ON NONTRIVIAL SOLUTIONS OF HOMOGENEOUS DIRICHLET PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS IN A LAYER

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ABSTRACT. We establish the necessary and sufficient conditions of existence of nontrivial quasi-polynomial solutions of the problem in a layer for homogeneous partial differential equation with $s + 1$ variables of second order in time variable and generally infinite order in other s (spatial) variables with Dirichlet boundary conditions in time. We apply the differential-symbol method for constructing such quasi-polynomial solutions. We also give examples of problems for which we construct other solutions besides of quasi-polynomial ones.

1. INTRODUCTION

The problems with conditions on the boundary of a layer for partial differential equations (PDE) are well-posed or ill-posed boundary value problems depending on the initial data of the problems (see, in particular, [1, 5, 7, 8]). The Dirichlet problem [7] in the layer (strip) $\{(t, x): t \in (0, h), x \in \mathbb{R}\}$, $h > 0$, for wave equation is an example of ill-posed problem. So, investigating the solutions of Dirichlet problem in a layer with homogeneous conditions on the layer boundary for a homogeneous PDE of second order with respect to time and generally infinite order with respect to other (spatial) variables, is an actual task. In the case if nontrivial solutions of the homogeneous problem exist, we use the differential-symbol method [3] for constructing the solutions of boundary value problems for PDE which allow separation of variables.

2. Problem Statement

The aim of this work is investigating, in the layer

$$L_{h,s} \equiv \{(t,x): \ t \in (0,h), \ x \in \mathbb{R}^s\}, \quad h > 0, s \in \mathbb{N} \setminus \{1\},$$

the set of solutions $U = U(t,x)$ of homogeneous Dirichlet problem

$$(2.1) \qquad L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)U \equiv \frac{\partial^2 U}{\partial t^2} + 2a\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t} + b\left(\frac{\partial}{\partial x}\right)U = 0,$$

$$(2.2) \qquad\qquad\qquad U(0,\,x) = U(h,\,x) = 0,$$

where $a\left(\partial/\partial x\right)$, $b\left(\partial/\partial x\right)$ are differential expressions with constant coefficients whose symbols are arbitrary entire functions $a = a(\nu)$, $b = b(\nu)$.

The solution of the problem (2.1), (2.2) is understood as entire function

$$U(t,x) = \sum_{\widehat{k} \in \mathbb{Z}_+^{s+1}} u_{\widehat{k}}\, t^{k_0}\, x^k, \ \ \widehat{k} = (k_0, k) = (k_0, k_1, \ldots, k_s), \quad u_{\widehat{k}} \in \mathbb{C},$$

of variables t and $x = (x_1, \ldots, x_s)$, where $x^k = x_1^{k_1} \ldots x_s^{k_s}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, that satisfies equation (2.1) and Dirichlet conditions (2.2). Thus, the action of differential expression $b\left(\partial/\partial x\right)$ with entire symbol $b\left(\nu\right) = \sum_{k \in \mathbb{Z}_+^s} b_k \nu^k$, $b_k \in \mathbb{C}$, $\nu^k = \nu_1^{k_1} \ldots \nu_s^{k_s}$ for $\nu \in \mathbb{C}^s$ onto function U is understood as

$$b\left(\frac{\partial}{\partial x}\right)U \equiv \sum_{k \in \mathbb{Z}_+^s} b_k \frac{\partial^{|k|}U}{\partial x_1^{k_1} \ldots \partial x_s^{k_s}},$$

where $|k| = k_1 + \ldots + k_s$. The action of differential expression $a\left(\partial/\partial x\right)$ onto $\partial U/\partial t$ is understood in a similar way.

The function that identically equals zero is the trivial solution of problem (2.1), (2.2). We will establish the existence of nontrivial solutions of problem (2.1), (2.2) and construct explicit solutions using the differential-symbol method [3]. Note, equation (2.1) contains as particular case a wave equation, Klein-Gordon-Fock equation, telegraph equation, bicalorical equation, elasticity theory equation. Equation (2.1) also describes the dynamic processes in the longitudinally moving media (see [9]).

3. Main Results

Consider the ordinary differential equation (ODE)

$$(3.1) \qquad L\left(\frac{d}{dt}, \nu\right)T(t,\nu) \equiv \left(\frac{d^2}{dt^2} + 2a(\nu)\frac{d}{dt} + b(\nu)\right)T(t,\nu) = 0$$

with vector-parameter $\nu \in \mathbb{C}^s$, constructed by means of the PDE (2.1).

Denote $D(\nu) = a^2(\nu) - b(\nu)$, where $4D(\nu)$ is the discriminant of the polynomial $L(\cdot, \nu)$, and

$$T_0(t,\nu) = e^{-a(\nu)t}\left\{a(\nu)\frac{\sinh\left[t\sqrt{D(\nu)}\right]}{\sqrt{D(\nu)}} + \cosh\left[t\sqrt{D(\nu)}\right]\right\},$$

$$T_1(t,\nu) = e^{-a(\nu)t}\frac{\sinh\left[t\sqrt{D(\nu)}\right]}{\sqrt{D(\nu)}}.$$

The functions $T_0(t,\nu)$, $T_1(t,\nu)$ are quasi-polynomials in t and form the normal in the point $t = 0$ fundamental system of solutions of equation (3.1).

Since $a(\nu)$, $b(\nu)$ are entire functions, so by the Poincare theorem [10, p. 59], both functions $T_0(t,\nu)$ and $T_1(t,\nu)$ are entire functions of the vector-parameter $\nu \in \mathbb{C}^s$. In particular,

$$T_0(t,\nu_0) = e^{-a(\nu_0)t}\left\{a(\nu_0)t + 1\right\}, \quad T_1(t,\nu_0) = te^{-a(\nu_0)t}$$

for all zeroes ν_0 of the function $D(\nu)$.

