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# SOME MONOTONICITY PROPERTIES AND INEQUALITIES FOR THE $(p,k)$-GAMMA FUNCTION

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Abstract. In this paper, the authors present some complete monotonicity properties and some inequalities involving the $(p,k)$-analogue of the Gamma function. The established results provide the $(p,k)$-generalizations for some results known in the literature.

## 1. Introduction

In a recent paper [12], the authors introduced a $(p,k)$-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt \\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)},\end{aligned} \tag{1.1}$$

and satisfying the basic properties:

$$\begin{aligned}\Gamma_{p,k}(x+k) &= \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \\ \Gamma_{p,k}(ak) &= \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+, \\ \Gamma_{p,k}(k) &= 1.\end{aligned} \tag{1.2}$$

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The $(p,k)$-analogue of the Digamma function is defined for $x>0$ as

$$\begin{aligned}\psi_{p,k}(x) &= \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x} \\ &= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt.\end{aligned} \tag{1.3}$$

Also, the $(p,k)$-analogue of the Polygamma functions are defined as

$$\begin{aligned}\psi_{p,k}^{(m)}(x) &= \frac{d^m}{dx^m}\psi_{p,k}(x) = \sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} \\ &= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt,\end{aligned} \tag{1.4}$$

where $m\in\mathbb{N}$ and $\psi_{p,k}^{(0)}(x)\equiv\psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{tikzcd} \Gamma_{p,k}(x) \arrow[r, "p\to\infty"] \arrow[d, "k\to1"'] & \Gamma_k(x) \arrow[d, "k\to1"] \\ \Gamma_p(x) \arrow[r, "p\to\infty"'] & \Gamma(x), \end{tikzcd} \qquad \begin{tikzcd} \psi_{p,k}(x) \arrow[r, "p\to\infty"] \arrow[d, "k\to1"'] & \psi_k(x) \arrow[d, "k\to1"] \\ \psi_p(x) \arrow[r, "p\to\infty"'] & \psi(x). \end{tikzcd}$$

From the identity (1.2), the following relations are established:

$$\psi_{p,k}(x+k)-\psi_{p,k}(x) = \frac{1}{x}-\frac{1}{x+pk+k}, \tag{1.5}$$

$$\psi_{p,k}^{(m)}(x+k)-\psi_{p,k}^{(m)}(x) = \frac{(-1)^m m!}{x^{m+1}}-\frac{(-1)^m m!}{(x+pk+k)^{m+1}}, \quad m\in\mathbb{N}. \tag{1.6}$$

It follows easily from (1.3) and (1.4) that for $x>0$,

(i) $\psi_{p,k}(x)$ is increasing;
(ii) $\psi_{p,k}^{(m)}(x)$ is positive and decreasing if $m$ is odd;
(iii) $\psi_{p,k}^{(m)}(x)$ is negative and increasing if $m$ is even.

Next, we recall the following definitions and lemmas which will be used in the paper.

A function $f$ is said to be *completely monotonic* on an interval $I$, if $f$ has derivatives of all order and satisfies

$$(-1)^m f^{(m)}(x)\geq 0, \quad \text{for} \quad x\in I, m\in\mathbb{N}_0. \tag{1.7}$$

If the inequality (1.7) is strict, then $f$ is said to be *strictly completely monotonic* on $I$.

A positive function $f$ is said to be *logarithmically completely monotonic* on an interval $I$, if $f$ satisfies

$$(-1)^m[\ln f(x)]^{(m)}\geq 0, \quad \text{for} \quad x\in I, m\in\mathbb{N}_0. \tag{1.8}$$

If the inequality (1.8) is strict, then $f$ is said to be *strictly logarithmically completely monotonic* on $I$.

**Lemma 1.1** ([1]). *If $h$ is completely monotonic on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic on $(0,\infty)$.*

**Lemma 1.2** ([1]). *Let $a_i$ and $b_i$, $i=1,2,\dots,n$ be real numbers such that $0<a_1\leq a_2\leq\cdots\leq a_n$, $0<b_1\leq b_2\leq\cdots\leq b_n$ and $\sum_{i=1}^{\lambda}a_i\leq\sum_{i=1}^{\lambda}b_i$ for $\lambda\in\mathbb{N}$. If $f$ is a decreasing and convex function on $\mathbb{R}$, then*

$$\sum_{i=1}^{n}f(b_i)\leq\sum_{i=1}^{n}f(a_i).$$

**Lemma 1.3** ([4]). *Let $f''(x)$ be completely monotonic on $(0,\infty)$. Then for $0\leq s\leq 1$, the functions*

$$\mu(x)=\exp\left(-\left(f(x+1)-f(x+s)-(1-s)f'\left(x+\frac{1+s}{2}\right)\right)\right),$$
$$\eta(x)=\exp\left(f(x+1)-f(x+s)-\frac{(1-s)}{2}\left(f'(x+1)+f'(x+s)\right)\right),$$

*are logarithmically completely monotonic on $(0,\infty)$.*

In this paper, our goal is to establish some complete monotonicity properties and some inequalities involving the $(p,k)$-analogue of the Gamma function. For additional information on results of this nature, one could refer to [3], [8] and the related references therein.

