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PERIODIC SOLUTIONS FOR IMPULSIVE NEUTRAL DYNAMIC EQUATIONS WITH INFINITE DELAY ON TIME SCALES

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Abstract. Let $\mathbb{T}$ be a periodic time scale. We use the Krasnoselskii's fixed point theorem to show that the impulsive neutral dynamic equations with infinite delay

$$x^{\Delta}(t) = -A(t)x^{\sigma}(t) + g^{\Delta}(t, x(t-h(t))) + \int_{-\infty}^{t} D(t,u)\, f(x(u))\triangle u, \quad t \neq t_j,\ t \in \mathbb{T},$$
$$x(t_j^{+}) = x(t_j^{-}) + I_j(x(t_j)), \quad j \in \mathbb{Z}^{+}$$

have a periodic solution. Under a slightly more stringent conditions we show that the periodic solution is unique using the contraction mapping principle.

1. Introduction

In 1988, Stephan Hilger [9] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [7,8,17] and references therein. The study of impulsive initial and boundary value problems is extensive. For the theory and classical results, we direct the reader to the monographs [6,16,18].

Recently Althubiti, Makhzoum and Raffoul [2] investigated the existence and uniqueness of periodic solutions for the neutral differential equation with infinite delay

$$x'(t) = -a(t)x(t) + \frac{d}{dt}g(t, x(t-h(t))) + \int_{-\infty}^{t} D(t,u)\, f(x(u))du.$$

By employing the Krasnoselskii's fixed point theorem and the contraction mapping principle, the authors obtained existence and uniqueness results for periodic solutions.

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The nonlinear impulsive dynamic equation

$$
\begin{aligned}
&x^{\Delta}(t) = -a(t)x^{\sigma}(t) + f(t, x(t)), \quad t \neq t_j,\ t \in \mathbb{T},\\
&x(t_j^+) = x(t_j^-) + I_j(t_j, x(t_j)), \quad j = 1, 2, \dots, n,
\end{aligned}
$$

has been investigated in [10]. By using Schaeffer's theorem, the existence of periodic solutions has been established.

In this article, we are interested in the analysis of qualitative theory of periodic solutions of impulsive neutral dynamic equations. Inspired and motivated by the works mentioned above and the papers [1–5, 10–15, 20–22] and the references therein, we are concerned with the system

$$
\begin{aligned}
x^{\Delta}(t) =& - A(t)x^{\sigma}(t) + g^{\Delta}(t, x(t - h(t)))\\
&+ \int_{-\infty}^{t} D(t, u) f(x(u))\triangle u, \quad t \neq t_j,\ t \in \mathbb{T},\\
x(t_j^+) =& x(t_j^-) + I_j(x(t_j)),\ j \in \mathbb{Z}^+,
\end{aligned} \tag{1.1}
$$

where $\mathbb{T}$ is an ω-periodic time scale, $0 \in \mathbb{T}$ and $x^{\sigma} = x \circ \sigma$. For each interval U of $\mathbb{R}$, we denote by $U_{\mathbb{T}} = U \cap \mathbb{T}$, $x(t_j^+)$ and $x(t_j^-)$ represent the right and the left limit of $x(t_j)$ in the sense of time scales, in addition, if t_j is left-scattered, then $x(t_j^-) = x(t_j)$, $A(t) = \operatorname{diag}(a_i(t))_{n\times n}(a_i \in \mathcal{R}^+)$ and $D(t, u) = \operatorname{diag}(D_i(t, u))_{n\times n}(D_i \in C(\mathbb{T}, \mathbb{R}))$ are diagonal matrices with continuous real-valued functions as its elements, $\mathcal{R}^+ = \{a \in C(\mathbb{T}, \mathbb{R}) : 1+ \mu(t)a(t) > 0\}$ where $\mu(t) = \sigma(t) - t$, $h \in C(\mathbb{T}, \mathbb{T})$, $g = (g_1, g_2, \dots, g_n) \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$, $f = (f_1, f_2, \dots, f_n) \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_j = (I_j^{(1)}, I_j^{(2)}, \dots, I_j^{(n)}) \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $A(t)$, $h(t)$, $g(t, x(t - h(t)))$ are all ω-periodic functions with respect to t, $D(t + \omega, u + \omega) = D(t, u)$, $\omega > 0$ is a constant. There exists a positive integer p such that $t_{j+p} = t_j + \omega$, $I_{j+p} = I_j$, $j \in \mathbb{Z}^+$, without loss of generality, we also assume that $[0, \omega)_{\mathbb{T}} \cap \{t_j,\ j \in \mathbb{Z}^+\} = \{t_1, t_2, \dots, t_p\}$.