According to the differential-symbol method [3, p. 106] we write down the set of formal solutions of the equation (2.1):

$$(3.2) \qquad U(t,x) = \varphi\left(\frac{\partial}{\partial\nu}\right)\left\{T_0(t,\nu)e^{\nu\cdot x}\right\}\Big|_{\nu=O} + \psi\left(\frac{\partial}{\partial\nu}\right)\left\{T_1(t,\nu)e^{\nu\cdot x}\right\}\Big|_{\nu=O},$$

where $\nu \cdot x = \nu_1 x_1 + \ldots + \nu_s x_s$, $O = (0,\ldots,0)$.

The differential expressions $\psi\left(\partial/\partial\nu\right)$, $\varphi\left(\partial/\partial\nu\right)$ are selected so that equality (3.2) determines a solution of the problem (2.1), (2.2).

Initially, the first condition (2.2) is satisfied for arbitrary entire function ψ and $\varphi = 0$. Since $T_0(0,\nu) = 1$, $T_1(0,\nu) = 0$, for all $\nu \in \mathbb{C}^s$,

$$U(0,x) = \varphi\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}\right\}\Big|_{\nu=O} = \varphi(x) = 0.$$

Therefore, the formal solutions of equation (2.1), that satisfy condition $U(0,x) = 0$ can be represented as follows:

$$(3.3) \qquad U(t,x) = \psi\left(\frac{\partial}{\partial\nu}\right)\left\{e^{-a(\nu)t+\nu\cdot x}\frac{\sinh\left[t\sqrt{D(\nu)}\right]}{\sqrt{D(\nu)}}\right\}\Big|_{\nu=O}.$$

Satisfying the second condition $U(h,x) = 0$, for finding the function ψ, we get such identity:

$$(3.4) \qquad \psi\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}T_1(h,\nu)\right\}\Big|_{\nu=O} \equiv 0.$$

Two cases are possible for discriminant $D(\nu)$. The case $D(\nu) \equiv D(0) = \text{const}$ was studied in [6]. The case $D(\nu) \not\equiv D(0)$ will be investigated in this work.

Let's consider the set

$$M = \left\{ \nu \in \mathbb{C}^s \colon \ \eta(\nu) \equiv \frac{\sinh\left[h\sqrt{D(\nu)}\right]}{\sqrt{D(\nu)}} = 0 \right\},$$

which in this case $D(\nu) \not\equiv D(0)$ is the unification of sets of zeros of entire functions $D_k(\nu)$ by the natural parameter k, where $D_k(\nu) \equiv h^2 D(\nu) + k^2\pi^2$. Note that $M \neq \varnothing$ and $M \neq \mathbb{C}^s$.

For $\alpha \in M$, we introduce the following sets [2] of multi-indexes:

$$\Omega_1(\alpha) = \left\{ \omega \in \mathbb{Z}_+^s \colon \ \left(\frac{\partial}{\partial \nu}\right)^\omega \eta(\nu)\bigg|_{\nu=\alpha} \equiv \eta^{(\omega)}(\alpha) \neq 0 \right\},$$

(3.5)
$$\Omega(\alpha) = \left\{ \widetilde{\omega} \in \mathbb{Z}_+^s \colon \ \widetilde{\omega} \geqslant \omega, \ \omega \in \Omega_1(\alpha) \right\},$$

$$\overline{\Omega}(\alpha) = \mathbb{Z}_+^s \backslash \Omega(\alpha).$$

It should be noted that inequalities $q \leqslant \omega$ for multi-indexes $q = (q_1, q_2, \ldots, q_s)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_s)$ are understood as follows: $q_k \leqslant \omega_k$ for $k = 1, 2, \ldots, s$.

The set $\overline{\Omega}(\alpha)$ has the following property: if $\omega \in \overline{\Omega}(\alpha)$ and $O \leqslant q \leqslant \omega$ then $\eta^{(q)}(\alpha) = 0$.

For $q = (q_1, q_2, \ldots, q_s), r = (r_1, r_2, \ldots, r_s) \in \mathbb{Z}_+^s$ and $r \geqslant q$ we will denote $C_r^q = \frac{r!}{q!(r-q)!}$, where $r! = \prod\limits_{k=1}^s r_k!$ ($r_k! = 1 \cdot 2 \cdot \ldots \cdot r_k$ and $0! = 1$).

Let's introduce the following classes of functions:
— K_M is a class of quasi-polynomials of the form

(3.6)
$$v(x) = \sum_{j=1}^m Q_j(x) e^{\alpha_j \cdot x}, \quad m \in \mathbb{N},$$

where $\alpha_1, \ldots, \alpha_m$ are pairwise distinct complex vectors from M and $Q_1(x), \ldots, Q_m(x)$ are arbitrary polynomials of vector x with complex coefficients;
— $K_{\mathbb{C},M}$ is a class of quasi-polynomials

$$f(t,x) = \sum_{j=1}^m \sum_{l=1}^N F_{lj}(t,x) e^{\beta_l t + \alpha_j \cdot x}, \quad m, N \in \mathbb{N},$$

where $F_{11}(t,x), \ldots, F_{Nm}(t,x)$ are polynomials of variables $t, x_1, \ldots, x_s$ with complex coefficients, $\beta_1, \ldots, \beta_N$ are pairwise distinct numbers from $\mathbb{C}$ and pairwise distinct complex vectors $\alpha_1, \ldots, \alpha_m$ belong to M;
— C_τ^l is a class of l times continuously differentiable τ-periodic functions in $\mathbb{R}$, where $l \in \mathbb{Z}_+$, $\tau > 0$;
— $C^l(\mathbb{R})$ is a class of l times continuously differentiable functions in $\mathbb{R}$, where $l \in \mathbb{Z}_+$.

