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DYNAMICAL SYSTEMS ON HILBERT MODULES OVER LOCALLY C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a locally C^*-algebra and $S(\mathcal{A})$ be the family of continuous C^*-seminorms and let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module. We prove that every dynamical system of unitary operators on $\mathcal{E}$ defines a dynamical system of automorphisms on the compact operators on $\mathcal{E}$ and show that under certain conditions, the converse is true. We define a generalized derivation on $\mathcal{E}$ and prove that if $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module and $\delta : \mathcal{E} \to \mathcal{E}$ is a bounded generalized derivation, then $\delta_p : \mathcal{E}_p \to \mathcal{E}_p$ is a generalized derivation on the Hilbert module $\mathcal{E}_p$ over the C^*-algebra $\mathcal{A}_p$ for each $p \in S(\mathcal{A})$.

1. Introduction

Locally C^*-algebras are generalizations of C^*-algebras. Instead of having a single C^*-norm, we have a given directed family of C^*-seminorms, which gives a topology. Recall that a C^*-seminorm on a topological $*$-algebra $\mathcal{A}$ is a seminorm p such that $p(ab) \leq p(a)p(b)$ and $p(aa^*) = p(a)^2$ for all $a, b \in \mathcal{A}$. A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$, whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_\gamma\}$ converges to zero if and only if the net $\{p(a_\gamma)\}_\gamma$ converges to 0, for every continuous C^*-seminorm p on $\mathcal{A}$. For example any C^*-algebra is a locally C^*-algebra and any closed subalgebra of a locally C^*-algebra is a locally C^*-algebra. The notion of locally C^*-algebra was first introduced by Inoue [6] and studied more by Phillips and Fragoulopoulou [3, 10]. See also the book of Joita [7].

Hilbert modules are essentially objects, which behave similar to Hilbert spaces by allowing the inner product to take values in a locally C^*-algebra rather than the field

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of complex numbers. They play an important role in the modern theory of operator algebras, in noncommutative geometry and in quantum groups, see [5] . The paper is organized as follows. In Section 2 we recall some facts about Hilbert module over locally C^*-algebras [7]. In Section 3 we extend results about dynamical system of unitary operators for Hilbert C^*-module from [4] in the context of Hilbert modules over locally C^*-algebra. In Section 4 we investigate generalized derivation on Hilbert modules over a locally C^*-algebra.

2. Preliminaries

Let $\mathcal{A}$ be a locally C^*-algebra and $S(\mathcal{A})$ be the set of all continuous C^*-seminorms on $\mathcal{A}$. For $p \in S(\mathcal{A})$, the quotient $*$-algebra $\mathcal{A}_p = \mathcal{A}/N_p$, where $N_p = \{a \in \mathcal{A} : p(a) = 0\}$ is a C^*-algebra with respect to the C^*-norm $\|\cdot\|_p$ induced by p (i.e. $\|a_p\|_p = p(a)$ for each $a \in \mathcal{A}$, where $a_p = a + N_p$). The canonical map from $\mathcal{A}$ onto $\mathcal{A}_p$ is denoted by π_p and $\pi_p(a) = a_p$ for all a in $\mathcal{A}$. For $p, q \in S(\mathcal{A})$ with $p \geq q$, the surjective canonical map ${\pi_{pq}}^{\mathcal{A}} : \mathcal{A}_p \to \mathcal{A}_q$ is defined by ${\pi_{pq}}^{\mathcal{A}}({\pi_p}^{\mathcal{A}}(a)) = {\pi_q}^{\mathcal{A}}(a)$ for all $a \in \mathcal{A}$. The $\{\mathcal{A}_p : \pi^{\mathcal{A}}_{pq}, p, q \in S(\mathcal{A}), p \geq q\}$ is an inverse system of C^*-algebras and $\varprojlim_p \mathcal{A}_p$ is a locally C^*-algebra which is identical with $\mathcal{A}$. Suppose that $\mathcal{A}$ is a locally C^*-algebra. A right $\mathcal{A}$-module $\mathcal{E}$, equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying the following conditions for all $x, y \in \mathcal{E}$, $a \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$:

