A REPRESENTATION FOR DERANGEMENT NUMBERS IN TERMS OF A TRIDIAGONAL DETERMINANT

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ABSTRACT. In the paper, the authors discover a representation for the derangement numbers in terms of a tridiagonal determinant. From the determinantal representation, the authors recover several identities and recurrence relations of the derangement numbers.

1. INTRODUCTION

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size n is called the derangement number and usually denoted by $!n$. The subfactorial function is a map from n to $!n$. The problem of counting derangements was first considered in 1708 and solved in 1713 by Pierre Raymond de Montmort, as did Nicholas Bernoulli at about the same time. The first eleven derangement numbers $!n$ for $0 \le n \le 10$ are

$$(1.1) \qquad 1, \quad 0, \quad 1, \quad 2, \quad 9, \quad 44, \quad 265, \quad 1854, \quad 14833, \quad 133496, \quad 1334961.$$

One of several expressions for computing the derangement numbers $!n$ is

$$(1.2) \qquad !n = n! \sum_{\ell=0}^{n} \frac{(-1)^{\ell}}{\ell!}.$$

The derangement numbers $!n$ have an exponential generating function

$$(1.3) \qquad D(x) = \frac{e^{-x}}{1-x} = \sum_{n=0}^{\infty} !n \frac{x^n}{n!}.$$

The derangement numbers $!n$ arise naturally in many different contexts. More generally, the number of derangements in various families of transitive permutation groups has been studied extensively in recent years. For more and detailed information on the derangement numbers $!n$, please refer to $[1, 2, 12, 13]$ and plenty of references therein.

In $[10]$, by studying the equation

$$D(-x) = \frac{e^x}{1+x} = \sum_{n=0}^{\infty} (-1)^n !n \frac{x^n}{n!},$$

the authors corrected and recovered two determinantal representations for derangement numbers $!n$. For more information, please refer to $[5, 8]$ and closely related references therein.

The aim of this paper is, by computing the nth derivative of the exponential generating function $D(x)$, to find a representation for the derangement numbers $!n$ in terms of a tridiagonal determinant.

Our main result can be summarized up as the following theorem.

Theorem 1.1. *For $n \in \{0\} \cup \mathbb{N}$, the derangement numbers $!n$ can be represented by a tridiagonal $(n+1) \times (n+1)$ determinant*

$$(1.4) \qquad !n = - \begin{vmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -(n-2) & n-2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1 \end{vmatrix}$$

$$= -|e_{ij}|_{(n+1)\times(n+1)},$$

where

$$e_{ij} = \begin{cases} 1, & i-j = -1, \\ i-2, & i-j = 0, \\ 2-i, & i-j = 1, \\ 0, & i-j \neq 0, \pm 1. \end{cases}$$

By virtue of the determinantal representation (1.4), we recover several identities and recurrence relations of the derangement numbers $!n$ in the form of remarks in the final section of this paper.

2. A Lemma

For supplying a concise proof for Theorem 1.1, we need the following lemma which was concluded in [6, Section 2.2, p. 849], [7, p. 94], [9, Remark 6], and [11, Lemma 2.1] from [3, p. 40, Exercise 5].

Lemma 2.1. *Let $u(x)$ and $v(x) \neq 0$ be differentiable functions, let $U_{(n+1) \times 1}(x)$ be an $(n+1) \times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \leq k \leq n+1$, let $V_{(n+1) \times n}(x)$ be an $(n+1) \times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \dbinom{i-1}{j-1} v^{(i-j)}(x), & i-j \geq 0, \\ 0, & i-j < 0, \end{cases}$$

for $1 \leq i \leq n+1$ and $1 \leq j \leq n$, and let $\left| W_{(n+1) \times (n+1)}(x) \right|$ denote the determinant of the $(n+1) \times (n+1)$ matrix

$$W_{(n+1) \times (n+1)}(x) = \begin{bmatrix} U_{(n+1) \times 1}(x) & V_{(n+1) \times n}(x) \end{bmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$(2.1) \qquad \frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{\left| W_{(n+1) \times (n+1)}(x) \right|}{v^{n+1}(x)}.$$

3. Proofs of Theorem 1.1

Now we are in a position to provide a concise proof for Theorem 1.1.