Remark 3.1. For each quasi-polynomial $v(x)$ of the form (3.6), we can put in correspondence the differential expression $v(\partial/\partial \nu)$ of finite order in the class of entire

functions $\Phi(\nu)$ that is

$$v\left(\frac{\partial}{\partial\nu}\right)\Phi(\nu) = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\nu}\right)e^{\alpha_j\cdot\frac{\partial}{\partial\nu}}\Phi(\nu) = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\nu}\right)\Phi(\nu+\alpha_j),$$

in particular,

$$v\left(\frac{\partial}{\partial\nu}\right)\Phi(\nu)\bigg|_{\nu=O} = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\nu}\right)\Phi(\nu)\bigg|_{\nu=\alpha_j} = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\alpha_j}\right)\Phi(\alpha_j).$$

Theorem 3.1. *Let the function $g(x)$ be nontrivial quasi-monomial from the class K_M, i.e.,*

$$(3.7) \qquad\qquad\qquad g(x) = G(x)e^{\alpha\cdot x},$$

where $G(x) = \sum\limits_{0\leqslant|r|\leqslant n} B_r x^r$ is a polynomial ($\sum\limits_{|r|=n} |B_r| > 0$, $n \in \mathbb{Z}_+$), whose coefficients B_r satisfy the homogeneous system of algebraic equations:

$$(3.8) \qquad\qquad \sum_{\substack{r\geqslant q,\,|r|\leqslant n,\\ r-q\in\Omega_1(\alpha)}} B_r C_r^q T_1^{(r-q)}(h,\alpha) = 0, \quad q \in \mathbb{Z}_+^s,\ |q| \leqslant n-1,$$

where $T_1^{(r-q)}(h,\alpha) = \frac{\partial^{r-q}}{\partial\nu^{r-q}}T_1(h,\nu)\Big|_{\nu=\alpha}$. Then the quasi-polynomial

$$(3.9) \qquad U(t,x) = g\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}T_1(t,\nu)\right\}\bigg|_{\nu=O} = G\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}T_1(t,\nu)\right\}\bigg|_{\nu=\alpha}$$

is a nontrivial solution of the problem (2.1), (2.2) *and belongs to the class $K_{\mathbb{C},M}$.*

    Vice versa, if a nontrivial monomial in x variable belongs to the class $K_{\mathbb{C},\mathbb{C}^s}$ and it is a solution of the problem (2.1), (2.2), *then it can be represented as* (3.9), *where $g(x)$ belongs to K_M, has the form* (3.7) *and the coefficients B_r of the polynomial $G(x)$ satisfy the system* (3.8).

Proof. To prove the sufficiency, we note that function (3.9) is a solution of equation (2.1) and satisfies the condition $U(0,x) = 0$. We will show that function of the form

(3.9) satisfies the condition $U(h,x) = 0$:

$$U(h,x) = \sum_{0 \leqslant |r| \leqslant n} B_r \left(\frac{\partial}{\partial \nu}\right)^r \left\{ e^{\nu \cdot x} T_1(h,\nu) \right\} \Big|_{\nu = \alpha}$$

$$= \sum_{0 \leqslant |r| \leqslant n} B_r \sum_{O \leqslant q \leqslant r} C_r^q x^q e^{\alpha \cdot x} T_1^{(r-q)}(h,\alpha)$$

$$= e^{\alpha \cdot x} \sum_{0 \leqslant |r| \leqslant n} \sum_{O \leqslant q \leqslant r} B_r C_r^q x^q T_1^{(r-q)}(h,\alpha)$$

$$= e^{\alpha \cdot x} \sum_{0 \leqslant |q| \leqslant n} \sum_{\substack{r \geqslant q, \\ 0 \leqslant |r| \leqslant n}} B_r C_r^q x^q T_1^{(r-q)}(h,\alpha)$$

$$= e^{\alpha \cdot x} \sum_{0 \leqslant |q| \leqslant n} x^q \sum_{r \geqslant q, |r| \leqslant n} B_r C_r^q T_1^{(r-q)}(h,\alpha).$$

The last expression equals zero if and only if

$$\sum_{\substack{r \geqslant q, \\ 0 \leqslant |r| \leqslant n}} B_r C_r^q T_1^{(r-q)}(h,\alpha) = 0,$$

for all $q \in \mathbb{Z}_+^s$, $|q| \leqslant n$.

Note that for $|q| = n$ all the equations $B_q T_1(h,\alpha) = 0$ of system (3.8) will be identities $0 = 0$ for all B_q. Therefore, in this system of equations, we can assume that $|q| \leqslant n-1$, $n \in \mathbb{N}$. Then for all $q \in \mathbb{Z}_+^s$, $|q| \leqslant n-1$ we obtain

$$\sum_{\substack{r \geqslant q, |r| \leqslant n, \\ r-q \in \Omega_1(\alpha)}} B_r C_r^q T_1^{(r-q)}(h,\alpha) = 0.$$

Besides, function (3.9) is the nontrivial solution, since $\partial U / \partial t \big|_{t=0} = g(x) \not\equiv 0$.

Let's prove the necessity. Let $U(t,x)$ be nontrivial solution of problem (2.1), (2.2), belongs to $K_{\mathbb{C}, \mathbb{C}^s}$ and has monomial form in variable x, i.e.,

$$(3.10) \qquad\qquad U(t,x) = F(t,x) e^{\alpha \cdot x}, \quad \alpha \in \mathbb{C}^s,$$

where $F(t,x) = \sum\limits_{l=1}^{N} F_l(t,x) e^{\beta_l t}$, moreover $F_1(t,x), \ldots, F_N(t,x)$ are polynomials of variables $t, x_1, \ldots, x_s$ with complex coefficients and $\beta_1, \ldots, \beta_N$ are pairwise distinct numbers from $\mathbb{C}$. Setting $t = 0$, we obtain $U(0,x) = F(0,x) e^{\alpha \cdot x} = 0$. Denote $\partial U / \partial t \big|_{t=0} = G(x) e^{\alpha \cdot x}$, where $G(x) = \partial F / \partial t \big|_{t=0}$ is nontrivial polynomial, since $U(t,x)$ is nontrivial solution. Let's construct the solution of equation (2.1) in the form

$$(3.11) \qquad\qquad U(t,x) = G\left(\frac{\partial}{\partial \nu}\right) \left\{ e^{\nu \cdot x} T_1(t,\nu) \right\} \Big|_{\nu = \alpha}.$$