(i) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0$ if and only if $x = 0$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$;
(iv) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;

is called a pre Hilbert $\mathcal{A}$-module. If $\mathcal{E}$ is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_{\mathcal{E}}\}_{p\in S(\mathcal{A})}$, where $\overline{p}_{\mathcal{E}}(x) = \sqrt{p(\langle x, x\rangle)}$, $x \in \mathcal{E}$, then $\mathcal{E}$ is called a Hilbert module over the locally C^*-algebra $\mathcal{A}$ (Hilbert $\mathcal{A}$-module). If $\mathcal{E}$ is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner-product $\langle\cdot,\cdot\rangle$, then for each $p \in S(\mathcal{A})$ and for all $x, y \in \mathcal{E}$ we have the Cauchy-Schwarz inequality $p(\langle x, y\rangle)^2 \leq p(\langle x, x\rangle)p(\langle y, y\rangle)$. Suppose $\mathcal{E}$ is a Hilbert $\mathcal{A}$-module and p belongs to $S(\mathcal{A})$. Then ${N_p}^{\mathcal{E}} = \{x \in \mathcal{E} : \overline{p}_{\mathcal{E}}(x) = 0\}$ is a closed submodule of $\mathcal{E}$ and $\mathcal{E}_p = \frac{\mathcal{E}}{{N_p}^{\mathcal{E}}}$ is a Hilbert module over C^*-algebra $\mathcal{A}_p$ via the module multiplication $(x + {N_p}^{\mathcal{E}})\pi_p(a) = xa + {N_p}^{\mathcal{E}}$ and the inner product $\left\langle x + {N_p}^{\mathcal{E}}, y + {N_p}^{\mathcal{E}}\right\rangle = \pi_p(\langle x, y\rangle)$. The canonical map from $\mathcal{E}$ onto $\mathcal{E}_p$ is denoted by ${\sigma_p}^{\mathcal{E}}$ and ${\sigma_p}^{\mathcal{E}}(x) = x_p$, $p \in S(A)$. For each $p, q \in S(A)$, with $p \geq q$, there is a canonical surjective linear map ${\sigma_{pq}}^{\mathcal{E}} : \mathcal{E}_p \to \mathcal{E}_q$ such that ${\sigma_{pq}}^{\mathcal{E}}(x_p) = x_q$ for all $x \in \mathcal{E}$. Then $\{\mathcal{E}_p; \mathcal{A}_p; \sigma^{\mathcal{E}}_{pq} : p \geq q, p, q \in S(\mathcal{A})\}$ is an inverse system of Hilbert C^*-modules in the following sense:

- ${\sigma_{pq}}^{\mathcal{E}}(x_p a_p) = {\sigma_{pq}}^{\mathcal{E}}(x_p)\pi_{pq}(a_p)$ for all $x_p \in \mathcal{E}_p$, $a_p \in \mathcal{A}_p$;
- $\left\langle {\sigma_{pq}}^{\mathcal{E}}(x_p), {\sigma_{pq}}^{\mathcal{E}}(y_p)\right\rangle = \pi_{pq}(\langle x_p, y_p\rangle)$ for all $x_p, y_p \in \mathcal{E}_p$;
- ${\sigma_{pk}}^{\mathcal{E}} = {\sigma_{qk}}^{\mathcal{E}} \circ {\sigma_{pq}}^{\mathcal{E}}$, for $p \geq q \geq k$;
- ${\sigma_{pp}}^{\mathcal{E}} = I_{\mathcal{E}_p}$;

and $\lim\limits_{\overleftarrow{p}} \mathcal{E}_p$ is a Hilbert $\mathcal{A}$-module with $((x_p)_p)((a_p)_p) = (x_p a_p)_p$ and $\langle (x_p)_p, (y_p)_p \rangle = (\langle x_p, y_p \rangle)_p$. Moreover,$\lim\limits_{\overleftarrow{p}} \mathcal{E}_p$ can be identified by $\mathcal{E}$.

