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THE HARMONIC INDEX OF EDGE-SEMITOTAL GRAPHS, TOTAL GRAPHS AND RELATED SUMS

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Abstract. For a connected graph G, there are several related graphs such as line graph $L(G)$, subdivision graph $S(G)$, vertex-semitotal graph $R(G)$, edge-semitotal graph $Q(G)$ and total graph $T(G)$ [I. Gutman, B. Furtula, Ž. K. Vukićević and G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem. **74** (2015), 5–16, W. Yan, B. -Y. Yang and Y. -N. Yeh, *The behavior of Wiener indices and polynomials of graphs under five graph decorations*, Appl. Math. Lett. **20** (2007), 290–295]. Let F be one of symbols S, R, Q or T. The F-sum $G_1 +_F G_2$ of two connected graphs G_1 and G_2 is a graph with vertex set $(V(G_1) \cup E(G_1)) \times V(G_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = u_2 \in V(G_1)$ and $v_1v_2 \in E(G_2)]$ or $[v_1 = v_2$ and $u_1u_2 \in E(F(G))]$ [M. Eliasi and B. Taeri, *Four new sums of graphs and their Wiener indices*, Discrete Appl. Math. **157** (2009), 794–803]. In this paper, we investigate the harmonic index of edge-semitotal graphs, total graphs and F-sum of graphs, where $F = Q$ or T.

1. Introduction

Throughout this paper, all graphs are finite, simple, undirected and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, the degree of a vertex u in G is denoted by $\deg_G(u)$. We will use P_n to denote the path of order n.

The edge-semitotal graph $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G (see Figure 1). The total graph $T(G)$ has as its vertices the edges and vertices of G; adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G (see Figure 1).

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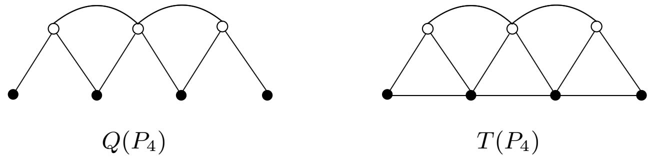

FIGURE 1. $Q(P_4)$ and $T(P_4)$

Let G_1 and G_2 be two graphs and $F \in \{Q, T\}$. Then, the F-sum $G_1 +_F G_2$ has $|V(G_2)|$ copies of the graph $F(G_1)$ and we can label these copies by vertices of G_2. The vertices in each copy have two situations: the vertices in $V(G_1)$ (black vertices) and the vertices in $E(G_1)$ (white vertices). Now, we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2 (see Figure 2).

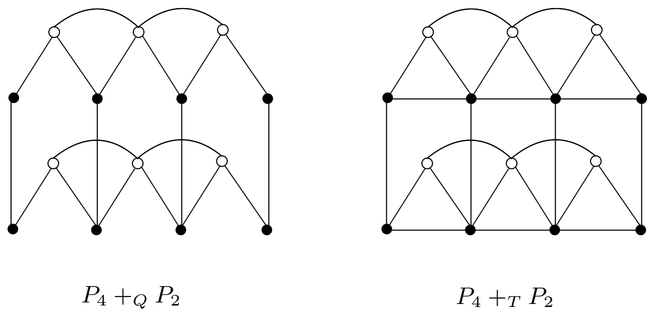

FIGURE 2. $P_4 +_Q P_2$ and $P_4 +_T P_2$

The first Zagreb index, the inverse degree and the harmonic index of a graph G are important vertex-degree-based indices related to G, where denoted by $M_1(G)$, $r(G)$ and $H(G)$, respectively, and defined as follows [3, 4]:

$$M_1(G) = \sum_{u \in V(G)} \deg_G^2(u), \quad r(G) = \sum_{u \in V(G)} \frac{1}{\deg_G(u)}, \quad H(G) = \sum_{uv \in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}.$$

The first Zagreb index can be expressed as $M_1(G) = \sum_{uv \in E(G)} (\deg_G(u) + \deg_G(v))$ [8].

In this paper, we consider the harmonic index. In recent years, this topological index has been extensively studied. Shwetha et al. [11] derived expressions for the harmonic index of the join, corona product, Cartesian product, composition and symmetric difference of graphs. Recently, Onagh investigated the harmonic index of product graphs, subdivision graphs, t-subdivision graphs and F-sum of graphs, where $F \in \{S, S_t\}$ [9, 10]. More results on the harmonic index can been found in [1, 6, 7, 12–15].