Functions (3.10), (3.11) are two solutions of equation (2.1), that satisfy the same initial conditions $U \big|_{t=0} = 0$, $\partial U / \partial t \big|_{t=0} = G(x) e^{\alpha \cdot x}$. Since the solution of the Cauchy

problem in the class analytical functions is unique, then quasi-polynomial solution of the problem (2.1), (2.2) can be represented in form (3.9), where $g(x) = G(x)e^{\alpha \cdot x}$. Since (3.11) satisfies condition $U(h, x) = 0$, we obtain $G(\partial/\partial\nu)\left\{e^{-a(\nu)h+\nu\cdot x}\eta(\nu)\right\}\Big|_{\nu=\alpha} \equiv 0$ or

$$G\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}T_1(h, \nu)\right\}\Big|_{\nu=\alpha} = e^{\alpha \cdot x}\left(\sum_{|r|=n} B_r x^r T_1(h, \alpha) + \dots\right) \equiv 0,$$

where three dots denote some polynomial of variable x of degree less than n. This implies that $B_r T_1(h, \alpha) \equiv B_r e^{-a(\alpha)h}\eta(\alpha) \equiv 0$ for all $|r| = n$, therefore $\eta(\alpha) \equiv 0$, that is $\alpha \in M$, moreover the coefficients of polynomial $G(x)$ satisfy system (3.8). This completes our proof. $\qquad\square$

Remark 3.2. In system (3.8), the coefficients $B_{\widetilde{r}}$, for which $\widetilde{r} \in \overline{\Omega}(\alpha)$, are neglected, because $B_{\widetilde{r}}\left(\frac{\partial}{\partial\nu}\right)^{\widetilde{r}}\left\{e^{\nu\cdot x}T_1(h, \nu)\right\}\Big|_{\nu=\alpha} \equiv 0$. Those coefficients $B_{\widetilde{r}}$ of the polynomial $G(x)$ are arbitrary. From equality (3.9) we obtain the formula

$$(3.12) \qquad U(t, x) = \sum_{\widetilde{r}\in\overline{\Omega}(\alpha)} B_{\widetilde{r}}\left(\frac{\partial}{\partial\nu}\right)^{\widetilde{r}}\left\{e^{\nu\cdot x}T_1(t, \nu)\right\}\Big|_{\nu=\alpha},$$

which also defines the nontrivial solution of the problem (2.1), (2.2) (see [6]).

Remark 3.3. In the case $\Omega_1(\alpha) = \varnothing$, the function

$$U(t, x) = G\left(\frac{\partial}{\partial\nu}\right)\left\{e^{\nu\cdot x}T_1(t, \nu)\right\}\Big|_{\nu=\alpha}$$

is nontrivial solution of problem (2.1), (2.2) for arbitrary polynomial $G(x)$.

Remark 3.4. We can also prove a similar theorem for the case if $v(x)$ is arbitrary m-term quasi-polynomial of form (3.6) from the class K_M. For this purpose, we can use the principle of linear superposition of solutions.

4. Examples

Example 4.1. Let's consider the Dirichlet problem for the equation of oscillations of a membrane in the layer $L_{\pi,2}$

$$(4.1) \qquad \left[\frac{\partial^2}{\partial t^2} - \Delta_2\right]U \equiv \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right]U(t, x_1, x_2) = 0,$$

$$(4.2) \qquad U(0, x_1, x_2) = U(\pi, x_1, x_2) = 0.$$

Problem (4.1), (4.2) is the problem (2.1), (2.2) in which $a(\nu) = 0$, $b(\nu) = -\nu_1^2-\nu_2^2 \equiv -||\nu||^2$, $D(\nu) = ||\nu||^2$, $h = \pi$, $\sqrt{||\nu||^2} \equiv ||\nu||$,

$$\eta(\nu) = T_1(\pi, \nu) = \frac{\sinh[\pi||\nu||]}{||\nu||}, \quad M = \left\{\nu \in \mathbb{C}^2: \ ||\nu||^2 = -k^2, \ k \in \mathbb{N}\right\}.$$

For $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$ the vectors $\alpha_k = (\pm i\sqrt{\mu^2 + k^2}, \mu)$ belong to set M. Let's calculate:

$$\frac{\partial \eta}{\partial \nu_1}(\alpha_k) = \pm \frac{\pi i (-1)^{k+1}}{k^2}\sqrt{\mu^2 + k^2}, \quad \frac{\partial \eta}{\partial \nu_2}(\alpha_k) = (-1)^{k+1}\frac{\pi \mu}{k^2},$$

$$\frac{\partial^2 \eta}{\partial \nu_1^2}(\alpha_k) = (-1)^k \frac{\pi(2k^2 + 3\mu^2)}{k^4}, \quad \frac{\partial^2 \eta}{\partial \nu_1 \partial \nu_2}(\alpha_k) = \pm \frac{(-1)^{k+1}3\pi i \mu \sqrt{\mu^2 + k^2}}{k^4},$$

$$\frac{\partial^2 \eta}{\partial \nu_2^2}(\alpha_k) = (-1)^{k+1}\frac{\pi(3\mu^2 + k^2)}{k^4}.$$

Let's construct nontrivial solutions of problem (4.1), (4.2), that correspond to the vectors $\alpha_k = (\pm ki, 0)$, $k \in \mathbb{N}$, for $\mu = 0$. We write the sets $\Omega_1(\alpha_k)$, $\Omega(\alpha_k)$, $\overline{\Omega}(\alpha_k)$. Therefore, among multi-indexes $r \in \mathbb{Z}_+^2$ for which $|r| \leqslant 2$ multi-indexes $(1,0)$, $(2,0)$, $(0,2)$ belong to the set $\Omega_1(\alpha_k)$. Then we obtain

$$\{(1,0), (2,0), (0,2), (1,1)\} \subset \Omega(\alpha_k), \quad \overline{\Omega}(\alpha_k) = \{(0,0), (0,1)\}.$$

We find the nontrivial solutions of problem (4.1), (4.2) using the set $\overline{\Omega}(\alpha_k)$ by formula (3.12) (see Remark 3.2):

$$U_k(t, x_1, x_2) = \left(B_k + C_k \frac{\partial}{\partial \nu_2}\right)\left\{e^{\nu \cdot x}\frac{\sinh[t\|\nu\|]}{\|\nu\|}\right\}\Bigg|_{\nu = \alpha_k} = \left(B_k + C_k x_2\right)e^{\pm ikx_1}\frac{\sin[kt]}{k},$$

where B_k, C_k are arbitrary nonzero constants, $k \in \mathbb{N}$, $|B_k|^2 + |C_k|^2 > 0$.

