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CONCIRCULAR CURVATURE TENSOR ON A P-SASAKIAN MANIFOLD ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study a Para-Sasakian manifold admitting a type of quarter-symmetric metric connection whose concircular curvature tensor satisfies certain curvature conditions.

1. Introduction

In 1977, Adati and Matsumoto [1] defined Para-Sasakian and Special Para-Sasakian manifolds which are special classes of an almost paracontact manifold introduced by Sato [14]. Para-Sasakian manifolds have been studied by De and Pathak [6], Matsumoto, Ianus and Mihai [13], De, Özgür, Arslan, Murathan and Yildiz [7], Desmukh and Ahmed [8], Yildiz, Turan and Acet [17], Barman ([2], [3]) and many others.

In 1924, Friedmann and Schouten [9] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\tilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X,\rho)$, for all vector fields X on $\chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1975, Golab [10] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A liner connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is called a quarter-symmetric connection [10] if its torsion tensor T satisfies $T(X,Y) = u(Y)\phi X - u(X)\phi Y$, where ϕ is a $(1,1)$ tensor field.

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In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [9]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) = 0,$$

for all X, Y, Z on $\chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection .

In a recent paper Mandal and De [12] studied a type of quarter-symmetric metric connection $\bar{\nabla}$ and it is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{1.1}$$

where ∇ is the Levi-Civita connection and ξ is the corresponding vector field metrically equivalent to the 1− form η defined by $\eta(X) = g(X, \xi)$.

Let $\bar{R}$ and R be the curvature tensors with respect to the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ respectively. Then we have from [12],

$$\begin{aligned} \bar{R}(X, Y)Z =& R(X, Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X \\ &+ \eta(Z)[\eta(X)Y - \eta(Y)X] \\ &- [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi, \end{aligned} \tag{1.2}$$

$$\begin{aligned} \bar{S}(Y, U) =& S(Y, U) + 2g(Y, U) - (n + 1)\eta(Y)\eta(U) \\ &- 3\,\mathrm{trace}\,\phi \quad g(\phi Y, U), \end{aligned} \tag{1.3}$$

$$\bar{S}(Y, \xi) = -2(n - 1)\eta(Y) \tag{1.4}$$

and

$$\bar{R}(\xi, Y)U = 2[\eta(U)Y - g(U, Y)\xi], \tag{1.5}$$

where $\bar{S}$ and S be the Ricci tensors with respect to the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ respectively.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([11], [15]). A concircular transformation is always a conformal transformation [11]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [5]). An interesing invariant of a concircular transformation is the concircular curvature tensor $\overline{\mathbb{W}}$. It is defined by ([15], [16])

$$\overline{\mathbb{W}}(X, Y)Z = \bar{R}(X, Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{1.6}$$

From (1.6), it follows that

$$\begin{aligned}\widetilde{\overline{\mathbb{W}}}(X,Y,Z,U) =&\tilde{\bar{R}}(X,Y,Z,U) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U)\\ &- g(X,Z)g(Y,U)]\end{aligned} \tag{1.7}$$

and $\widetilde{\overline{\mathbb{W}}}(X,Y,Z,U) = g(\overline{\mathbb{W}}(X,Y)Z,U)$ $\tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$, where X, Y, Z, U on M and $\overline{\mathbb{W}}$ is the concircular curvature tensor and $\bar{r}$ is the scalar curvature with respect to the quarter-symmetric metric connection respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Putting $X = \xi$ in (1.6) and using (1.5), we obtain

$$\overline{\mathbb{W}}(\xi,Y)Z = \left[2 + \frac{\bar{r}}{n(n-1)}\right][\eta(Z)Y - g(Y,Z)\xi]. \tag{1.8}$$

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y).R = 0$, where $R(X,Y)$ acts on R as a derivation. A Riemannian manifold M is said to be Ricci-semisymmetric manifold if the relation $\bar{R}(X,Y).\bar{S} = 0$ holds, where $\bar{R}(X,Y)$ is the curvature operator.

In this paper we study a type of quarter-symmetric metric connection due to Mandal and De [12] on P-Sasakian manifolds. The paper is organized as follows: After introduction Section 2 is equipped with some prerequisites of a P-Sasakian manifold. Section 3 is devoted to study ξ-concircularly flat in a P-Sasakian manifold with respect to the quarter-symmetric metric connection. ϕ-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric metric connection have been studied in Section 4. Next section we investigate Ricci-semisymmetric manifolds with respect to the quarter-symmetric metric connection of a P-Sasakian manifold. Finally, we construct an example of a 5-dimensional P-Sasakian manifolds admitting the quarter-symmetric metric connections which verify the results of Section 3 and Section 4.