## 2. Main Results

We now present our findings in this section.

**Theorem 2.1.** *Let $p\in\mathbb{N}$, $k>0$ and $m\in\mathbb{N}_0$. Then the function $\psi'_{p,k}(x)$ is strictly completely monotonic on $(0,\infty)$.*

*Proof.* It follows directly from (1.4) that

$$\begin{aligned}(-1)^m\left(\psi'_{p,k}(x)\right)^{(m)}&=(-1)^m\psi_{p,k}^{(m+1)}(x)\\&=(-1)^m\sum_{n=0}^{p}\frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}}\\&=(-1)^{2m+2}\sum_{n=0}^{p}\frac{(m+1)!}{(nk+x)^{m+2}}>0,\end{aligned}$$

which concludes the proof. □

*Remark* 2.1. It follows from Lemma 1.1 that $\exp(-\psi_{p,k}(x))$ is also completely monotonic.

**Theorem 2.2.** *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0,1)$. Then the function*
$$Q(x) = \psi_{p,k}(x+a) - \psi_{p,k}(x),$$
*is strictly completely monotonic on $(0,\infty)$. In particular, $Q$ is decreasing and convex.*

*Proof.* By direct computation, we obtain
$$\begin{aligned}(-1)^m (Q(x))^{(m)} &= (-1)^m \left[\psi_{p,k}^{(m)}(x+a) - \psi_{p,k}^{(m)}(x)\right] \\ &= (-1)^m \left[\sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x+a)^{m+1}} - \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x)^{m+1}}\right] \\ &= (-1)^{2m+1} m! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+a)^{m+1}} - \frac{1}{(nk+x)^{m+1}}\right] \\ &> 0,\end{aligned}$$
which establishes the result. In particular, $Q'(x) = \psi'_{p,k}(x+a) - \psi'_{p,k}(x) \leq 0$ since $\psi'_{p,k}(x)$ is decreasing. Hence $Q$ is decreasing. Furthermore, $Q''(x) = \psi''_{p,k}(x+a) - \psi''_{p,k}(x) \geq 0$ implying that $Q$ is convex. □

*Remark* 2.2. Theorem 2.2 generalizes the the previous result [10, Theorem 1].

In the following theorem, we prove a generalization of the results of Mortici [11].

**Theorem 2.3.** *Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \in (0,1)$. Then the function*
$$T(x) = \psi_{p,k}(x+\alpha) - \psi_{p,k}(x) - \frac{\alpha}{x},$$
*is strictly completely monotonic on $(0,\infty)$. Particularly, $T$ is decreasing and convex.*

*Proof.* Similarly, by direct computation, we obtain
$$\begin{aligned}&(-1)^m (T(x))^{(m)} \\ =&(-1)^m \left[\psi_{p,k}^{(m)}(x+\alpha) - \psi_{p,k}^{(m)}(x) - (-1)^{m+1}(m-1)!\frac{\alpha}{x^{m+1}}\right] \\ =&(-1)^m \left[\sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x+\alpha)^{m+1}} - \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk+x)^{m+1}} + \frac{(-1)^{m+2}\alpha(m-1)!}{x^{m+1}}\right] \\ =&(-1)^{2m+1} m! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+\alpha)^{m+1}} - \frac{1}{(nk+x)^{m+1}}\right] + (-1)^{2m+2}\frac{\alpha(m-1)!}{x^{m+1}} \\ >&0.\end{aligned}$$
Hence $T$ is strictly completely monotonic on $(0,\infty)$. In particular,
$$\begin{aligned}T'(x) &= \psi'_{p,k}(x+\alpha) - \psi'_{p,k}(x) + \frac{\alpha}{x^2} \\ &= -\frac{1}{x^2} + \frac{1}{(x+p\alpha+\alpha)^2} + \frac{\alpha}{x^2} \\ &= -\frac{1-\alpha}{x^2} + \frac{1}{(x+p\alpha+\alpha)^2} \leq 0,\end{aligned}$$

as a result of (1.6). Thus $T$ is decreasing. Next,

$$\begin{aligned}
T''(x) &= \psi''_{p,k}(x+\alpha) - \psi''_{p,k}(x) - \frac{2\alpha}{x^3}\\
&= \frac{2}{x^3} - \frac{2}{(x+p\alpha+\alpha)^3} - \frac{2\alpha}{x^3}\\
&= 2\left(\frac{1-\alpha}{x^3} - \frac{1}{(x+p\alpha+\alpha)^3}\right) \geq 0.
\end{aligned}$$