From the solutions $U_k(t, x_1, x_2)$, by the differential-symbol method [3], we can obtain the solution of problem (4.1), (4.2) in the form:

$$(4.3) \qquad U(t, x_1, x_2) = \varphi_1(x_1 + t) - \varphi_1(x_1 - t) + x_2\{\varphi_2(x_1 + t) - \varphi_2(x_1 - t)\},$$

where the functions $\varphi_1(x)$ and $\varphi_2(x)$ belong to the class $C_{2\pi}^2$.

In fact, setting $B_k = 2ik$, $ik = \xi$ in $U_{k1}(t, x_1, x_2) = B_k e^{\pm ikx_1}\frac{\sin[kt]}{k}$ we obtain

$$U(t, \xi, x_1, x_2) = e^{\xi(x_1 + t)} - e^{\xi(x_1 - t)}.$$

Acting onto this function with respect to ξ by arbitrary differential expression with entire symbol $\varphi_1(x)$ and setting $\xi = 0$ after that, we find the solution of problem (4.1), (4.2) in the form

$$U_1(t, x_1, x_2) = \varphi_1(x_1 + t) - \varphi_1(x_1 - t).$$

It is easy to verify that last equality defines solution of the problem (4.1), (4.2) if $\varphi_1 \in C_{2\pi}^2$.

Analogously, from $U_{k2}(t, x_1, x_2) = C_k x_2 e^{\pm ikx_1}\frac{\sin[kt]}{k}$ we can find the solution of problem (4.1), (4.2) in the form:

$$U_2(t, x_1, x_2) = x_2\{\varphi_2(x_1 + t) - \varphi_2(x_1 - t)\},$$

where $\varphi_2 \in C_{2\pi}^2$.

Due to the symmetry of variables x_1 and x_2 in the equation and the Dirichlet conditions for one-parameter series $(0, ki)$, $k \in \mathbb{N}$, of the roots of the equation $||\nu||^2 = -k^2$, from (4.3) we find the following solutions of problem (4.1), (4.2):

$$U(t, x_1, x_2) = \varphi_3(x_2 + t) - \varphi_3(x_2 - t) + x_1 \left\{ \varphi_4(x_2 + t) - \varphi_4(x_2 - t) \right\},$$

where φ_3, $\varphi_4 \in C_{2\pi}^2$.

Note that in recent years, the techniques of differentiation are widely used for integration and integral transforms such as Fourier and Laplace transforms (see [4]).

For vectors $\alpha_k = (\pm ik, 0) \in M$, now we find the solutions of problem (4.1), (4.2) due to Theorem 3.1 according to subsets $\Omega(\alpha_k)$ and $\Omega_1(\alpha_k)$, moreover the solutions are constructed along the quasi-polynomial in the form

$$g(x_1, x_2) = \left\{ B_{(0,0)} + B_{(0,1)}x_2 + B_{(1,0)}x_1 + B_{(2,0)}x_1^2 + B_{(0,2)}x_2^2 + B_{(1,1)}x_1 x_2 \right\} e^{\pm ikx_1},$$

where $B_{(1,0)}$, $B_{(2,0)}$, $B_{(0,2)}$, $B_{(1,1)}$ are unknown coefficients, $B_{(0,0)}$ and $B_{(0,1)}$ are arbitrary nonzero coefficients.

We form the system of equations (3.8):

$$\begin{cases} B_{(1,0)} C_{(1,0)}^{(0,0)} \dfrac{\partial \eta}{\partial \nu_1}(\alpha_k) + B_{(2,0)} C_{(2,0)}^{(0,0)} \dfrac{\partial^2 \eta}{\partial \nu_1^2}(\alpha_k) + B_{(0,2)} C_{(0,2)}^{(0,0)} \dfrac{\partial^2 \eta}{\partial \nu_2^2}(\alpha_k) = 0, \\[2mm] B_{(2,0)} C_{(2,0)}^{(1,0)} \dfrac{\partial \eta}{\partial \nu_1}(\alpha_k) = 0, \\[2mm] B_{(1,1)} C_{(1,1)}^{(0,1)} \dfrac{\partial \eta}{\partial \nu_1}(\alpha_k) = 0. \end{cases}$$

Solving the obtained system, we find $B_{(1,0)}(-1)^{k+1} \frac{\pm \pi i}{k} + B_{(0,2)}(-1)^{k+1} \frac{\pi}{k^2} = 0$, or $B_{(0,2)} = \mp ik B_{(1,0)}$ and $B_{(2,0)} = B_{(1,1)} = 0$. Thus,

$$g(x_1, x_2) = \left\{ B_{(0,0)} + B_{(0,1)}x_2 + B_{(1,0)}x_1 \mp ki B_{(1,0)}x_2^2 \right\} e^{\pm ikx_1}$$

$$= \left\{ B_{(0,0)} + B_{(0,1)}x_2 + B_{(1,0)} \left\{ x_1 \mp ik x_2^2 \right\} \right\} e^{\pm ikx_1}.$$

Besides solutions (4.3), we obtain solutions of problem (4.1), (4.2) by formula (3.9), which correspond to the quasi-polynomial $g(x_1, x_2) = B_{(1,0)} \left\{ x_1 \mp ik x_2^2 \right\} e^{\pm ikx_1}$:

$$U_{k3}(t, x_1, x_2) = A_k \left\{ \frac{\partial}{\partial \nu_1} \mp ik \frac{\partial^2}{\partial \nu_2^2} \right\} \left. \left(e^{\nu \cdot x} \frac{\sinh [t||\nu||]}{||\nu||} \right) \right|_{\nu = \alpha_k}$$

$$= A_k \left\{ x_1 \mp ik x_2^2 \right\} e^{\pm ikx_1} \frac{\sin[kt]}{k},$$

where A_k are arbitrary nonzero coefficients.