Let $\mathcal{A}$ be a locally C^*-algebra and $\mathcal{E}, \mathcal{F}$ be two Hilbert $\mathcal{A}$-modules, a map $T : \mathcal{E} \to \mathcal{F}$ is said to be adjointable if there exists a map $T^* : \mathcal{F} \to \mathcal{E}$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in \mathcal{E}$ and $y \in \mathcal{F}$. We use $L(\mathcal{E}, \mathcal{F})$ for the set of all adjointable module maps. If $\mathcal{E} = \mathcal{F}$ we write $L(\mathcal{E})$. A map $T : \mathcal{E} \to \mathcal{F}$ is called a bounded $\mathcal{A}$-module map if for each $p \in S(\mathcal{A})$ there exists $K_p \geq 0$ such that $\overline{p}_{\mathcal{F}}(Tx) \leq K_p \overline{p}_{\mathcal{E}}(x)$. The space of all bounded $\mathcal{A}$-module maps between $\mathcal{E}$ and $\mathcal{F}$ is denoted by $B(\mathcal{E}, \mathcal{F})$. It is easy to see that $\widetilde{P}(T) = \sup\{\overline{p}_{\mathcal{F}}(Tx) : \overline{p}_{\mathcal{E}}(x) \leq 1\}$ is a seminorm on $B(\mathcal{E}, \mathcal{F})$. For $y \in \mathcal{E}$ and $x \in \mathcal{F}$, $\theta_{x,y} : \mathcal{E} \to \mathcal{F}$ is defined by $\theta_{x,y}(z) = x \langle y, z \rangle$ for each $z \in \mathcal{E}$. We have $\theta^*_{x,y} = \theta_{y,x}$. The closed subspace of $L(\mathcal{E}, \mathcal{F})$ generated by $\{\theta_{x,y} : y \in \mathcal{E}, x \in \mathcal{F}\}$ is denoted by $K(\mathcal{E}, \mathcal{F})$. When $\mathcal{E} = \mathcal{F}$ we use $K(\mathcal{E})$ instead of $K(\mathcal{E}, \mathcal{E})$. An element in $K(\mathcal{E}, \mathcal{F})$ is called a compact operator from $\mathcal{E}$ to $\mathcal{F}$ and $K(\mathcal{E})$ is a two-sided $*$-ideal of $L(\mathcal{E})$. In fact $L(\mathcal{E})$ is a locally C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\widetilde{P}$ for each $p \in S(\mathcal{A})$ (see [7, Theorem 2.2.6]) and $K(\mathcal{E})$ is a locally C^*-subalgebra of $L(\mathcal{E})$.

A Hilbert $\mathcal{A}$-module $\mathcal{E}$ is called *full* if the closed linear span $\{\langle x, y \rangle : x, y \in \mathcal{E}\}$ denoted by $\langle \mathcal{E}, \mathcal{E} \rangle$, coincides with $\mathcal{A}$. For each $p \in S(\mathcal{A})$ and $\theta_{x,y} \in K_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, we have $\widetilde{P}(\theta_{x,y}) \leq \overline{p}_{\mathcal{F}}(x) \overline{p}_{\mathcal{E}}(y)$, since for each $z \in \mathcal{E}$, $p \in S(\mathcal{A})$ such that $\overline{p}_{\mathcal{E}}(z) \leq 1$, it follows from [7, Corollary 1.2.3] that $\overline{p}_{\mathcal{F}}(\theta_{x,y}(z)) = \overline{p}_{\mathcal{F}}(x \langle y, z \rangle) \leq \overline{p}_{\mathcal{F}}(x) p(\langle y, z \rangle) \leq \overline{p}_{\mathcal{F}}(x) \overline{p}_{\mathcal{E}}(y) \overline{p}(z) \leq \overline{p}_{\mathcal{F}}(x) \overline{p}_{\mathcal{E}}(y)$.

Throughout this paper, we assume that $\mathcal{A}$ is a locally C^*-algebra and $\mathcal{E}$, $\mathcal{F}$ are two Hilbert $\mathcal{A}$-modules. An adjointable operator u from $\mathcal{E}$ to $\mathcal{F}$ is said to be a unitary if $u^* u = I_{\mathcal{E}}$ and $u u^* = I_{\mathcal{F}}$, where $I_{\mathcal{E}}$ and $I_{\mathcal{F}}$ are identity operators on $\mathcal{E}$ and $\mathcal{F}$, respectively.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two locally C^*-algebras. A morphism from $\mathcal{A}$ to $\mathcal{B}$ is a continuous linear map $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = (\varphi(a))^*$ for all $a, b \in \mathcal{A}$. Two locally C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if there is an isomorphism (bijective morphism) from $\mathcal{A}$ to $\mathcal{B}$.

Joita in [7] characterized the unitary operators on Hilbert modules over locally C^*-algebras by proving the following proposition.

Proposition 2.1. [7, Proposition 2.5.3] *Let $u : \mathcal{E} \to \mathcal{F}$ be a linear map. Then the following statements are equivalent:*

(*i*) *u is a unitary operator from $\mathcal{E}$ to $\mathcal{F}$;*
(*ii*) *u is surjective and $\langle ux, ux \rangle = \langle x, x \rangle$ for all $x \in \mathcal{E}$;*
(*iii*) *$\overline{p}_{\mathcal{F}}(u(x)) = \overline{p}_{\mathcal{E}}(x)$ for all $x \in \mathcal{E}$, $p \in S(\mathcal{A})$ and u is a surjective module homomorphism from $\mathcal{E}$ to $\mathcal{F}$.*

Remark 2.1. If $\varphi : \mathcal{A} \to \mathcal{A}$ is an automorphism of locally C^*-algebras, then $\varphi_p : \mathcal{A}_p \to \mathcal{A}_p$ is a well-defined automorphism of C^*-algebras for each $p \in S(\mathcal{A})$. Thus $p(\varphi(a)) = \|(\varphi(a))_p\|_p = \|\varphi_p(a_p)\|_p = \|a_p\|_p = p(a)$ for each $a \in \mathcal{A}$ and $p \in S(\mathcal{A})$.