2. P-Sasakian Manifolds

An n-dimensional differentiable manifold M is said to be an almost para-contact structure (ϕ,ξ,η,g),if there exists ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M which satisfy the conditions

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.4}$$

for any vector fields X, Y on M.

If moreover, (ϕ, ξ, η, g) satisfy the conditions

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

In a P-Sasakian manifold the following relations hold ([1], [14]):

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3. ξ-concircularly Flat P-Sasakian Manifolds with Respect to the Quarter-symmetric Metric Connection

Definition 3.1. A P-Sasakian manifold is said to be ξ-concircularly flat [4] with respect to the quarter-symmetric metric connection if $\overline{\mathbb{W}}(X, Y)\xi = 0$, where $X, Y \in \chi(M)$.

Theorem 3.1. *A P-Sasakian manifold admitting a quarter-symmetric metric connection is ξ-concircularly flat if and only if the scalar curvature $\bar{r}$ with respect to the quarter-symmetric metric connection is equal to $-2n(n-1)$.*

Proof. Combining (1.2) and (1.6), it follows that

$$\begin{aligned}\overline{\mathbb{W}}(X, Y,)Z =& R(X, Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X \\ &+ \eta(Z)[\eta(X)Y - \eta(Y)X] - [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi \\ &- \frac{\bar{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) and using (2.1), we have

$$\begin{aligned}\overline{\mathbb{W}}(X, Y,)\xi =& R(X, Y)\xi + [\eta(X)Y - \eta(Y)X] \\ &+ \frac{\bar{r}}{n(n-1)}[\eta(X)Y - \eta(Y)X].\end{aligned} \tag{3.2}$$

By making use of (2.10) and (3.2), we get

$$\overline{\mathbb{W}}(X,Y,)\xi = \left[2 + \frac{\bar{r}}{n(n-1)}\right] R(X,Y)\xi. \tag{3.3}$$

If $\overline{\mathbb{W}}(X,Y,)\xi = 0$, then $\bar{r} = -2n(n-1)$ or $R(X,Y)\xi = \eta(Y)X - \eta(X)Y = 0$, implies that $\eta(X) = 0$ which is not admissible.

Conversly, if $\bar{r} = -2n(n-1)$, then from (3.3), it follows that $\overline{\mathbb{W}}(X,Y,)\xi = 0$. This completes the proof. □

Theorem 3.2. *If P-Sasakian manifolds satisfying $\bar{R}(\xi,Y)\cdot\overline{\mathbb{W}} = 0$ with respect to a quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection and the scalar curvature $\bar{r}$ with respect to the quarter-symmetric metric connection is a negative constant.*

Proof. Now we can state the following lemma.

Lemma 3.1. [12] *If a P-Sasakian manifold is semisymmetric with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection and the scalar curvature $\bar{r}$ with respect to the quarter-symmetric metric connection is a negative constant.*

From the definition of concircular curvature tensor, it follows that

$$\bar{R}(X,Y)\cdot\overline{\mathbb{W}} = \bar{R}(X,Y)\cdot\bar{R}.$$

Thus, using Lemma 3.1 we obtain Theorem 3.2. □

4. ϕ-concircularly Flat in P-Sasakian Manifolds with Respect to the Quarter-symmetric Metric Connection

Theorem 4.1. *If a P-Sasakian manifold admitting a quarter-symmetric metric connection is ϕ-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an η-Einstein manifold.*

Definition 4.1. A P-Sasakian manifold is said to be ϕ-concircularly flat [4] with respect to the quarter-symmetric metric connection if $\widetilde{\overline{\mathbb{W}}}(\phi X,\phi Y,\phi Z,\phi U) = 0$, where $X,Y,Z,U \in \chi(M)$.

Definition 4.2. A P-Sasakian manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on the manifold.

Proof. In view of (1.2) and (1.7) yields

$$\begin{aligned}\widetilde{\widetilde{\mathbb{W}}}(X,Y,Z,U) =&\widetilde{R}(X,Y,Z,U) + 3g(\phi X,Z)g(\phi Y,U) - 3g(\phi Y,Z)g(\phi X,U)\\ &+ \eta(Z)[\eta(X)g(Y,U) - \eta(Y)g(X,U)]\\ &- [\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\eta(U)\\ &- \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],\end{aligned} \tag{4.1}$$

where $\widetilde{R}(X,Y,Z,U) = g(R(X,Y)Z,U)$.