From those solutions, by the differential-symbol method, we find the solution of problem (4.1), (4.2) in the form:

$$U(t, x_1, x_2) = x_1 \left[\varphi_5(x_2 + t) - \varphi_5(x_2 - t) \right] - x_2^2 \left[\varphi_5'(x_2 + t) - \varphi_5'(x_2 - t) \right],$$

where $\varphi_5 \in C_{2\pi}^3$.

Due to the symmetry of variables x_1 and x_2 in the equation and Dirichlet conditions we shall write the solution of the problem (4.1), (4.2) that correspond to multi-indexes $(0, \pm ki)$:

$$U(t, x_1, x_2) = x_2\Big[\varphi_6(x_1 + t) - \varphi_6(x_1 - t)\Big] - x_1^2\Big[\varphi_6'(x_1 + t) - \varphi_6'(x_1 - t)\Big],$$

where $\varphi_6 \in C_{2\pi}^3$.

Now consider the vectors $\beta_k = (\pm i\sqrt{\mu^2 + k^2},\, \mu)$ from M for $k \in \mathbb{N}$, $\mu \neq 0$, $\mu \neq \pm ik$. In this case only multi-index $(0, 0)$ belongs to the set $\overline{\Omega}(\beta_k)$, then, according to the Remark 3.2 we find such nontrivial solutions of problem (4.1), (4.2):

$$(4.4) \quad U_{k4}(t, \mu, x_1, x_2) = A_{0k}\left\{ e^{\nu \cdot x}\frac{\sinh[t\|\nu\|]}{\|\nu\|}\right\}\Bigg|_{\nu = \beta_k} = A_{0k}e^{\pm i\sqrt{\mu^2+k^2}x_1 + \mu x_2}\frac{\sin[kt]}{k},$$

where A_{0k} are arbitrary nonzero coefficients, $k \in \mathbb{N}$.

Now we find the nontrivial solutions of problem (4.1), (4.2) by Theorem 3.1 using quasi-polynomial of the form

$$g_k(x_1, x_2) = \Big\{ A_k x_1 + B_k x_2 \Big\} e^{\pm i\sqrt{\mu^2+k^2}x_1 + \mu x_2},$$

where A_k, B_k are unknown coefficients.

Let's write system (3.8), which is one equation:

$$A_k C_{(1,0)}^{(0,0)}\frac{\partial \eta}{\partial \nu_1}(\beta_k) + B_k C_{(0,1)}^{(0,0)}\frac{\partial \eta}{\partial \nu_2}(\beta_k) = 0,$$

that is $\pm A_k \frac{\pi i(-1)^{k+1}}{k^2}\sqrt{\mu^2 + k^2} + B_k \frac{\pi \mu}{k^2}(-1)^{k+1} = 0$, from which $A_k = \pm \frac{\mu i}{\sqrt{\mu^2 + k^2}}B_k$.

Therefore,

$$g_k(x_1, x_2) = B_k\left\{ \pm \frac{\mu i}{\sqrt{\mu^2 + k^2}}x_1 + x_2\right\} e^{\pm i\sqrt{\mu^2+k^2}x_1 + \mu x_2},$$

where B_k are arbitrary nonzero coefficients, $k \in \mathbb{N}$.

According to Theorem 3.1 we find

$$(4.5) \qquad U_{k5}(t, \mu, x_1, x_2) = B_k\left\{ \pm\frac{\mu i}{\sqrt{\mu^2 + k^2}}\frac{\partial}{\partial \nu_1} + \frac{\partial}{\partial \nu_2}\right\}\left(e^{\nu \cdot x}\frac{\sinh[t\|\nu\|]}{\|\nu\|}\right)\Bigg|_{\nu = \beta_k}$$

$$= B_k\left\{ \pm \frac{\mu i}{\sqrt{\mu^2 + k^2}}x_1 + x_2\right\} e^{\pm i\sqrt{\mu^2+k^2}x_1 + \mu x_2}\frac{\sin[kt]}{k}.$$

Note that solutions (4.5) may be obtained from (4.4) by differentiating with respect to μ.

For one-parameter set $\big((4 + 3i)m, (2 - 6i)m\big)$, $m \in \mathbb{N}$, of roots of equations $\|\nu\|^2 = -k^2$, $k = 5m$, $m \in \mathbb{N}$, from (4.4) we obtain the following solutions of problem (4.1), (4.2):

$$U_m(t, x_1, x_2) = A_{0m}e^{(4+3i)mx_1 + (2-6i)mx_2}\sin[5mt], \quad m \in \mathbb{N}.$$

According to the differential-symbol method, from those solutions we obtain such solution of problem (4.1), (4.2):

$$U(t, x_1, x_2) = \varphi_7\Big((3 - 4i)x_1 + (-6 - 2i)x_2 + 5t\Big) - \varphi_7\Big((3 - 4i)x_1 + (-6 - 2i)x_2 - 5t\Big),$$

where $\varphi_7 \in C_{10\pi}^2$.

Example 4.2. Let's consider the finding of nontrivial solutions of Dirichlet problem in the layer $L_{\pi,2}$

(4.6)
$$\left[\frac{\partial^2}{\partial t^2} + 2\frac{\partial^3}{\partial t \partial x_1 \partial x_2} + 4\right] U(t, x_1, x_2) = 0,$$

(4.7)
$$U(0, x_1, x_2) = U(\pi, x_1, x_2) = 0.$$

Consider problem (4.6), (4.7) as problem (2.1), (2.2) in which $a(\nu) = \nu_1\nu_2$, $b(\nu) = 4$, $h = \pi$,

$$\eta(\nu) = \frac{\sinh\left[\pi\sqrt{\nu_1^2\nu_2^2 - 4}\right]}{\sqrt{\nu_1^2\nu_2^2 - 4}}, \quad T_1(\pi, \nu) = e^{-\pi\nu_1\nu_2}\frac{\sinh\left[\pi\sqrt{\nu_1^2\nu_2^2 - 4}\right]}{\sqrt{\nu_1^2\nu_2^2 - 4}}.$$