3. Dynamical Systems on Hilbert Modules

In this section we generalized the definitions of dynamical systems on Hilbert modules over locally C^*-algebras.

Definition 3.1. Let $\mathcal{E}$ be a Hilbert $\mathcal{A}$-module and $U(\mathcal{E})$ be the set of all unitary operators on $\mathcal{E}$. A mapping $\alpha : \mathbb{R} \to U(\mathcal{E})$, $t \mapsto \alpha_t$ is said to be a one-parameter group of unitaries if for each $t, s \in \mathbb{R}$

(*i*) $\alpha_0 = I$;
(*ii*) $\alpha_{t+s} = \alpha_t \alpha_s$.

We say that α is a strongly continuous one-parameter group (C_0-group) of unitaries if, in addition, $\lim\limits_{t\to 0}\alpha_t(x) = x$ in $\mathcal{E}$. In this case, α is called a dynamical system of unitary operators on $\mathcal{E}$.

The infinitesimal generator of α is the mapping $\delta : D(\delta) \subseteq \mathcal{E} \to \mathcal{E}$, where $D(\delta) = \left\{x \in \mathcal{E} : \lim\limits_{t\to 0}\frac{\alpha_t(x) - x}{t} \text{ exists}\right\}$ and $\delta(x) = \lim\limits_{t\to 0}\frac{\alpha_t(x) - x}{t}$ for each $x \in D(\delta)$. The above limit is taken in the topology on $\mathcal{E}$.

Remark 3.1. Let $\mathcal{A}$ be a locally C^*-algebra and $\mathrm{Aut}(\mathcal{A})$ be the set of all automorphisms on $\mathcal{A}$, then, similar to Definition 3.1, we can define a dynamical system of automorphisms on $\mathcal{A}$.

In the following theorem, we show that every dynamical system of unitary operators on Hilbert $\mathcal{A}$-module $\mathcal{E}$, defines a dynamical system of automorphisms on locally C^*-algebra $K(\mathcal{E})$.

Theorem 3.1. *Let α be a dynamical system of unitary operators on $\mathcal{E}$ and $u : \mathbb{R} \to \mathrm{Aut(K(\mathcal{E}))}$ defined by $u_t(T) = \alpha_t T \alpha_t^*$ for each $T \in K(\mathcal{E})$, then u is a dynamical system of automorphism on $K(\mathcal{E})$.*

Proof. Obviously $u_0 = I$ and $u_{t+s} = u_t u_s$. It is enough to show that $\lim\limits_{t\to 0}u_t(T) = T$ for each $T \in K(\mathcal{E})$. Put $S = \theta_{x,y}$ for some $x, y \in \mathcal{E}$. Then

$$\begin{aligned}
\widetilde{P}(u_t(S) - S) &= \widetilde{P}(\alpha_t S \alpha_t^* - S) = \widetilde{P}(\theta_{\alpha_t(x),\alpha_t(y)} - \theta_{x,y}) \\
&= \widetilde{P}(\theta_{\alpha_t(x),\alpha_t(y)} - \theta_{x,y} - \theta_{x,\alpha_t(y)} + \theta_{x,\alpha_t(y)}) \\
&= \widetilde{P}(\theta_{\alpha_t(x)-x,\alpha_t(y)} + \theta_{x,\alpha_t(y)-y}) \\
&\leq \widetilde{P}(\theta_{\alpha_t(x)-x,\alpha_t(y)}) + \widetilde{P}(\theta_{x,\alpha_t(y)-y}) \\
&\leq \overline{p}_{\mathcal{E}}(\alpha_t(x) - x)\overline{p}_{\mathcal{E}}(\alpha_t(y)) + \overline{p}_{\mathcal{E}}(x)\overline{p}_{\mathcal{E}}(\alpha_t(y) - y).
\end{aligned}$$