Applying $u(x) = e^{-x}$ and $v(x) = 1 - x$ in Lemma 2.1 gives

$$u_{k,1} = (e^{-x})^{(k-1)} = (-1)^{k-1} e^{-x} \to (-1)^{k-1},$$

for $1 \leq k \leq n+1$ as $x \to 0$ and

$$v_{i,j} = \binom{i-1}{j-1}(1-x)^{(i-j)} = \begin{cases} \dbinom{i-1}{j-1}(1-x), & i-j = 0, \\ -\dbinom{i-1}{j-1}, & i-j = 1, \\ 0, & i-j \neq 0, 1, \end{cases}$$

$$= \begin{cases} 1-x, & i-j = 0, \\ 1-i, & i-j = 1, \\ 0, & i-j \neq 0, 1, \end{cases} \to \begin{cases} 1, & i-j = 0, \\ 1-i, & i-j = 1, \\ 0, & i-j \neq 0, 1, \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$ as $x \to 0$. Consequently, by virtue of the formula (2.1), we have

$$\frac{\mathrm{d}^n D(x)}{\mathrm{d}\, x^n} = \frac{(-1)^n}{(1-x)^{n+1}} \begin{vmatrix} e^{-x} & 1-x & 0 & \cdots & 0 & 0 \\ -e^{-x} & -1 & 1-x & \cdots & 0 & 0 \\ e^{-x} & 0 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2}e^{-x} & 0 & 0 & \cdots & 1-x & 0 \\ (-1)^{n-1}e^{-x} & 0 & 0 & \cdots & -(n-1) & 1-x \\ (-1)^n e^{-x} & 0 & 0 & \cdots & 0 & -n \end{vmatrix}$$

$$\to (-1)^n \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -2 & 1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2} & 0 & 0 & 0 & \cdots & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & 0 & \cdots & -(n-1) & 1 \\ (-1)^n & 0 & 0 & 0 & \cdots & 0 & -n \end{vmatrix}$$

as $x \to 0$ for $n \ge 0$. Therefore, since $D(x)$ is a generating function of $!n$, as showed in (1.3), we obtain

$$(3.1) \quad !n = \lim_{x \to 0} \frac{\mathrm{d}^n D(x)}{\mathrm{d}\, x^n} = (-1)^n \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -2 & 1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{n-2} & 0 & 0 & 0 & \cdots & 1 & 0 \\ (-1)^{n-1} & 0 & 0 & 0 & \cdots & -(n-1) & 1 \\ (-1)^n & 0 & 0 & 0 & \cdots & 0 & -n \end{vmatrix}.$$

Adding the nth row to the $(n+1)$th row, then the $(n-1)$th row to the nth row, $\ldots$, then the 1st row to the 2nd row of the above determinant yield

$$!n = (-1)^n \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -2 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2-n & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1-n & 1-n \end{vmatrix}_{(n+1) \times (n+1)}.$$

Using -1 to multiply all even rows and all odd columns of the above determinant immediately results in (1.4). The proof of Theorem 1.1 is complete.

4. REMARKS

After supplying a concise proof for Theorem 1.1, we show the significance of the determinantal representation (1.4) by listing several remarks below.

Remark 4.1. By expanding the determinant in (3.1) according to the first row or the first column consecutively, we can recover the expression (1.2).

Remark 4.2. By expanding the determinant in (3.1) according to the nth row or according to the nth column, we can easily recover the recurrence relation

$$!n = (-1)^n + n \times !(n-1), \quad n \in \mathbb{N}.$$

Remark 4.3. By expanding the determinant in (1.4) according to the nth row or according to the nth column, we can recover the recurrence relation

$$!n = (n-1)[!(n-1)+!(n-2)], \quad n \geq 2.$$

Remark 4.4. By expanding the determinant in (1.4) according to the first row or according to the first column, we can obtain

$$!n = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & n-2 & 1 \\ 0 & 0 & 0 & \cdots & -(n-1) & n-1 \end{vmatrix}_{n \times n}, \quad n \in \mathbb{N}.$$