In this paper, we study the harmonic index of edge-semitotal graphs, total graphs and F-sum of graphs, where $F = Q$ or T.

2. The Harmonic Index of Edge-semitotal and Total Graphs

In this section, we give some upper bounds for the harmonic index of edge-semitotal and total graphs. Hereafter, we deal with nontrivial graphs.

Note that $\deg_{Q(G)}(u) = \deg_G(u)$ for all $u \in V(G)$ and $\deg_{Q(G)}(w) = \deg_G(u) + \deg_G(v) = 2 + \deg_{L(G)}(w)$ for all $w \in V(Q(G)) - V(G)$, where w is the vertex inserted into the edge uv of G.

Theorem 2.1. *Let G be a graph of order n and size m. Then*

$$H(Q(G)) < \frac{1}{2}H(G) + \frac{1}{4}H(L(G)) + \frac{1}{16}M_1(G) + \frac{1}{2}n - \frac{1}{8}m.$$

Proof. By definition of the harmonic index, we have

$$\begin{aligned}
&H(Q(G))\\
&= \sum_{uv\in E(G)} \left(\frac{2}{\deg_{Q(G)}(u) + \deg_{Q(G)}(x)} + \frac{2}{\deg_{Q(G)}(x) + \deg_{Q(G)}(v)} \right)\\
&\quad \text{(where } x \text{ is the vertex inserted into the edge } uv \text{ of } G\text{)}\\
&\quad + \sum_{ww'\in E(L(G))} \frac{2}{\deg_{Q(G)}(w) + \deg_{Q(G)}(w')}\\
&= \sum_{uv\in E(G)} \left(\frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} + \frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} \right)\\
&\quad + \sum_{ww'\in E(L(G))} \frac{2}{\big(2 + \deg_{L(G)}(w)\big) + \big(2 + \deg_{L(G)}(w')\big)}\\
&= \sum_{uv\in E(G)} \left(\frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} + \frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} \right)\\
&\quad + \sum_{ww'\in E(L(G))} \frac{2}{4 + \big(\deg_{L(G)}(w) + \deg_{L(G)}(w')\big)}\\
&:= \sum 1 + \sum 2.
\end{aligned}$$

By Jensen's inequality, for every $uv \in E(G)$, we have

$$\begin{aligned}
\frac{2}{\deg_G(u) + (\deg_G(u) + \deg_G(v))} &< \frac{1}{2}\frac{1}{\deg_G(u)} + \frac{1}{4}\frac{2}{\deg_G(u) + \deg_G(v)},\\
\frac{2}{(\deg_G(u) + \deg_G(v)) + \deg_G(v)} &< \frac{1}{4}\frac{2}{\deg_G(u) + \deg_G(v)} + \frac{1}{2}\frac{1}{\deg_G(v)}.
\end{aligned}$$

So,

$$\sum 1 < \frac{1}{2}\sum_{uv\in E(G)} \left(\frac{1}{\deg_G(u)} + \frac{1}{\deg_G(v)} \right) + \frac{1}{2}\sum_{uv\in E(G)} \frac{2}{\deg_G(u) + \deg_G(v)}$$

$$= \frac{1}{2}\sum_{u\in V(G)}\left(\frac{1}{\deg_G(u)}\times \deg_G(u)\right)+\frac{1}{2}H(G)$$

$$= \frac{1}{2}n+\frac{1}{2}H(G).$$

Similarly, for every $ww' \in E(L(G))$,

$$\frac{2}{4+\left(\deg_{L(G)}(w)+\deg_{L(G)}(w')\right)} \leq \frac{1}{8}+\frac{1}{4}\frac{2}{\deg_{L(G)}(w)+\deg_{L(G)}(w')},$$

with equality if and only if $\deg_{L(G)}(w)+\deg_{L(G)}(w') = 4$. This implies that

$$\begin{aligned}\sum 2 &\leq \frac{1}{8}|E(L(G))|+\frac{1}{4}H(L(G))\\ &= \frac{1}{8}\left(\frac{1}{2}M_1(G)-m\right)+\frac{1}{4}H(L(G))\\ &= \frac{1}{16}M_1(G)-\frac{1}{8}m+\frac{1}{4}H(L(G)).\end{aligned}$$