Since α_t is a unitary operator, by Proposition 2.1 we have $\overline{p}_{\mathcal{E}}(\alpha_t(y)) = \overline{p}_{\mathcal{E}}(y)$, so the right of above inequality tends to zero. We know that $T = \lim_{t\to\infty} T_n$, where each T_n is of the form $T_n = \sum_{i=1}^{k^n} \lambda_i^n \theta_{x_i^n, y_i^n}$, where $\lambda_i^n \in \mathbb{C}$, $x_i^n, y_i^n \in \mathcal{E}$. By continuity of seminorns, $\lim_{t\to 0} \widetilde{P}(u_t(T) - T) = \lim_{t\to 0} \widetilde{P}(\alpha_t T \alpha_t^* - T) = 0$. Hence $\lim_{t\to 0} u_t(T) = T$. □

The converse of Theorem 3.1 is not true in general, we want to show that under some mild conditions on a dynamical system α of automorphism on $K(\mathcal{E})$, there is a dynamical system u of unitary operators on $\mathcal{E}$ such that $\alpha_t(T) = u_t T u_t^*$ for each $T \in K(\mathcal{E})$.

Theorem 3.2. *Let α be a dynamical system of automorphisms on $K(\mathcal{E})$. If there is $x \in \mathcal{E}$ such that $\langle x, x\rangle = 1$ and $\alpha_t(\theta_{x,x}) = \theta_{x,x}$ for each $t \in \mathbb{R}$, then there is a dynamical system u of unitary operators on $\mathcal{E}$ such that $\alpha_t(T) = u_t T u_t^*$ for each $T \in K(\mathcal{E})$.*

Proof. For each $T \in K(\mathcal{E})$, $x \in \mathcal{E}$ with $\langle x, x\rangle = 1$, let us define $u_t : \mathcal{E} \to \mathcal{E}$ by $u_t(Tx) = \alpha_t(T)x$. Then

$$\begin{aligned}\overline{p}_{\mathcal{E}}(Tx) &= \overline{p}_{\mathcal{E}}(Tx\ 1) = \overline{p}_{\mathcal{E}}(Tx\ \langle x,x\rangle) = \overline{p}_{\mathcal{E}}(T(x\langle x,x\rangle)) = \overline{p}_{\mathcal{E}}(T\theta_{x,x}(x))\\ &\le \widetilde{P}(T\theta_{x,x})\overline{p}_{\mathcal{E}}(x) \le \widetilde{P}(T\theta_{x,x}) = \widetilde{P}(\theta_{Tx,x}) \le \overline{p}_{\mathcal{E}}(Tx)\overline{p}_{\mathcal{E}}(x) \le \overline{p}_{\mathcal{E}}(Tx).\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{p}_{\mathcal{E}}(Tx) &= \widetilde{P}(\theta_{Tx,x})\\ &= \widetilde{P}(\alpha_t(\theta_{Tx,x})) \quad (\alpha_t \text{ is an automorphism and by Remark 2.1})\\ &= \widetilde{P}(\alpha_t(T\theta_{x,x})) = \widetilde{P}(\alpha_t(T)\alpha_t(\theta_{x,x})) = \widetilde{P}(\alpha_t(T)\theta_{x,x}) = \widetilde{P}(\theta_{\alpha_t(T)x,x})\\ &= \overline{p}_{\mathcal{E}}(\alpha_t(T)x) = \overline{p}_{\mathcal{E}}(u_t(Tx)).\end{aligned}$$

Thus, $\overline{p}_{\mathcal{E}}(Tx) = \overline{p}_{\mathcal{E}}(u_t(Tx))$ for each x with $\langle x, x\rangle = 1$. Let y be an arbitrary element in $\mathcal{E}$. Then $y = y \cdot 1 = y\langle x, x\rangle = \theta_{y,x}(x)$. Put $T_0 = \theta_{y,x}$. Then $T_0 \in K(\mathcal{E})$ and $y = T_0 x$. Now put $z = \alpha_t^{-1}(T_0)x$ and let $T' = \alpha_t^{-1}(T_0)$. Then

$$\begin{aligned}u_t(T'x) &= \alpha_t(T')x = \alpha_t(\alpha_t^{-1}(T_0))x = \alpha_t(\alpha_{-t}(T_0))x\\ &= \alpha_t\alpha_{-t}(T_0)x = \alpha_0(T_0)x = T_0x = y.\end{aligned}$$

