As to (1), if (X, τ_1, τ_2) is a compact object of $L\text{-BiQTop}$, that is, for all $S \subseteq (\tau_1 \vee \tau_2)$, $\bigvee S = \underline{1}$, exists $F(\text{finite}) \subseteq S$, $\bigvee F = \underline{1}$ if and only if $(\tau_1 \vee \tau_2)$ is a compact semi-quantale if and only if $(\tau_1 \vee \tau_2, \tau_1, \tau_2)$ is a compact coupled semi-quantale.

As to (2), let $Q = (Q_0, Q_1, Q_2)$ be spatial, then the mapping

$$\varepsilon_Q^{op} : Q \rightarrow \mathcal{O}_L(LPT(Q_0, Q_1, Q_2))$$

is a coupled semi-quantale isomorphism, that is, $Q \approx \Phi_L^{\rightarrow}(Q)$.

Compactness of $(Q_0, Q_1, Q_2) \Leftrightarrow Q_0$ is compact

$$\Leftrightarrow LPT(Q_0) = (LPT(Q_0), \Phi_L^{\rightarrow}(Q_0)) \text{ is compact}$$

$$\Leftrightarrow (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1) \vee \Phi_L^{\rightarrow}(Q_2)) \text{ is compact.}$$

$$\Leftrightarrow LPT(Q) = (LPT(Q_0), \Phi_L^{\rightarrow}(Q_1), \Phi_L^{\rightarrow}(Q_2))$$

is pairwise compact and this completes the proof. □

5. CONCLUSION

The concept of coupled semi-quantales is introduced as a pointfree analogues of lattice-valued bitopological (or biquasi-topological spaces). An adjunction between the category of coupled semi-quantales and the category of lattice-valued biquasi-topological spaces is established. Through such adjunction topological and the lattice-theoretic concepts of regularity and compactness are defined and studied for both lattice-valued biquasi-topological spaces and coupled semi-quantales, respectively.

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