To reach our desired end we have to transform the system (1.1) into an integral system and then use Krasnoselskii's fixed point theorem to show the existence of periodic solutions. The obtained integral system is the sum of two mappings, one is a contraction and the other is a compact. Also, transforming system (1.1) to an integral system enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary results needed in later sections, then we give the Green's function of (1.1), which plays an important role in this paper. In Section 3, we establish our main results for periodic solutions by applying the Krasnoselskii's fixed point theorem and the contraction mapping principle.

2. Preliminaries

In this section, we shall recall some basic definitions and lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense points and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $x(t)$, $x^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood $U_\mathbb{T}$ of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U_\mathbb{T}$.

If x is continuous, then x is right-dense continuous, and if x is delta differentiable at t, then x is continuous at t.

Remark 2.1. $x : \mathbb{T} \to \mathbb{R}^n$ is delta derivable or right-dense continuous or continuous if each entry of x is delta derivable or right-dense continuous or continuous.

Let x be right-dense continuous. If $X^\Delta(t) = x(t)$, then we define the delta integral by

$$\int_a^t x(s)\Delta s = X(t) - X(a).$$

Definition 2.1 ([12])**.** We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$ and ω is the smallest positive number such that $f(t + \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.2. According to [12], if $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t + np) = \sigma(t) + np$ and the graininess function μ is a periodic function with period p.

Definition 2.2 ([8]). An $n \times n$-matrix-valued function A on time scale $\mathbb{T}$ is called regressive (respect to $\mathbb{T}$) provided

$$I + \mu(t)A(t),$$

is invertible for all $t \in \mathbb{T}^k$.

Let $A, B : \mathbb{T} \to R^{n\times n}$ be two $n \times n$-matrix-valued regressive functions on T, we define

$$\begin{aligned}(A \oplus B)(t) &:= A(t) + B(t) + \mu(t)A(t)B(t),\\ (\ominus A)(t) &:= -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1},\\ (A(t)) \ominus (B(t)) &:= (A(t)) \oplus (\ominus(B(t))),\end{aligned}$$

for all $t \in \mathbb{T}^k$.

Theorem 2.1 ([8]). *Let A be an regressive and rd-continuous $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^\Delta = A(t)y + f(t), \quad y(t_0) = y_0,$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$.

Definition 2.3 ([8]). Let $t_0 \in \mathbb{T}$ and assume that A is an regressive and rd-continuous $n \times n$-matrix-valued function. The unique matrix-valued solution of the initial value problem

$$x^\Delta(t) = A(t)x(t), \quad x(t_0) = I,$$

where I denotes as usual the $n \times n$-identity matrix, is called the matrix exponential function (at t_0), and it is denoted by $e_A(\cdot, t_0)$.

Remark 2.3. Assume that A is a constant $n \times n$-matrix. If $\mathbb{T} = \mathbb{R}$, then

$$e_A(t, t_0) = e^{A(t-t_0)},$$

while if $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then

$$e_A(t, t_0) = (I + A)^{t-t_0}.$$

In the following lemma, we give some properties of the matrix exponential function.

Lemma 2.1 ([8]). *Assume that $A, B : \mathbb{T} \to \mathbb{R}^{n\times n}$ are regressive and rd-continuous matrix-valued functions on $\mathbb{T}$. Then*

(i) $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$;
(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
(iii) $e_A^{-1}(t, s) = e^*_{\ominus A^*}(t, s)$;
(iv) $e_A(t, s) = e_A^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$;
(v) $e_A(t, s)e_A(s, r) = e_A(t, r)$;
(vi) $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$, *if $A(t)$ and $B(t)$ commute,*

where A^* denotes the conjugate transpose of A.