Therefore,

$$H(Q(G)) < \frac{1}{2}H(G)+\frac{1}{4}H(L(G))+\frac{1}{16}M_1(G)+\frac{1}{2}n-\frac{1}{8}m.$$

This completes the proof. □

Example 2.1. For any $n \geq 4$,

$$H(Q(P_n)) = \left(\frac{9}{5}+\frac{2}{3}(n-3)\right)+\left(\frac{4}{7}+\frac{1}{4}(n-4)\right) = \frac{83}{35}+\frac{2}{3}(n-3)+\frac{1}{4}(n-4).$$

Note that $\deg_{T(G)}(u) = 2\deg_G(u)$ for all $u \in V(G)$ and $\deg_{T(G)}(w) = \deg_G(u) + \deg_G(v) = 2+\deg_{L(G)}(w)$ for all $w \in V(T(G))-V(G)$, where w is the vertex inserted into the edge uv of G.

Theorem 2.2. *Let G be a graph of order n and size m. Then*

$$H(T(G)) \leq H(G)+\frac{1}{4}H(L(G))+\frac{1}{16}M_1(G)+\frac{1}{4}n-\frac{1}{8}m,$$

with equality if and only if $G \cong C_n$.

Proof. Note that

$$\begin{aligned}&H(T(G))\\ &= \sum_{uv\in E(G)}\frac{2}{\deg_{T(G)}(u)+\deg_{T(G)}(v)}\\ &\quad+\sum_{uv\in E(G)}\left(\frac{2}{\deg_{T(G)}(u)+\deg_{T(G)}(x)}+\frac{2}{\deg_{T(G)}(x)+\deg_{T(G)}(v)}\right)\end{aligned}$$

(where x is the vertex inserted into the edge uv of G)

$$
\begin{aligned}
&+\sum_{ww'\in E(L(G))}\frac{2}{\deg_{T(G)}(w)+\deg_{T(G)}(w')}\\
=&\sum_{uv\in E(G)}\frac{2}{2\deg_G(u)+2\deg_G(v)}\\
&+\sum_{uv\in E(G)}\Bigg(\frac{2}{2\deg_G(u)+(\deg_G(u)+\deg_G(v))}\\
&+\frac{2}{(\deg_G(u)+\deg_G(v))+2\deg_G(v)}\Bigg)\\
&+\sum_{ww'\in E(L(G))}\frac{2}{\big(2+\deg_{L(G)}(w)\big)+\big(2+\deg_{L(G)}(w')\big)}\\
=&\frac{1}{2}H(G)\\
&+\sum_{uv\in E(G)}\Bigg(\frac{2}{2\deg_G(u)+(\deg_G(u)+\deg_G(v))}\\
&+\frac{2}{(\deg_G(u)+\deg_G(v))+2\deg_G(v)}\Bigg)\\
&+\sum_{ww'\in E(L(G))}\frac{2}{4+\big(\deg_{L(G)}(w)+\deg_{L(G)}(w')\big)}\\
:=&\frac{1}{2}H(G)+\sum 1+\sum 2.
\end{aligned}
$$

By Jensen's inequality, for every $uv\in E(G)$, we have

$$
\frac{2}{2\deg_G(u)+(\deg_G(u)+\deg_G(v))}\le\frac{1}{4}\frac{1}{\deg_G(u)}+\frac{1}{4}\frac{2}{\deg_G(u)+\deg_G(v)}, \tag{2.1}
$$

with equality if and only if $\deg_G(u)=\deg_G(v)$. Also,

$$
\frac{2}{(\deg_G(u)+\deg_G(v))+2\deg_G(v)}\le\frac{1}{4}\frac{2}{\deg_G(u)+\deg_G(v)}+\frac{1}{4}\frac{1}{\deg_G(v)}, \tag{2.2}
$$

with equality if and only if $\deg_G(u)=\deg_G(v)$. Thus,

$$
\begin{aligned}
\sum 1&\le\frac{1}{4}\sum_{uv\in E(G)}\left(\frac{1}{\deg_G(u)}+\frac{1}{\deg_G(v)}\right)+\frac{1}{2}\sum_{uv\in E(G)}\frac{2}{\deg_G(u)+\deg_G(v)}\\
&=\frac{1}{4}n+\frac{1}{2}H(G).
\end{aligned}
$$