Hence u_t is onto. Since $\overline{p}_{\mathcal{E}}(T_0x) = \overline{p}_{\mathcal{E}}(u_t(T_0x))$ or $\overline{p}_{\mathcal{E}}(y) = \overline{p}_{\mathcal{E}}(u_t(y))$ for each $y \in \mathcal{E}$, so by Proposition 2.1, u_t is unitary and $u_t^* = u_t^{-1}$. The equations $\alpha_{-t}(u_t(Tx)) = \alpha_{-t}(\alpha_t(T)x) = \alpha_{-t}(\alpha_t(T))x = \alpha_0(T)x = Tx$ and $u_t(\alpha_{-t}(T)x) = Tx$ show that $(u_t)^{-1}(Tx) = \alpha_{-t}(T)x$. Let $z, y \in \mathcal{E}$, then there exist $T_0, S_0 \in K(\mathcal{E})$ such that $z = T_0x$ and $y = S_0x$. Hence

$$\begin{aligned}\langle u_s u_t(z), y\rangle &= \langle u_s u_t(T_0x), S_0x\rangle = \langle u_t(T_0x), (u_s)^*(S_0x)\rangle\\ &= \langle \alpha_t(T_0)x, \alpha_{-s}(S_0)x\rangle = \langle (\alpha_{-s}(S_0))^*\alpha_t(T_0)x, x\rangle\\ &= \langle \alpha_{-s}(S_0^*)\alpha_t(T_0)x, x\rangle = \langle \alpha_{-s}(S_0^*\alpha_{t+s}(T_0))x, x\rangle = \langle\ u_{-s}(S_0^*\alpha_{t+s}(T_0)x), x\rangle\\ &= \langle S_0^*\alpha_{t+s}(T_0)x, (u_{-s})^*\theta_{x,x}(x)\rangle \quad (x = x1 = x\langle x,x\rangle = \theta_{x,x}(x))\end{aligned}$$

$$
\begin{aligned}
&= \langle S_0^*\alpha_{t+s}(T_0)x, \alpha_s(\theta_{x,x})x\rangle = \langle S_0^*\alpha_{t+s}(T_0)x, x\rangle \\
&= \langle u_{t+s}(T_0x), S_0(x)\rangle = \langle u_{t+s}(z), y\rangle,
\end{aligned}
$$

whence $u_{t+s} = u_t u_s$.

Also $u_0(y) = u_0(S_0x) = \alpha_0(S_0)x = S_0x = y$, so $u_0 = I$. Hence $\overline{p}_{\mathcal{E}}((u_t)y - y) = \overline{p}_{\mathcal{E}}((u_t)(Tx) - Tx) = \overline{p}_{\mathcal{E}}((\alpha_t(T) - T)x) \leq \widetilde{p}(\alpha_t(T) - T)\overline{p}_{\mathcal{E}}(x)$. Therefore, $\lim_{t\to 0} u_t(y) = y$, so u is a dynamical system of unitary on $\mathcal{E}$. Also $u_tTu_t^*(z) = u_tTu_t^*(T_0x) = u_tT(\alpha_t)^{-1}(T_0x) = \alpha_t(T\alpha_t^{-1}(T_0))x = \alpha_t(T)(\alpha_t\alpha_t^{-1}(T_0)x) = \alpha_t(T)T_0x = \alpha_t(T)z$ so $\alpha_t(T) = u_tTu_t^*$. ☐

Theorem 3.3. *Let α be an automorphism on $K(\mathcal{E})$ such that $\alpha(\theta_{x,x}) = \theta_{y,y}$, where $x \in \mathcal{E}$ and $\langle y, y\rangle = 1$. Then, there is a unitary operator u in $K(\mathcal{E})$ such that $\alpha(T) = uTu^*$ for each $T \in K(\mathcal{E})$.*

Proof. For each $T \in K(\mathcal{E})$ we define $u(Tx) = \alpha(T)y$. Then, by the some reasoning as in the proof of Theorem 3.2, we have

$$
\begin{aligned}
\overline{p}_{\mathcal{E}}(u(Tx)) &= \overline{p}_{\mathcal{E}}(\alpha(T)y) = \widetilde{p}(\theta_{\alpha(T)y,y}) = \widetilde{p}(\alpha(T)\theta_{y,y}) = \widetilde{p}(\alpha(T)\alpha(\theta_{x,x})) \\
&= \widetilde{p}(\alpha(T\theta_{x,x}) = \widetilde{p}(\alpha(\theta_{Tx,x})) = \widetilde{p}(\theta_{Tx,x}) = \overline{p}_{\mathcal{E}}(Tx).
\end{aligned}
$$