Now putting $X = \phi X, Y = \phi Y, Z = \phi Z, U = \phi U$ in (4.1) and using (2.1) and (2.2), we derive that

$$\begin{aligned}\widetilde{\widetilde{\mathbb{W}}}(\phi X,\phi Y,\phi Z,\phi U) =&\widetilde{R}(\phi X,\phi Y,\phi Z,\phi U) + 3g(X,\phi Z)g(Y,\phi U)\\ &- 3g(Y,\phi Z)g(\phi X,\phi U)\\ &- \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].\end{aligned} \tag{4.2}$$

Let $\{e_1,\ldots,e_{n-1},\xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1,\ldots,\phi e_{n-1},\xi\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (4.2) and summing over $i = 1$ to $n-1$, we obtain

$$\begin{aligned}\widetilde{\widetilde{\mathbb{W}}}(\phi e_i,\phi Y,\phi Z,\phi e_i) =&S(\phi Y,\phi Z) + 3g(Y,Z) - 3\alpha g(Y,\phi Z)\\ &- \frac{(n-2)\bar{r}}{n(n-1)}g(\phi Y,\phi Z),\end{aligned} \tag{4.3}$$

where $\alpha = g(\phi e_i, e_i)$.

Using (2.3) and (2.12) in (4.3), it follows that

$$\begin{aligned}\widetilde{\widetilde{\mathbb{W}}}(\phi e_i,\phi Y,\phi Z,\phi e_i) =&S(Y,Z) + (n-1)\eta(Y)\eta(Z) + 3g(Y,Z) - 3\alpha g(Y,\phi Z)\\ &- \frac{(n-2)\bar{r}}{n(n-1)}[g(Y,Z) - \eta(Y)\eta(Z)].\end{aligned} \tag{4.4}$$

By virtue of (1.3) and (4.4) yields

$$\begin{aligned}\widetilde{\widetilde{\mathbb{W}}}(\phi e_i,\phi Y,\phi Z,\phi e_i) =&\bar{S}(Y,Z) + \left[1 - \frac{(n-2)\bar{r}}{n(n-1)}\right]g(Y,Z)\\ &+ \left[2n + \frac{(n-2)\bar{r}}{n(n-1)}\right]\eta(Y)\eta(Z).\end{aligned} \tag{4.5}$$

If $\widetilde{\widetilde{\mathbb{W}}}(\phi e_i,\phi Y,\phi Z,\phi e_i) = 0$, then

$$\bar{S}(Y,Z) = -\left[1 - \frac{(n-2)\bar{r}}{n(n-1)}\right]g(Y,Z) - \left[2n + \frac{(n-2)\bar{r}}{n(n-1)}\right]\eta(Y)\eta(Z),$$

where $a = -\left[1 - \frac{(n-2)\bar{r}}{n(n-1)}\right]$ and $b = 2n + \frac{(n-2)\bar{r}}{n(n-1)}$.

From which it follows that the manifold is an η-Einstein manifold with respect to the quarter-symmetric metric connection. Hence the proof of Theorem 4.1 is completed. □

5. P-Sasakian Manifolds Satisfying $\overline{\mathbb{W}} \cdot \bar{S} = 0$ with Respect to a Quarter-symmetric Metric Connection

Theorem 5.1. *If P-Sasakian manifolds satisfying $\overline{\mathbb{W}} \cdot \bar{S} = 0$ with respect to a quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to a quarter-symmetric metric connection.*

Proof. We consider P-Sasakian manifolds with respect to a quarter-symmetric metric connection $\bar{\nabla}$ satisfying the curvature condition $\overline{\mathbb{W}} \cdot \bar{S} = 0$. Then

$$(\overline{\mathbb{W}}(X,Y) \cdot \bar{S})(U,V) = 0.$$

So,

$$\bar{S}(\overline{\mathbb{W}}(X,Y)U,V) + \bar{S}(U,\overline{\mathbb{W}}(X,Y)V) = 0. \tag{5.1}$$

Definition 5.1. A P-Sasakian manifold ($n > 2$) is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = \tau g(X,Y),$$

where τ is a constant on the manifold.