On the other hand,

$$
\sum 2\le\frac{1}{4}H(L(G))+\frac{1}{16}M_1(G)-\frac{1}{8}m, \tag{2.3}
$$

with equality if and only if $\deg_{L(G)}(w) + \deg_{L(G)}(w') = 4$ for all $ww' \in E(L(G))$. Therefore,

$$H(T(G)) \leq H(G) + \frac{1}{4}H(L(G)) + \frac{1}{16}M_1(G) + \frac{1}{4}n - \frac{1}{8}m.$$

One can see that equality holds in above inequality if and only if the inequalities (2.1), (2.2) and (2.3) be equalities, i.e., G is a k-regular such that $\deg_{L(G)}(w) + \deg_{L(G)}(w') = 4$ $(*)$ for all $ww' \in L(G)$. Since $L(G)$ is $(2k-2)$-regular, by $(*)$, we can get $k = 2$.

This completes the proof. ☐

Example 2.2. For any $n \geq 4$,

$$\begin{aligned} H(T(P_n)) &= \left(\frac{2}{3} + \frac{1}{4}(n-3)\right) + \left(\frac{48}{35} + \frac{1}{2}(n-3)\right) + \left(\frac{4}{7} + \frac{1}{4}(n-4)\right) \\ &= \frac{274}{105} + \frac{3}{4}(n-3) + \frac{1}{4}(n-4). \end{aligned}$$

3. The Harmonic Index for Q-sum and T-sum of Graphs

In the following theorem, we give an upper bound for the harmonic index of $G_1 +_Q G_2$ in terms of $H(Q(G_1))$, $H(L(G_1))$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 3.1. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned} H(G_1 +_Q G_2) <& \frac{1}{4}n_2 H(Q(G_1)) + \frac{3}{16}n_2 H(L(G_1)) + \frac{1}{4}n_1 H(G_2) \\ &+ \frac{3}{64}n_2 M_1(G) + \frac{1}{4}m_2 r(G_1) + m_1 r(G_2) - \frac{3}{32}n_2 m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u, v) = \deg_{G_1 +_Q G_2}(u, v)$ be the degree of a vertex (u, v) in $G_1 +_Q G_2$. Then,

$$\begin{aligned} H(G_1 +_Q G_2) =& \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\deg(u, v_1) + \deg(u, v_2)} \\ &+ \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(Q(G_1))} \frac{2}{\deg(u_1, v) + \deg(u_2, v)} \\ :=& \sum 1 + \sum 2. \end{aligned}$$

Note that

$$\begin{aligned} \sum 1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{\left(\deg_{Q(G_1)}(u) + \deg_{G_2}(v_1)\right) + \left(\deg_{Q(G_1)}(u) + \deg_{G_2}(v_2)\right)} \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{2}{2\deg_{Q(G_1)}(u) + \left(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\right)} \end{aligned}$$

$$= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{2}{2\deg_{G_1}(u) + \big(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\big)}.$$

By Jensen's inequality, for every $u \in V(G_1)$ and every $v_1v_2 \in E(G_2)$, we have

(3.1)

$$\frac{2}{2\deg_{G_1}(u) + \big(\deg_{G_2}(v_1) + \deg_{G_2}(v_2)\big)} \leq \frac{1}{4}\frac{1}{\deg_{G_1}(u)} + \frac{1}{4}\frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)},$$

with equality if and only if $2\deg_{G_1}(u) = \deg_{G_2}(v_1) + \deg_{G_2}(v_2)$. Thus,

$$\begin{aligned}\sum 1 &\leq \frac{1}{4}\sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4}\sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{2}{\deg_{G_2}(v_1) + \deg_{G_2}(v_2)}\\ &= \frac{1}{4}\sum_{u\in V(G_1)}\left(m_2 \times \frac{1}{\deg_{G_1}(u)}\right) + \frac{1}{4}\sum_{u\in V(G_1)} H(G_2)\\ &= \frac{1}{4}m_2 r(G_1) + \frac{1}{4}n_1 H(G_2).\end{aligned}$$