The set M has the form $M = \{\nu \in \mathbb{C}^2 \colon \nu_1^2\nu_2^2 - 4 = -k^2, k \in \mathbb{N}\}$, and, obviously, contains the vector $\alpha = (0, 0)$ when $k = 2$. It is easy to verify that $\overline{\Omega}(\alpha) = \mathbb{Z}_+^2$. According to Remark 3.3 for an arbitrary polynomial $Q(x_1, x_2)$, we find the solutions of problem (4.6), (4.7) as follows:

$$U(t, x_1, x_2) = Q\left(\frac{\partial}{\partial \nu_1}, \frac{\partial}{\partial \nu_2}\right)\left\{e^{-t\nu_1\nu_2 + \nu_1 x_1 + \nu_2 x_2}\frac{\sinh\left[t\sqrt{\nu_1^2\nu_2^2 - 4}\right]}{\sqrt{\nu_1^2\nu_2^2 - 4}}\right\}\Bigg|_{\nu_1 = 0, \nu_2 = 0}.$$

In particular, if $Q(x_1, x_2) = B_{(0,0)} + B_{(1,0)}x_1 + B_{(0,1)}x_2 + B_{(2,0)}x_1^2 + B_{(1,1)}x_1x_2 + B_{(2,0)}x_2^2$ is arbitrary second degree polynomial then solution of problem (4.6), (4.7) can be represented in the form

$$U(t, x_1, x_2) = \Big(B_{(0,0)} + B_{(1,0)}x_1 + B_{(0,1)}x_2 + B_{(2,0)}x_1^2$$

$$+ B_{(1,1)}(x_1x_2 - t) + B_{(2,0)}x_2^2\Big)\frac{\sin[2t]}{2}$$

$$= \Big\{Q(x_1, x_2) - B_{(1,1)}t\Big\}\frac{\sin[2t]}{2}.$$

In the case $k = 2$, the vector $\alpha = (\mu, 0)$ for $\mu \in \mathbb{C}\backslash\{0\}$ also belongs to set M. Let's construct nontrivial solutions of problem (4.6), (4.7), that correspond to this vector. Let's calculate:

$$\frac{\partial \eta}{\partial \nu_1}(\mu, 0) = 0, \quad \frac{\partial \eta}{\partial \nu_2}(\mu, 0) = 0,$$

$$\frac{\partial^2 \eta}{\partial \nu_1^2}(\mu, 0) = 0, \quad \frac{\partial^2 \eta}{\partial \nu_1 \partial \nu_2}(\mu, 0) = 0, \quad \frac{\partial^2 \eta}{\partial \nu_2^2}(\mu, 0) = \frac{(-1)^{k+1}\pi i}{2}\mu^2.$$

Therefore, among multi-indexes $r \in \mathbb{Z}_+^2$, for which $|r| \leq 2$, only the multi-index $(0,2)$ belongs to the set $\Omega_1(\alpha)$. Then we obtain

$$\{(0,2)\} \subset \Omega_1(\alpha), \quad \{(0,0),(0,1),(1,0),(2,0),(1,1)\} \subset \overline{\Omega}(\alpha).$$

For the vector $\alpha = (\mu, 0)$ we find the nontrivial solutions of problem (4.6), (4.7) according to Theorem 3.1 using quasi-polynomial of the form:

$$g(x_1, x_2) = \left\{ B_{(0,0)} + B_{(0,1)}x_2 + B_{(1,0)}x_1 + B_{(2,0)}x_1^2 + B_{(0,2)}x_2^2 + B_{(1,1)}x_1x_2 \right\} e^{\mu x_1},$$

where $B_{(0,0)}$, $B_{(1,0)}$, $B_{(0,1)}$, $B_{(2,0)}$, $B_{(1,1)}$ are arbitrary coefficients, $B_{(0,2)}$ is unknown coefficient. From system (3.8), we obtain $B_{(0,2)} = 0$. Thus,

$$g(x_1, x_2) = \left\{ B_{(0,0)} + B_{(0,1)}x_2 + B_{(1,0)}x_1 + B_{(2,0)}x_1^2 + B_{(1,1)}x_1x_2 \right\} e^{\mu x_1}.$$

By formula (3.9), we obtain:

$$
\begin{aligned}
U(t, \mu, x_1, x_2) =& g\left(\frac{\partial}{\partial\nu_1}, \frac{\partial}{\partial\nu_2}\right) \left\{ e^{-\nu_1\nu_2 t + \nu\cdot x} \frac{\sinh\left[t\sqrt{\nu_1^2\nu_2^2 - 4}\right]}{\sqrt{\nu_1^2\nu_2^2 - 4}} \right\} \Bigg|_{(\mu,0)} \\
=& \Big\{ B_{(0,0)} + B_{(0,1)}(x_2 - t\mu) + B_{(1,0)}x_1 + B_{(2,0)}x_1^2 \\
& + B_{(1,1)}(x_1x_2 - t\mu x_1 - t) \Big\} e^{\mu x_1} \frac{\sin[2t]}{2},
\end{aligned}
$$

where $B_{(0,0)}$, $B_{(1,0)}$, $B_{(0,1)}$, $B_{(2,0)}$, $B_{(1,1)}$ are arbitrary (one of them is nonzero) coefficients.

From those solutions by differential-symbol method and due to the symmetry of variables x_1 and x_2 in the equation and the Dirichlet conditions, we obtain the solutions of problem (4.6), (4.7) in such forms:

$$U(t, x_1, x_2) = \left\{ \varphi_1(x_1) + x_2\varphi_2(x_1) - t\varphi_2'(x_1) + (x_1x_2 - t)\varphi_3(x_1) - tx_1\varphi_3'(x_1) \right\} \sin[2t],$$

$$U(t, x_1, x_2) = \left\{ \varphi_4(x_2) + x_1\varphi_5(x_2) - t\varphi_5'(x_2) + (x_1x_2 - t)\varphi_6(x_2) - tx_2\varphi_6'(x_2) \right\} \sin[2t],$$

where φ_1, $\varphi_4 \in C^2(\mathbb{R})$, φ_2, φ_3, φ_5, $\varphi_6 \in C^3(\mathbb{R})$.