Also, u is onto since for each $z \in \mathcal{E}$ there exists $T_0 \in K(\mathcal{E})$ such that $z = T_0x$. One can see $u(\alpha^{-1}(T_0)y) = \alpha(\alpha^{-1}T_0)y = T_0y = z$. So u is well-defined, onto and $\overline{p}_{\mathcal{E}}(u(Tx)) = \overline{p}_{\mathcal{E}}(Tx)$. Hence, by Proposition 2.1, u is a unitary operator. Let $S \in K(\mathcal{E})$ and $x \in \mathcal{E}$ be arbitrary, then

$$uTu^*(Sx) = uT\alpha^{-1}(S)x = \alpha(T\alpha^{-1}(S))x = \alpha(T)Sx,$$

which implies that $uTu^* = \alpha(T)$. ☐

4. Generalized Derivations on Hilbert Modules

Let $\mathcal{A}$ be an algebra, a linear mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$, where $D(d)$ is a dense subalgebra of $\mathcal{A}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for each $a, b \in D(d)$. We introduce the notion of a generalized derivation on Hilbert modules over locally C^*-algebras. This definition is similar to that of a generalized derivation on Hilbert C^*-modules introduced in [1].

Definition 4.1. Let $\mathcal{E}$ be a full Hilbert $\mathcal{A}$-module. A linear map $\delta : D(\delta) \subseteq \mathcal{E} \to \mathcal{E}$, where $D(\delta)$ is a dense subspace of $\mathcal{E}$, is called a generalized derivation if there exists a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$, where $D(d)$ is a dense subalgebra of $\mathcal{A}$ such that $D(\delta)$ is an algebraic left $D(d)$-module and $\delta(xa) = \delta(x)a + xd(a)$ for each $x \in D(\delta)$ and $a \in D(d)$.

Recall that if $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module and $a \in \mathcal{A}$ such that $xa = 0$ for each $x \in \mathcal{E}$, then $a = 0$ (see [8, Remark 2.1]).

Proposition 4.1. *Let $\mathcal{A}$ be a locally C^*-algebra and $\mathcal{E}$ be a full Hilbert $\mathcal{A}-$module and $\delta : \mathcal{E} \to \mathcal{E}$ be a bounded generalized derivation. Then $\delta_p : \mathcal{E}_p \to \mathcal{E}_p$ defined by $\delta_p(x + {N_p}^{\mathcal{E}}) = \delta(x) + {N_p}^{\mathcal{E}}$ is a generalized derivation for each $p \in S(\mathcal{A})$. Conversely, if δ_p is a generalized derivation for each $p \in S(\mathcal{A})$ then there is a generalized derivation on $\mathcal{E}$.*

Proof. Let $\delta : \mathcal{E} \to \mathcal{E}$ be a bounded generalized derivation. There then exists a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(xa) = \delta(x)a + xd(a)$ for all $x \in \mathcal{E}$ and $a \in D(d)$. For any $a, b \in \mathcal{A}$ and $x \in \mathcal{E}$ we have $\delta(xab) = \delta(x)(ab) + xd(ab)$. Also,

$$\begin{aligned}\delta(xab) &= \delta((xa)b) = \delta(xa)b + (xa)d(b)\\ &= (\delta(x)a + xd(a))b + (xa)d(b) = \delta(x)(ab) + xd(a)b + xad(b).\end{aligned}$$

So $xd(ab) = xd(a)b + xad(b)$ or $x(d(ab) - (d(a)b + ad(b))) = 0$ for all $x \in \mathcal{E}$. Since $\mathcal{E}$ is full, we have $d(ab) = ad(b) + d(a)b$. It means that d is a derivation. Similarly, we can show that d is linear. For each $p \in S(\mathcal{A})$ consider the mapping $d_p : D(d_p) \subseteq \mathcal{A}_p \to \mathcal{A}_p$, defined by $d_p(a + N_p) = d(a) + N_p$. We show that d_p is well-defined. Indeed, if $a \in N_p$, then by [2] there exist elements $b_1, b_2, b_3, b_4 \in N_p$ such that $a = \sum_{k=1}^{4} i^k b_k^2$ and $p(b_k) = 0$ for $k = 1, 2, 3, 4$ and

$$\begin{aligned}p(d(a)) &= p\left(d\left(\sum_{k=1}^{4} i^k b_k^2\right)\right)\\ &= p\left(\sum_{k=1}^{4} i^k d(b_k^2)\right)\\ &= p\left(\sum_{k=1}^{4} i^k (b_k d(b_k) + d(b_k)b_k)\right)\\ &\leq \sum_{k=1}^{4} p(b_k)p(d(b_k)) + p(d(b_k))p(b_k) = 0.\end{aligned}$$