Also,

$$\begin{aligned}\sum 2 &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1))\\ u_1\in V(G_1),\ u_2\in V(Q(G_1))-V(G_1)}} \frac{2}{\deg(u_1,v) + \deg(u_2,v)}\\ &\quad + \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1))\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} \frac{2}{\deg(u_1,v) + \deg(u_2,v)}\\ &= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1))\\ u_1\in V(G_1),\ u_2\in V(Q(G_1))-V(G_1)}} \frac{2}{\big(\deg_{Q(G_1)}(u_1) + \deg_{G_2}(v)\big) + \deg_{Q(G_1)}(u_2)}\\ &\quad + \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(Q(G_1))\\ u_1,u_2\in V(Q(G_1))-V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)}.\end{aligned}$$

For every $v \in V(G_2)$ and every $u_1u_2 \in E(Q(G_1))$ with $u_1 \in V(G_1)$ and $u_2 \in V(Q(G_1)) - V(G_1)$, we have

$$\text{(3.2)}\qquad \begin{aligned}\frac{2}{\big(\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)\big) + \deg_{G_2}(v)} \leq &\frac{1}{4}\frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)}\\ &+ \frac{1}{2}\frac{1}{\deg_{G_2}(v)},\end{aligned}$$

with equality if and only if $\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2) = \deg_{G_2}(v)$. Thus,

$$\sum_{\substack{u_1u_2\in E(Q(G_1))\\ u_1\in V(G_1),\ u_2\in V(Q(G_1))-V(G_1)}} \frac{2}{\big(\deg_{Q(G_1)}(u_1) + \deg_{G_2}(v)\big) + \deg_{Q(G_1)}(u_2)}$$

$$
\begin{aligned}
&\leq \frac{1}{4} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1),\ u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&\quad + \frac{1}{2} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1),\ u_2 \in V(Q(G_1)) - V(G_1)}} \frac{1}{\deg_{G_2}(v)} \\
&= \frac{1}{4} \sum_{u_1u_1' \in E(G_1)} \left(\frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} + \frac{2}{\deg_{Q(G_1)}(u_2) + \deg_{Q(G_1)}(u_1')} \right) \\
&\quad \text{(where } u_2 \text{ is the vertex inserted into the edge } u_1u_1' \text{ of } G_1) \\
&\quad + \frac{m_1}{\deg_{G_2}(v)} \\
&= \frac{1}{4} \left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right) + \frac{m_1}{\deg_{G_2}(v)}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\sum 2 &\leq \frac{1}{4} \sum_{v \in V(G_2)} \left(H(Q(G_1)) - \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \right) \\
&\quad + \sum_{v \in V(G_2)} \frac{m_1}{\deg_{G_2}(v)} \\
&\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) \\
&\quad - \frac{1}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \\
&\quad + n_2 \sum_{u_1u_2 \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(u_1) + \deg_{Q(G_1)}(u_2)} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) + \frac{3}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{\deg_{Q(G_1)}(w) + \deg_{Q(G_1)}(w')} \\
&= \frac{1}{4} n_2 H(Q(G_1)) + m_1 r(G_2) \\
&\quad + \frac{3}{4} n_2 \sum_{ww' \in E(L(G_1))} \frac{2}{4 + \left(\deg_{L(G_1)}(w) + \deg_{L(G_1)}(w') \right)}.
\end{aligned}
$$

On the other hand,

(3.3)
$$\sum_{ww'\in E(L(G_1))} \frac{2}{4+\Big(\deg_{L(G_1)}(w)+\deg_{L(G_1)}(w')\Big)} \le \frac{1}{4}H(L(G_1))+\frac{1}{16}M_1(G_1)-\frac{1}{8}m_1,$$

with equality if and only if $\deg_{L(G_1)}(w)+\deg_{L(G_1)}(w') = 4$ for all $ww' \in E(L(G_1))$. Therefore,

$$\begin{aligned} H(G_1 +_Q G_2) \le & \frac{1}{4}n_2H(Q(G_1)) + \frac{3}{16}n_2H(L(G_1)) + \frac{1}{4}n_1H(G_2) \\ & + \frac{3}{64}n_2M_1(G) + \frac{1}{4}m_2r(G_1) + m_1r(G_2) - \frac{3}{32}n_2m_1. \end{aligned}$$

Now, suppose that there exist two graphs G_1 and G_2 such that equality holds in above inequality. Then, the inequalities (3.1), (3.2) and (3.3) must be equalities. So, G_1 and G_2 are k_1-regular and k_2-regular graphs, respectively, such that $2k_1 = k_2 + k_2$, $k_1 + (k_1 + k_1) = k_2$ and $4 = (2k_1 - 2) + (2k_1 - 2)$, a contradiction.