Consider now the vector $\alpha_k = \left(\mu, \pm\frac{\sqrt{4-k^2}}{\mu}\right)$ from M, when $\mu \neq 0$ and $k \in \mathbb{N}\backslash\{2\}$. Let's calculate

$$\frac{\partial\eta}{\partial\nu_1}(\alpha_k) = (-1)^{k+1}\frac{\pi(4-k^2)}{\mu k^2} \neq 0, \quad \frac{\partial\eta}{\partial\nu_2}(\alpha_k) = \pm(-1)^{k+1}\frac{\pi\mu\sqrt{4-k^2}}{k^2} \neq 0.$$

For vectors α_k the set $\overline{\Omega}(\alpha_k)$ only consists of multi-index $(0,0)$. From Remark 3.2 we obtain

$$U_k(t, \mu, \nu_1, \nu_2) = B_k\left\{ T_1(t, \nu)e^{\nu\cdot x} \right\}\Bigg|_{\nu=\alpha_k} = B_k e^{\mp\sqrt{4-k^2}t + \mu x_1 \pm \frac{\sqrt{4-k^2}}{\mu}x_2} \frac{\sin[kt]}{k},$$

where B_k are arbitrary nonzero constants.

Another solutions of problem (4.6), (4.7) can be constructed by differentiating the solution $U_k(t, \mu, \nu_1, \nu_2)$ with respect to parameter μ.

Let's give an example of Dirichlet problem for the equation with infinite order differential expressions.

Example 4.3. Let's find nontrivial solutions of the Dirichlet problem in the layer $L_{1,2}$ for differential-functional equation

$$(4.8) \qquad \frac{\partial^2 U(t, x_1, x_2)}{\partial t^2} = U(t, x_1 + 1, x_2 - 1) - (\pi^2 + 1)U(t, x_1, x_2),$$

$$(4.9) \qquad U(0, x_1, x_2) = U(1, x_1, x_2) = 0.$$

The differential-functional equation is represented in the form

$$\left[\frac{\partial^2}{\partial t^2} - e^{\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}} + \pi^2 + 1 \right] U(t, x_1, x_2) = 0.$$

Consider the problem (4.8), (4.9) as problem (2.1), (2.2) in which $b(\nu) = -e^{\nu_1 - \nu_2} + \pi^2 + 1$, $a(\nu) = 0$, $D(\nu) = e^{\nu_1 - \nu_2} - \pi^2 - 1$,

$$\eta(\nu_1, \nu_2) = T_1(1, \nu_1, \nu_2) = \frac{\sinh \sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1}}{\sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1}},$$

$$M = \left\{ \nu \in \mathbb{C}^2 : \quad e^{\nu_1 - \nu_2} - \pi^2 - 1 = -\pi^2 k^2, \ k \in \mathbb{N} \right\}.$$

The vectors $\alpha_{1m} = (\mu + 2\pi mi, \mu)$ for $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}$ belong to the set M in the case $k = 1$. We calculate

$$\frac{\partial \eta}{\partial \nu_1}(\alpha_{1m}) = \frac{1}{2\pi^2}, \quad \frac{\partial \eta}{\partial \nu_2}(\alpha_{1m}) = -\frac{1}{2\pi^2}.$$

Therefore, the set $\overline{\Omega}(\alpha_{1m})$ only contains of the multi-index $(0, 0)$. According to Remark 3.2, we find

$$U_{1m}(t, \mu, x_1, x_2) = B_{1m} \left\{ e^{\nu_1 x_1 + \nu_2 x_2} \frac{\sinh \left[t\sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1} \right]}{\sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1}} \right\} \Bigg|_{\nu = \alpha_{1m}}$$

$$= B_{1m} e^{\mu(x_1 + x_2)} \frac{\sin[\pi t]}{\pi},$$

where B_{1m} are arbitrary nonzero constants.

By the differential-symbol method [3], we obtain the solutions of the problem in the following form:

$$U(t, x_1, x_2) = \varphi_1(x_1 + x_2) \sin[\pi t],$$

where $\varphi_1 \in C(\mathbb{R})$.

For vectors $\alpha_{km} = (\mu + \ln(\pi^2 k^2 - \pi^2 - 1) + (2m+1)\pi i, \ \mu)$ from M, where $\mu \in \mathbb{C}$, $m \in \mathbb{Z}$, $k \in \mathbb{N}\backslash\{1\}$, we calculate:

$$\frac{\partial \eta}{\partial \nu_1}(\alpha_{km}) = \frac{(-1)^{k+1}(\pi^2 + 1 - \pi^2 k^2)}{2\pi^2 k^2} \neq 0,$$

$$\frac{\partial \eta}{\partial \nu_2}(\alpha_{km}) = \frac{(-1)^{k}(\pi^2 + 1 - \pi^2 k^2)}{2\pi^2 k^2} \neq 0.$$

Due to Remark 3.2, we find

$$U_{km}(t, \mu, x_1, x_2) = C_{km}\left\{ e^{\nu_1 x_1 + \nu_2 x_2} \frac{\sinh\left[t\sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1}\right]}{\sqrt{e^{\nu_1 - \nu_2} - \pi^2 - 1}} \right\}\Bigg|_{\nu = \alpha_{km}}$$

$$= C_{km} e^{(\mu + \ln(\pi^2 k^2 - \pi^2 - 1) + \pi i)x_1 + \mu x_2} \frac{\sin[\pi k t]}{\pi k},$$

where C_{km} are arbitrary nonzero constants.

Using differential-symbol method [3], we obtain the solutions of problem (4.8), (4.9) in the following form:

$$U_k(t, x_1, x_2) = \varphi_k(x_1 + x_2) e^{(\ln(\pi^2 k^2 - \pi^2 - 1) + \pi i)x_1} \sin[\pi k t], \quad k \in \mathbb{N}\backslash\{1\},$$

where $\varphi_k \in C(\mathbb{R})$. ▲

5. Conclusions

In the present work, we have studied the existence of nontrivial solutions of Dirichlet problem in a layer in the case when the discriminant of characteristic polynomial does not equal to constant. The differential-symbol method is used for constructing quasi-polynomial and other solutions. We also have shown by examples the results of investigating the null space of Dirichlet problem in a layer.

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