Hence $T$ is convex. □

*Remark* 2.3. By letting $p \to \infty$ and $k = 1$ in Theorem 2.3, we obtain the main result of [11].

**Theorem 2.4.** *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$, $a_i$ and $b_i$, $i = 1, 2, \dots, n$, be such that $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, $0 < b_1 \leq b_2 \leq \cdots \leq b_n$ and $\sum_{i=1}^{\lambda} a_i \leq \sum_{i=1}^{\lambda} b_i$ for $\lambda \in \mathbb{N}$. Then the function*

$$H(x) = \prod_{i=1}^{n} \frac{\Gamma_{p,k}(x+a_i)}{\Gamma_{p,k}(x+b_i)},$$

*is completely monotonic on $(0, \infty)$.*

*Proof.* Let $h$ be defined by $h(x) = \sum_{i=1}^{n} [\ln \Gamma_{p,k}(x+b_i) - \ln \Gamma_{p,k}(x+a_i)]$. Then for $m \in \mathbb{N}_0$, we have

$$\begin{aligned}
(-1)^m \left(h'(x)\right)^{(m)} =&(-1)^m \sum_{i=1}^{n} \left[\psi^{(m)}_{p,k}(x+b_i) - \psi^{(m)}_{p,k}(x+a_i)\right]\\
=&(-1)^m \sum_{i=1}^{n} \left[(-1)^m \sum_{s=0}^{p} \frac{m!}{(sk+x+b_i)^{m+1}}\right.\\
&\left. - (-1)^m \sum_{s=0}^{p} \frac{m!}{(sk+x+a_i)^{m+1}}\right]\\
=&(-1)^{2m+1} m! \sum_{i=1}^{n} \sum_{s=0}^{p} \left[\frac{1}{(sk+x+b_i)^{m+1}} - \frac{1}{(sk+x+a_i)^{m+1}}\right].
\end{aligned}$$

Since $\frac{1}{x^m}$ is decreasing and convex on $\mathbb{R}$ for $m \in \mathbb{N}_0$, then by Lemma 1.2 we obtain

$$\sum_{i=1}^{n} \left[\frac{1}{(sk+x+b_i)^{m+1}} - \frac{1}{(sk+x+a_i)^{m+1}}\right] \leq 0.$$

Thus, $(-1)^m \left(h'(x)\right)^{(m)} \geq 0$ for $m \in \mathbb{N}_0$. Hence $h'(x)$ is completely monotonic on $(0, \infty)$. Then by Lemma 1.1,

$$\exp(-h(x)) = \prod_{i=1}^{n} \frac{\Gamma_{p,k}(x+a_i)}{\Gamma_{p,k}(x+b_i)} = H(x),$$

is completely monotonic on $(0, \infty)$. □

*Remark* 2.4. By letting $p \to \infty$ in Theorem 2.4, we obtain the result of [6, Theorem 2.6].

*Remark* 2.5. By letting $k = 1$ in Theorem 2.4, we obtain the result of [7, Theorem 13].

*Remark* 2.6. By letting $p \to \infty$ and $k = 1$ in Theorem 2.4, we obtain the result of [1, Theorem 10].

**Theorem 2.5.** *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0,1)$. Then the inequality*

$$0 < \psi_{p,k}(x+a) - \psi_{p,k}(x) \leq \frac{a(p+1)}{1+a(p+1)},$$

*is satisfied for $x \in [1, \infty)$.*

*Proof.* Let $Q$ be defined as in Theorem 2.2. Since $Q$ is decreasing, then for $x \in [1, \infty)$, we obtain

$$0 = \lim_{x\to\infty} Q(x) < Q(x) \leq Q(1) = \psi_{p,k}(a+1) - \psi_{p,k}(1),$$

which by (1.5) yields the desired result. □

**Theorem 2.6.** *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) - \frac{1}{x} + \frac{1}{x+pk+k} \leq \psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right), \tag{2.1}$$