Further expanding the above determinant according to the first row or according to the first column, we can obtain

$$(4.1) \quad !n = \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -3 & 3 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -4 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -(n-2) & n-2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1 \end{vmatrix}_{(n-2) \times (n-2)}, \quad n \geq 3.$$

Remark 4.5. In general, for all integers k with $k \leq n - 1$, define the tridiagonal determinant

$$
D_n(k) = \begin{vmatrix}
k & 1 & 0 & \cdots & 0 & 0 & 0 \\
-(k+1) & k+1 & 1 & \cdots & 0 & 0 & 0 \\
0 & -(k+1) & k+2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\
0 & 0 & 0 & \cdots & -(n-2) & n-2 & 1 \\
0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1
\end{vmatrix}.
$$

It is clear that $D_n(-1) = -!n$ for $n \geq 0$, $D_n(0) = !n$ for $n \geq 1$, and $D_n(2) = !n$ for $n \geq 3$. Moreover, the determinant $D_n(k)$ satisfies

$$
D_n(n-1) = n - 1,
$$

$$
D_n(n-2) = \begin{vmatrix} n-2 & 1 \\ -(n-1) & n-1 \end{vmatrix} = (n-1)^2,
$$

$$
D_n(n-3) = \begin{vmatrix} n-3 & 1 & 0 \\ -(n-2) & n-2 & 1 \\ 0 & -(n-1) & n-1 \end{vmatrix} = (n-1)(n^2 - 3n + 1),
$$

$$
D_n(n-4) = \begin{vmatrix} n-4 & 1 & 0 & 0 \\ 3-n & n-3 & 1 & 0 \\ 0 & 2-n & n-2 & 1 \\ 0 & 0 & 1-n & n-1 \end{vmatrix} = (n-1)(n^3 - 6n^2 + 9n - 1),
$$

and the recurrence relation

$$
D_n(k) = k D_n(k+1) + (k+1) D_n(k+2), \quad k \leq n - 1.
$$

Remark 4.6. Directly combining (1.2) with (4.1) arrives at

$$
\begin{vmatrix}
2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-3 & 3 & 1 & \cdots & 0 & 0 & 0 \\
0 & -4 & 4 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\
0 & 0 & 0 & \cdots & -(n-2) & n-2 & 1 \\
0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1
\end{vmatrix}_{(n-2)\times(n-2)} = n! \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!},
$$

for $n \geq 3$.

Remark 4.7. From the first proof of Theorem 1.1, we can conclude that the nth derivative of the generating function $D(x)$ can be computed by

$$\frac{\mathrm{d}^n D(x)}{\mathrm{d}\, x^n} = \frac{(-1)^n e^{-x}}{(1-x)^{n+1}} \begin{vmatrix} -x & 1-x & 0 & \cdots & 0 & 0 \\ -1 & -1-x & 1-x & \cdots & 0 & 0 \\ 0 & -2 & -2-x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-x & 0 \\ 0 & 0 & 0 & \cdots & 2-n-x & 1-x \\ 0 & 0 & 0 & \cdots & 1-n & 1-n-x \end{vmatrix}$$

$$= \frac{(-1)^n e^{-x}}{(1-x)^{n+1}} |e_{ij}(x)|_{n \times n},$$

where $n \in \mathbb{N}$ and

$$e_{ij}(x) = \begin{cases} 1-x, & i-j = -1, \\ 1-i-x, & i-j = 0, \\ 1-i, & i-j = 1, \\ 0, & i-j \neq 0, \pm 1. \end{cases}$$

Remark 4.8. In [4], it was deduced that

$$(4.2) \qquad !(r+1) = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 3 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & r-1 & r \\ 0 & 0 & 0 & 0 & \cdots & -1 & r \end{vmatrix},$$

for $r \in \mathbb{N}$. By this determinantal expression for the derangement numbers $!(r+1)$, we figure out that $!2 = 1$, $!3 = 2$, $!4 = 6$, and $!5 = 24$. But, the latter two values do not coincide with their corresponding ones in (1.1). This means that the expression (4.2) is wrong.

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