Putting $X = \xi$ in (5.1) and using (1.8), we get

$$\begin{aligned}&[2 + \frac{\bar{r}}{n(n-1)}][\eta(U)\bar{S}(Y,V) + 2(n-1)\eta(V)g(U,Y) + \eta(V)\bar{S}(Y,U)\\ &+2(n-1)\eta(U)g(V,Y)] = 0.\end{aligned} \tag{5.2}$$

Again putting $U = \xi$ in (5.2), implies that

$$\bar{S}(Y,V) = -2(n-1)g(Y,V).$$

Therefore, $S(Y,Z) = \tau g(Y,Z)$, where $\tau = -2(n-1)$.

This means that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection. This completes the proof. □

6. Example

In this section we construct an example on P-Sasakian manifold with respect to the quarter-symmetric metric connections $\bar{\nabla}$ [12] which verify the results of Section 3 and Section 4.

We consider the 5-dimensional manifold $\{(x,y,z,u,v) \in R^5\}$, where (x,y,z,u,v) are the standard coordinates in R^5. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = e^{-x}\frac{\partial}{\partial y}, \quad e_3 = e^{-x}\frac{\partial}{\partial z}, \quad e_4 = e^{-x}\frac{\partial}{\partial u}, \quad e_5 = e^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \end{cases}$$

where $i, j = 1, 2, 3, 4, 5$.

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_1),$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = 0, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_3, \quad \phi(e_4) = e_4, \quad \phi(e_5) = e_5.$$

Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1, \quad \phi^2 Z = Z - \eta(Z)e_1$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any vector fields Z and U on M. Thus for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M. Then we have

$$\begin{aligned} [e_1, e_2] =& - e_2,\ [e_1, e_3] = -e_3,\ [e_1, e_4] = -e_4,\ [e_1, e_5] = -e_5, \\ [e_2, e_3] =& [e_2, e_4] = 0,\ [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0. \end{aligned}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula, we get the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0, \\ &\nabla_{e_2} e_1 = e_2,\ \nabla_{e_2} e_2 = -e_1,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0, \\ &\nabla_{e_3} e_1 = e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_1,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0, \\ &\nabla_{e_4} e_1 = e_4,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_1,\ \nabla_{e_4} e_5 = 0, \\ &\nabla_{e_5} e_1 = e_5,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = -e_1. \end{aligned}$$

In view of the above relations, we see that $\nabla_X \xi = \phi X$, $(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi$, for all $e_5 = \xi$.

Therefore the manifold is a P-Sasakian manifold with the structure (ϕ, ξ, η, g) [12]. Using (1.1) in above equations, we obtain

$$\begin{aligned}
&\bar{\nabla}_{e_1}e_1 = 0,\ \bar{\nabla}_{e_1}e_2 = 0,\ \bar{\nabla}_{e_1}e_3 = 0,\ \bar{\nabla}_{e_1}e_4 = 0,\ \bar{\nabla}_{e_1}e_5 = 0,\\
&\bar{\nabla}_{e_2}e_1 = 2e_2,\ \bar{\nabla}_{e_2}e_2 = -2e_1,\ \bar{\nabla}_{e_2}e_3 = 0,\ \bar{\nabla}_{e_2}e_4 = 0,\ \bar{\nabla}_{e_2}e_5 = 0,\\
&\bar{\nabla}_{e_3}e_1 = 2e_3,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_3}e_3 = -2e_1,\ \bar{\nabla}_{e_3}e_4 = 0,\ \bar{\nabla}_{e_3}e_5 = 0,\\
&\bar{\nabla}_{e_4}e_1 = 2e_4,\ \bar{\nabla}_{e_4}e_2 = 0,\ \bar{\nabla}_{e_4}e_3 = 0,\ \bar{\nabla}_{e_4}e_4 = -2e_1,\ \bar{\nabla}_{e_4}e_5 = 0,\\
&\bar{\nabla}_{e_5}e_1 = 2e_5,\ \bar{\nabla}_{e_5}e_2 = 0,\ \bar{\nabla}_{e_5}e_3 = 0,\ \bar{\nabla}_{e_5}e_4 = 0,\ \bar{\nabla}_{e_5}e_5 = -2e_1.
\end{aligned}$$