*holds for $x > 0$.*

*Proof.* It follows from (1.2) that $\ln\Gamma_{p,k}(x+k) - \ln\Gamma_{p,k}(x) = \ln\left(\frac{pkx}{x+pk+k}\right)$. Let $g(x) = \ln\Gamma_{p,k}(x)$. Then by the classical mean value theorem, the exits a $\lambda \in (x, x+k)$ such that

$$\frac{g(x+k)-g(x)}{k} = \frac{\ln\Gamma_{p,k}(x+k) - \ln\Gamma_{p,k}(x)}{k} = \psi_{p,k}(\lambda).$$

Since $\psi_{p,k}(x)$ is increasing, then for $\lambda \in (x, x+k)$, we have

$$\psi_{p,k}(x) \leq \psi_{p,k}(\lambda) \leq \psi_{p,k}(x+k),$$

which implies

$$\psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) \leq \psi_{p,k}(x+k).$$

Then by (1.5) we obtain

$$\psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) \leq \psi_{p,k}(x) + \frac{1}{x} - \frac{1}{x+pk+k},$$

yielding the result (2.1). □

*Remark* 2.7. Let $p \to \infty$ and $k = 1$ in (2.1). Then we obtain the result

$$\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x, \tag{2.2}$$

for the classical digamma function, $\psi(x)$.

**Theorem 2.7.** *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x) \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) + \frac{1}{x^2} - \frac{1}{(x+pk+k)^2}, \tag{2.3}$$

*holds for $x > 0$.*

*Proof.* Consider the function $\psi_{p,k}(x)$ on the interval $(x, x+k)$. By the mean value theorem, the exits a $c \in (x, x+k)$ such that

$$\frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) = \frac{\psi_{p,k}(x+k) - \psi_{p,k}(x)}{k} = \psi'_{p,k}(c).$$

Since $\psi'_{p,k}(x)$ is decreasing, then for $c \in (x, x+k)$, we have

$$\psi'_{p,k}(x+k) \leq \psi'_{p,k}(c) \leq \psi'_{p,k}(x),$$

which implies

$$\psi'_{p,k}(x+k) \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x).$$

Then by (1.6), we obtain

$$\psi'_{p,k}(x) - \frac{1}{x^2} + \frac{1}{(x+pk+k)^2} \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x),$$

which results to (2.3). □

*Remark* 2.8. Let $p \to \infty$ and $k = 1$ in (2.3). Then we obtain the result

$$\frac{1}{x} \leq \psi'(x) \leq \frac{1}{x} + \frac{1}{x^2}, \tag{2.4}$$

for the trigamma function, $\psi'(x)$.

*Remark* 2.9. The right side of (2.2) and the left side of (2.4) are however weaker than the results obtained in [5, Theorem 3].

**Theorem 2.8.** *Let $p \in \mathbb{N}$, $k > 0$ and $0 \leq s \leq 1$. Then the functions*

$$u(x) = \frac{\Gamma_{p,k}(x+s)}{\Gamma_{p,k}(x+1)} \exp\left((1-s)\psi_{p,k}\left(x + \frac{1-s}{2}\right)\right),$$
$$w(x) = \frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \exp\left(-\frac{1-s}{2}\left(\psi_{p,k}(x+1) + \psi_{p,k}(x+s)\right)\right),$$

*are logarithmically completely monotonic on $(0, \infty)$.*

*Proof.* Let $f(x) = \ln \Gamma_{p,k}(x)$ and recall that $f''(x) = \psi'_{p,k}(x)$ is completely monotonic on $(0, \infty)$ (See Theorem 2.1). Then the results follow directly from Lemma 1.3. □

**Theorem 2.9.** *Let $p \in \mathbb{N}$, $k > 0$ and $0 \leq s \leq 1$. Then the inequality*

$$\begin{aligned} \exp\left(\frac{1-s}{2}\left(\psi_{p,k}(x+s)+\psi_{p,k}(x+1)\right)\right) &\leq \frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \\ &\leq \exp\left((1-s)\psi_{p,k}\left(x+\frac{1+s}{2}\right)\right), \end{aligned} \tag{2.5}$$

*is satisfied for $x > 0$.*

*Proof.* We employ the Hermite-Hadamard's inequality which states that: if $f(x)$ is convex on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let $f(x) = -\psi_{p,k}(x)$, $a = x + s$ and $b = x + 1$. Then we have

$$-\psi_{p,k}\left(x+\frac{1+s}{2}\right) \leq -\frac{1}{1-s}\int_{x+s}^{x+1}\psi_{p,k}(t)\,dt \leq -\frac{\psi_{p,k}(x+s)+\psi_{p,k}(x+1)}{2},$$

which implies

$$\frac{\psi_{p,k}(x+s)+\psi_{p,k}(x+1)}{2} \leq \frac{1}{1-s}\ln\frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \leq \psi_{p,k}\left(x+\frac{1+s}{2}\right).$$

Then by taking exponents, we obtain the desired result. □

*Remark* 2.10. By letting $p \to \infty$ in Theorems 2.8 and 2.9, we respectively obtain the results of Theorems 2.1 and 2.3 of [6].