Lemma 2.2 ([8])**.** *Suppose A and B are regressive matrix-valued functions, then*

(i) A^* *is regressive;*
(ii) $\ominus A^* = (\ominus A)^*$*;*
(iii) $(A^*)^\Delta = (A^\Delta)^*$ *holds for any differential matrix-valued function A.*

Next, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution of (1.1). For its proof we refer the reader to [19].

Theorem 2.2 (Krasnoselskii)**.** *Let M be a closed convex nonempty subset of Banach space $(B, \|\cdot\|)$. Suppose that Φ and Ψ map M into B such that*

(i) $x, y \in M$ *imply* $\Phi x + \Psi y \in M$*;*
(ii) Ψ *is compact and continuous;*
(iii) Φ *is a contraction mapping.*

Then there exists $z \in M$ with $z = \Phi z + \Psi z$.

Lemma 2.3. *A function x is an ω-periodic solution of* (1.1) *if and only if x is an ω-periodic solution of the equation*

$$\begin{aligned} x(t) =& g(t, x(t-h(t))) + \int_t^{t+\omega} G(t,s) \left[\int_{-\infty}^{s} D\left(s,u\right) f(x(u)) \triangle u \right. \\ & \left. - A(s) g^\sigma(s, x(s-h(s))) \right] \Delta s + \sum_{j: t_j \in [t, t+\omega)} G(t, t_j) I_j(x(t_j)), \end{aligned}$$

where

$$\begin{aligned} G(t,s) &= \operatorname{diag}(G_i(t,s))_{n\times n},\ G_i(t,s) = \left(1 - e_{\ominus a_i}(\omega, 0)\right)^{-1} e_{\ominus a_i}(t+\omega, s), \\ A(t) &= \operatorname{diag}(a_i(t))_{n\times n},\ e_{\ominus a_i}(t,s) = \frac{1}{e_{a_i}(t,s)}, \\ \ominus a_i\left(t\right) &= -\frac{a_i\left(t\right)}{1+\mu\left(t\right) a_i\left(t\right)},\ g^\sigma(t, x(t-h(t))) = g(\sigma\left(t\right), x^\sigma(t-h(t))). \end{aligned}$$

Proof. If x is an ω-periodic solution of (1.1). For any $t \in \mathbb{T}$, there exists $j \in \mathbb{Z}$ such that t_j is the first impulsive point after t. Then for $i = 1, 2, \ldots, n$, x_i is an ω-periodic solution of the equation

$$x_i^\Delta(t) + a_i(t) x_i^\sigma(t) = g_i^\Delta(t, x_i(t-h(t))) + \int_{-\infty}^{t} D_i\left(t,u\right) f_i(x(u)) \triangle u. \tag{2.1}$$

Multiply both sides of (2.1) by $e_{a_i}(t,0)$ and then integrate from t to $s \in [t, t_j]_{\mathbb{T}}$, we obtain

$$\begin{aligned} &\int_t^s [e_{a_i}(\tau,0) x_i(\tau)]^\Delta \Delta\tau \\ &= \int_t^s e_{a_i}(\tau,0) \left[g_i^\Delta(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i\left(\tau,u\right) f_i(x_i(u)) \triangle u \right] \Delta\tau, \end{aligned}$$

or

$$e_{a_i}(s,0)x_i(s) = e_{a_i}(t,0)x_i(t) + \int_t^s e_{a_i}(\tau,0)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau,$$

then

$$x_i(s) = e_{\ominus a_i}(s,t)x_i(t) + \int_t^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau, \quad i = 1,2,\ldots,n,$$

hence

$$x_i(t_j) = e_{\ominus a_i}(t_j,t)x_i(t) + \int_t^{t_j} e_{\ominus a_i}(t_j,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau, \quad i = 1,2,\ldots,n. \tag{2.2}$$

Similarly, for $s \in (t_j, t_{j+1}]$, we have

$$\begin{aligned}
x_i(s) &= e_{\ominus a_i}(s,t_j)x_i(t_j^+) + \int_{t_j}^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau)))\right.\\
&\qquad \left. + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau\\
&= e_{\ominus a_i}(s,t_j)x_i(t_j^-) + \int_{t_j}^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau)))\right.\\
&\qquad \left. + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau + e_{\ominus a_i}(s,t_j)I_j^{(i)}(x_i(t_j))\\
&= e_{\ominus a_i}(s,t_j)x_i(t_j) + \int_{t_j}^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau)))\right.\\
&\qquad \left. + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau + e_{\ominus a_i}(s,t_j)I_j^{(i)}(x_i(t_j)),
\end{aligned}$$

for $i = 1,2,\ldots,n$. Substituting (2.2) in the above equality, we obtain

$$x_i(s) = e_{\ominus a_i}(s,t)x_i(t) + \int_t^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\right]\Delta\tau + e_{\ominus a_i}(s,t_j)I_j^{(i)}(x_i(t_j)).$$