Now, we can easily obtain the non-zero components of the curvature tensors [12] as follows:

$$\begin{aligned}
&R(e_1,e_2)e_1 = e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_1 = e_3,\ R(e_1,e_3)e_3 = -e_1,\\
&R(e_1,e_4)e_1 = e_4,\ R(e_1,e_4)e_4 = -e_1,\ R(e_1,e_5)e_1 = e_5,\ R(e_1,e_5)e_5 = -e_1,\\
&R(e_2,e_3)e_2 = e_3,\ R(e_2,e_3)e_3 = -e_2,\ R(e_2,e_4)e_2 = e_4,\ R(e_2,e_4)e_4 = -e_2,\\
&R(e_2,e_5)e_2 = e_5,\ R(e_2,e_5)e_5 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_3,e_4)e_4 = -e_3,\\
&R(e_3,e_5)e_3 = e_5,\ R(e_3,e_5)e_5 = -e_3,\ R(e_4,e_5)e_4 = e_5,\ R(e_4,e_5)e_5 = -e_4.
\end{aligned}$$

and

$$\begin{aligned}
&\bar{R}(e_1,e_2)e_1 = 2e_2,\ \bar{R}(e_1,e_2)e_2 = -2e_1,\ \bar{R}(e_1,e_3)e_1 = 2e_3,\ \bar{R}(e_1,e_3)e_3 = -2e_1,\\
&\bar{R}(e_1,e_4)e_1 = 2e_4,\ \bar{R}(e_1,e_4)e_4 = -2e_1,\ \bar{R}(e_1,e_5)e_1 = 2e_5,\ \bar{R}(e_1,e_5)e_5 = -2e_1,\\
&\bar{R}(e_2,e_3)e_2 = 2e_3,\ \bar{R}(e_2,e_3)e_3 = -2e_2,\ \bar{R}(e_2,e_4)e_2 = 2e_4,\ \bar{R}(e_2,e_4)e_4 = -2e_2,\\
&\bar{R}(e_2,e_5)e_2 = 2e_5,\ \bar{R}(e_2,e_5)e_5 = -2e_2,\ \bar{R}(e_3,e_4)e_3 = 2e_4,\ \bar{R}(e_3,e_4)e_4 = -2e_3,\\
&\bar{R}(e_3,e_5)e_3 = 2e_5,\ \bar{R}(e_3,e_5)e_5 = -2e_3,\ \bar{R}(e_4,e_5)e_4 = 2e_5,\ \bar{R}(e_4,e_5)e_5 = -2e_4.
\end{aligned}$$

With the help of the above curvature tensors with respect to the quarter-symmetric metric connection we find the Ricci tensors [12] as follows:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_4,e_4) = \bar{S}(e_5,e_5) = -8.$$

Also it follows that the scalar curvature tensor [12] with respect to the quarter-symmetric metric connection is $\bar{r} = -40$.

Let X, Y, Z and U be any four vector fields given by

$$\begin{aligned}
X =& a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5, \quad Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5\\
Z =& c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 \quad W = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5,
\end{aligned}$$

where a_i, b_i, c_i, d_i, for all $i = 1, 2, 3, 4, 5$ are all non-zero real numbers.

Using the above curvature tensors and the scalar curvature tensors of the quarter-symmetric metric connection, we obtain

$$\begin{aligned}
\overline{\mathbb{W}}(X,Y)\xi =& [(2e_2 - 2e_2)(a_1b_2) + (2e_5 - 2e_5)(a_1b_5)\\
&+ (2e_4 - 2e_4)(a_1b_4) + (2e_3 - 2e_3)(a_1b_3)] = 0,
\end{aligned}$$

which verifies the result of Section 3.

Now we see that the ϕ-concircularly flat with respect to the quarter-symmetric metric connections from the above relations as follow:

$$\begin{aligned}\overline{\mathbb{W}}(\phi X, \phi Y, \phi Z, \phi U) =& 2a_2b_3(c_2d_3 - c_3d_2) + 2a_2b_5(c_2d_5 - c_5d_2) + 2a_3b_4(c_3d_4 - c_4d_3) \\ &+ 2a_4b_5(c_4d_5 - c_5d_4) + 2a_2b_4(c_2d_4 - c_4d_2) \\ &+ 2a_3b_5(c_3d_5 - c_5d_3) = 0.\end{aligned}$$

Hence P-Sasakian manifolds will be ϕ-concircularly flat with respect to the quarter-symmetric metric connections if $\frac{c_2}{d_2} = \frac{c_3}{d_3} = \frac{c_4}{d_4} = \frac{c_5}{d_5}$.

The above arguments tell us that the 5-dimensional P-Sasakian manifolds with respect to the quarter-symmetric metric connections under consideration agrees with the Section 4.

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