Therefore $a \in N_p$ implies that $p(d(a)) = 0$. Now, if $a + N_p = a' + N_p$, then $a - a' \in N_p$ so $p(d(a - a')) = 0$ thus $d(a) + N_p = d(a') + N_p$. It means that $(d(a))_p = (d(a'))_p$. So d_p is well defined. Obviously, the mapping d_p is a derivation. Also,

$$\begin{aligned}\delta_p(x_p a_p) &= \delta_p((xa)_p)\\ &= \delta(xa) + {N_p}^{\mathcal{E}}\\ &= (\delta(x)a + xd(a)) + {N_p}^{\mathcal{E}}\\ &= (\delta(x)a + {N_p}^{\mathcal{E}}) + (xd(a) + {N_p}^{\mathcal{E}})\\ &= \delta_p(x_p)a_p + x_p d_p(a_p),\end{aligned}$$

hence δ_p is a generalized derivation. Now suppose that δ_p is a generalized derivation for each $p \in S(\mathcal{A})$, then there exists a mapping $d_p : \mathcal{A}_p \to \mathcal{A}_p$ such that δ_p is d_p-derivation.

Now, if we define $\delta : \varprojlim_{p} \mathcal{E}_p \to \varprojlim_{p} \mathcal{E}_p$ by $\delta((a_p)_p) = (\delta_p(a_p))_p$ and $d : \varprojlim_{p} \mathcal{A}_p \to \varprojlim_{p} \mathcal{A}_p$ by $d((a_P)_P) = (d_p(a_P))_p$, then δ is $d-$generalized derivation. Indeed,

$$\begin{aligned}\delta(x)a + xd(a) &= (\delta_p(x_p))_p(a_p)_p + (x_p)_p(d_p(a_p))_p\\ &= (\delta_p(x_p)a_p)_p(x_pd_p(a_p))_p\\ &= (\delta_p(x_p)a_p + x_pd_p(a_p))_p\\ &= (\delta_p(x_pa_p))_p\\ &= \delta(xa),\end{aligned}$$

which is stated. □

Proposition 4.2. *Suppose that $\mathcal{E}$ is a full Hilbert $\mathcal{A}$-module, α is a dynamical system of unitaries on $\mathcal{E}$ and δ is the infinitesimal generator of α such that $D(\delta)$ is a dense subspace of $\mathcal{E}$. Then there exists a derivation $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(xa) = \delta(x)a + xd(a)$ for all $a \in D(d)$, $x \in D(\delta)$.*

Proof. Since α is a dynamical system of unitaries on $\mathcal{E}$, the mapping $\alpha_t : \mathcal{E} \to \mathcal{E}$ is a unitary for each $t \in \mathbb{R}$. It follows from [8, Corollary 3.6] there is an isomorphism of locally C^*-algebras $\alpha'_t : \mathcal{A} \to \mathcal{A}$ such that $\alpha'_t(\langle x, y\rangle) = \langle \alpha_t(x), \alpha_t(y)\rangle$ and

$$\begin{aligned}&\langle \alpha_t(xa) - \alpha_t(x)\alpha'_t(a), \alpha_t(xa) - \alpha_t(x)\alpha'_t(a)\rangle\\ =&\alpha'_t(\langle xa, xa\rangle) - \alpha'_t(\langle xa, x\rangle)\alpha'_t(a) - \alpha'_t(a^*)\alpha'_t(\langle x, xa\rangle) + \alpha'_t(a^*)\alpha'_t(\langle x, x\rangle)\alpha'_t(a) = 0.\end{aligned}$$

Whence $\alpha_t(xa) = \alpha_t(x)\alpha'_t(a)$. In addition,

$$\begin{aligned}\overline{p}_{\mathcal{E}}(x\alpha'_t(a) - xa) &= \overline{p}_{\mathcal{E}}(x\alpha'_t(a) - \alpha_t(x)\alpha'_t(a) + \alpha_t(x)\alpha'_t(a) - xa)\\ &\le \overline{p}_{\mathcal{E}}(x\alpha'_t(a) - \alpha_t(x)\alpha'_t(a) + \overline{p}_{\mathcal{E}}(\alpha_t(x)\alpha'_t(a) - xa)\\ &\le p(\alpha'_t(a))\overline{p}_{\mathcal{E}}(\alpha_t(x) - x) + \overline{p}_{\mathcal{E}}(\alpha_t(x)\alpha'_t(a) - xa).\end{aligned}$$

Hence $\lim_{t\to 0} x\alpha'_t(a) - xa = 0$ for each $x \in \mathcal{E}$. Thus, $\lim_{t\to 0}\alpha'_t(a) = a$ for each $a \in \mathcal{A}$. Therefore $\alpha' : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ is a dynamical system of automorphisms on $\mathcal{A}$. The rest of proof is similar to [1, Theorem 4.3] and we remove it. □

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