This completes the proof. □

Example 3.1. For any $n \ge 4$,

$$\begin{aligned} H(P_n +_Q P_2) &= \Big(1+\frac{1}{3}(n-2)\Big) + 2\left(\frac{22}{15}+\frac{4}{7}(n-3)+\frac{4}{7}+\frac{1}{4}(n-4)\right) \\ &= \frac{533}{105}+\frac{1}{3}(n-2)+\frac{8}{7}(n-3)+\frac{1}{2}(n-4). \end{aligned}$$

Now, we obtain an upper bound for the harmonic index of $G_1 +_T G_2$ in terms of $H(T(G_1))$, $H(L(G_1))$, $H(G_2)$, $M_1(G_1)$, $r(G_1)$ and $r(G_2)$.

Theorem 3.2. *Let G_1 and G_2 be two graphs. Then*

$$\begin{aligned} H(G_1 +_T G_2) < & \frac{1}{4}n_2H(T(G_1)) + \frac{3}{16}n_2H(L(G_1)) + \frac{1}{4}n_1H(G_2) \\ & + \frac{3}{64}n_2M_1(G) + \frac{1}{8}m_2r(G_1) + \frac{5}{4}m_1r(G_2) - \frac{3}{32}n_2m_1, \end{aligned}$$

where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$, $i = 1, 2$.

Proof. Let $\deg(u,v) = \deg_{G_1+_T G_2}(u,v)$ be the degree of a vertex (u,v) in $G_1 +_T G_2$. By definition of harmonic index, we have

$$\begin{aligned} H(G_1 +_T G_2) = & \sum_{u\in V(G_1)}\ \sum_{v_1v_2\in E(G_2)} \frac{2}{\deg(u,v_1)+\deg(u,v_2)} \\ & + \sum_{v\in V(G_2)}\ \sum_{u_1u_2\in E(T(G_1))} \frac{2}{\deg(u_1,v)+\deg(u_2,v)} \\ := & \sum 1 + \sum 2. \end{aligned}$$

Note that

$$\sum 1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{2}{\Big(\deg_{T(G_1)}(u)+\deg_{G_2}(v_1)\Big)+\Big(\deg_{T(G_1)}(u)+\deg_{G_2}(v_2)\Big)}$$

$$= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{2}{2\deg_{T(G_1)}(u)+\Big(\deg_{G_2}(v_1)+\deg_{G_2}(v_2)\Big)}$$

$$= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \frac{2}{4\deg_{G_1}(u)+\Big(\deg_{G_2}(v_1)+\deg_{G_2}(v_2)\Big)}.$$

By similar argument as in the proof of Theorem 3.1, one can show that

$$\sum 1 \leq \frac{1}{8}m_2 r(G_1) + \frac{1}{4}n_1 H(G_2), \tag{3.4}$$

with equality if and only if $4\deg_{G_1}(u) = \deg_{G_2}(v_1)+\deg_{G_2}(v_2)$ for all $u\in V(G_1)$ and all $v_1v_2\in E(G_2)$.

On the other hand,

$$\sum 2 = \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(G_1)}} \frac{2}{\deg(u_1,v)+\deg(u_2,v)}$$

$$+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1\in V(G_1),\ u_2\in V(T(G_1))-V(G_1)}} \frac{2}{\deg(u_1,v)+\deg(u_2,v)}$$

$$+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(T(G_1))-V(G_1)}} \frac{2}{\deg(u_1,v)+\deg(u_2,v)}$$

$$= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(G_1)}} \frac{2}{\Big(\deg_{T(G_1)}(u_1)+\deg_{G_2}(v)\Big)+\Big(\deg_{T(G_1)}(u_2)+\deg_{G_2}(v)\Big)}$$