*Remark* 2.11. By letting $k = 1$ in Theorems 2.8 and 2.9, we respectively obtain the results of Theorems 2.3 and 2.4 of [9].

*Remark* 2.12. The $q$-analogue of these results can also be found in [4].

The following theorem is a $(p, k)$-generalization of Lemma 2.1 of [2]. We derive our results by using similar techniques.

**Theorem 2.10.** *Let $p \in \mathbb{N}$ and $k > 0$. Then the function*

$$f(x) = \frac{1}{[\Gamma_{p,k}(x+k)]^{\frac{1}{x}}},$$

*is logarithmically completely monotonic on $(0, \infty)$.*

*Proof.* We employ the Leibniz's rule for $n$-times differentiable functions $u$ and $v$, which is given by

$$[u(x)v(x)]^{(n)} = \sum_{s=0}^{n}\binom{n}{s}u^{(s)}(x)v^{(n-s)}(x).$$

That is,

$$
\begin{aligned}
(\ln f(x))^{(n)} &= \left[\left(\frac{1}{x}\right)(-\ln\Gamma_{p,k}(x+k))\right]^{(n)} \\
&= \sum_{s=0}^{n}\binom{n}{s}\left(\frac{1}{x}\right)^{(s)}(-\ln\Gamma_{p,k}(x+k))^{(n-s)} \\
&= -\frac{1}{x^{n+1}}\sum_{s=0}^{n}\binom{n}{s}(-1)^s s! x^{n-s}\psi_{p,k}^{(n-s-1)}(x+k) \\
&\triangleq -\frac{1}{x^{n+1}}\phi(x).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\phi'(x) =& \sum_{s=0}^{n}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\
&+ \sum_{s=0}^{n}\binom{n}{s}(-1)^s s! x^{n-s}\psi_{p,k}^{(n-s)}(x+k) \\
=& \sum_{s=0}^{n-1}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\
&+ x^n\psi_{p,k}^{(n)}(x+k) + \sum_{s=1}^{n}\binom{n}{s}(-1)^s s! x^{n-s}\psi_{p,k}^{(n-s)}(x+k) \\
=& \sum_{s=0}^{n-1}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\
&+ x^n\psi_{p,k}^{(n)}(x+k) + \sum_{s=0}^{n-1}\binom{n}{s+1}(-1)^{s+1}(s+1)! x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\
=& \sum_{s=0}^{n-1}\left[\binom{n}{s}(n-s) - \binom{n}{s+1}(s+1)\right](-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\
&+ x^n\psi_{p,k}^{(n)}(x+k) \\
=& x^n\psi_{p,k}^{(n)}(x+k) \\
=& x^n(-1)^{n+1}\sum_{s=0}^{p}\frac{n!}{(k(s+1)+x)^{n+1}}.
\end{aligned}
$$

Suppose that $n$ is odd. Then,

$$\phi'(x) > 0 \implies \phi(x) > \phi(0) = 0 \implies (\ln f(x))^{(n)} < 0.$$

Thus $(-1)^n (\ln f(x))^{(n)} > 0$. Also, suppose that $n$ is even. Then

$$\phi'(x) < 0 \implies \phi(x) < \phi(0) = 0 \implies (\ln f(x))^{(n)} > 0,$$

yielding $(-1)^n (\ln f(x))^{(n)} > 0$. Therefore, for every $n \in \mathbb{N}$, we have

$$(-1)^n (\ln f(x))^{(n)} > 0,$$

which concludes the proof. □

*Remark* 2.13. By letting $p \to \infty$ in Theorem 2.10, we recover the results of Theorem 2.8 of [6].

*Remark* 2.14. By letting $k = 1$ in Theorem 2.10, we recover the results of Theorem 2.1 of [9].

## 3. Conclusion

In the study, the authors established some complete monotonicity properties and some inequalities involving the $(p, k)$-analogue of the Gamma function which was recently introduced in [12]. The established results provide the $(p, k)$-generalizations for some results known in the literature.

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