$$+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1\in V(G_1),\ u_2\in V(T(G_1))-V(G_1)}} \frac{2}{\Big(\deg_{T(G_1)}(u_1)+\deg_{G_2}(v)\Big)+\deg_{T(G_1)}(u_2)}$$

$$+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(T(G_1))-V(G_1)}} \frac{2}{\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)}$$

$$= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1,u_2\in V(G_1)}} \frac{2}{\Big(\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)\Big)+2\deg_{G_2}(v)}$$

$$+ \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1\in V(G_1),\ u_2\in V(T(G_1))-V(G_1)}} \frac{2}{\Big(\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)\Big)+\deg_{G_2}(v)}$$

$$+\sum_{v\in V(G_2)}\sum_{u_1u_2\in E(L(G_1))}\frac{2}{\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)}$$

$$=\sum_{v\in V(G_2)}\sum_{u_1u_2\in E(G_1)}\frac{2}{\Big(2\deg_{G_1}(u_1)+2\deg_{G_1}(u_2)\Big)+2\deg_{G_2}(v)}$$

$$+\sum_{v\in V(G_2)}\sum_{\substack{u_1u_2\in E(T(G_1))\\ u_1\in V(G_1),\ u_2\in V(T(G_1))-V(G_1)}}\frac{2}{\Big(\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)\Big)+\deg_{G_2}(v)}$$

$$+\sum_{v\in V(G_2)}\sum_{u_1u_2\in E(L(G_1))}\frac{2}{\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)}.$$

One can prove that

$$\begin{aligned}\sum 2 \le&\frac{1}{8}n_2H(G_1)+\frac{1}{4}m_1r(G_2)\\ &+\frac{1}{4}n_2H(T(G_1))-\frac{1}{8}n_2H(G_1)+m_1r(G_2)\\ &+\frac{3}{4}n_2\left(\frac{1}{4}H(L(G_1))+\frac{1}{16}M_1(G_1)-\frac{1}{8}m_1\right).\end{aligned}$$

Equality holds in above inequality if and only if

$$\deg_{G_1}(u_1)+\deg_{G_1}(u_2)=\deg_{G_2}(v), \tag{3.5}$$

for all $v\in V(G_2)$ and all $u_1u_2\in E(G_1)$,

$$\deg_{T(G_1)}(u_1)+\deg_{T(G_1)}(u_2)=\deg_{G_2}(v), \tag{3.6}$$

for all $v\in V(G_2)$ and all $u_1u_2\in E(T(G_1))$ with $u_1\in V(G_1)$ and $u_2\in V(T(G_1))-V(G_1)$, and

$$\deg_{L(G_1)}(w)+\deg_{L(G_1)}(w')=4, \tag{3.7}$$

for all $ww'\in E(L(G_1))$. Therefore,

$$\begin{aligned}H(G_1+_T G_2)\le&\frac{1}{4}n_2H(T(G_1))+\frac{3}{16}n_2H(L(G_1))+\frac{1}{4}n_1H(G_2)\\ &+\frac{3}{64}n_2M_1(G)+\frac{1}{8}m_2r(G_1)+\frac{5}{4}m_1r(G_2)-\frac{3}{32}n_2m_1.\end{aligned} \tag{3.8}$$

Moreover, equality holds in (3.8) if and only the inequalities (3.4), (3.5), (3.6) and (3.7) be equalities, i.e., G_1 and G_2 are k_1-regular and k_2-regular graphs, respectively, such that $4k_1=k_2+k_2$, $k_1+k_1=k_2$, $2k_1+(k_1+k_1)=k_2$ and $(2k_1-2)+(2k_1-2)=4$, a contradiction. This completes the proof. □

Example 3.2. For any $n\ge 4$,

$$\begin{aligned}H(P_n+_T P_2)&=\left(\frac{2}{3}+\frac{1}{5}(n-2)\right)+2\left(\frac{1}{2}+\frac{1}{5}(n-3)+\frac{7}{6}+\frac{4}{9}(n-3)+\frac{4}{7}+\frac{1}{4}(n-4)\right)\\ &=\frac{36}{7}+\frac{1}{5}(n-2)+\frac{58}{45}(n-3)+\frac{1}{2}(n-4).\end{aligned}$$

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