Repeating the above process for $s \in [t, t+\omega]_{\mathbb{T}}$, we have

$$x_i(s) = e_{\ominus a_i}(s,t)x_i(t) + \int_t^s e_{\ominus a_i}(s,\tau)\left[g_i^{\Delta}(\tau, x_i(\tau - h(\tau)))\right.$$

$$+\int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\Bigg]\Delta\tau+\sum_{j:t_j\in[t,t+\omega)} e_{\ominus a_i}(s,t_j)I_j^{(i)}(x_i(t_j)),$$

for $i=1,2,\ldots,n$. Let $s=t+\omega$ in the above equality, we have

$$\begin{aligned}x_i(t+\omega)=\,&e_{\ominus a_i}(t+\omega,t)x_i(t)+\int_t^{t+\omega} e_{\ominus a_i}(t+\omega,\tau)\Bigg[g_i^{\Delta}(\tau,x_i(\tau-h(\tau)))\\&+\int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\Bigg]\Delta\tau+\sum_{j:t_j\in[t,t+\omega)} e_{\ominus a_i}(t+\omega,t_j)I_j^{(i)}(x_i(t_j)),\end{aligned}$$

$i=1,2,\ldots,n$. Noticing that $x_i(t+\omega)=x_i(t)$ and $e_{\ominus a_i}(t+\omega,t)=e_{\ominus a_i}(\omega,0)$, we obtain

$$\begin{aligned}(1-e_{\ominus a_i}(\omega,0))x_i(t)=&\int_t^{t+\omega} e_{\ominus a_i}(t+\omega,\tau)\Bigg[g_i^{\Delta}(\tau,x_i(\tau-h(\tau)))\\&+\int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\Bigg]\Delta\tau\\&+\sum_{j:t_j\in[t,t+\omega)} e_{\ominus a_i}(t+\omega,t_j)I_j^{(i)}(x_i(t_j)),\end{aligned}\tag{2.3}$$

for $i=1,2,\ldots,n$. Notice that

$$\begin{aligned}&\int_t^{t+\omega} e_{\ominus a_i}(t+\omega,\tau)g_i^{\Delta}(\tau,x_i(\tau-h(\tau)))\Delta\tau\\&=e_{\ominus a_i}(t+\omega,t+\omega)g_i(t+\omega,x_i(t+\omega-h(t+\omega)))\\&\quad-e_{\ominus a_i}(t+\omega,t)g_i(t,x_i(t-h(t)))\\&\quad-\int_t^{t+\omega} e_{\ominus a_i}(t+\omega,\tau)a_i(\tau)g_i^{\sigma}(\tau,x_i(\tau-h(\tau)))\Delta\tau\\&=[1-e_{\ominus a_i}(\omega,0)]g_i(t,x_i(t-h(t)))\\&\quad-\int_t^{t+\omega} e_{\ominus a_i}(t+\omega,\tau)a_i(\tau)g_i^{\sigma}(\tau,x_i(\tau-h(\tau)))\Delta\tau,\quad i=1,2,\ldots,n.\end{aligned}\tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}x_i(t)=\,&g_i(t,x_i(t-h(t)))+\int_t^{t+\omega}[1-e_{\ominus a_i}(\omega,0)]^{-1}e_{\ominus a_i}(t+\omega,\tau)\\&\times\left[\int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u-a_i(\tau)g_i^{\sigma}(\tau,x_i(\tau-h(\tau)))\right]\Delta\tau\\&+\sum_{j:t_j\in[t,t+\omega)}[1-e_{\ominus a_i}(\omega,0)]^{-1}e_{\ominus a_i}(t+\omega,t_j)I_j^{(i)}(x_i(t_j))\\=\,&g_i(t,x_i(t-h(t)))+\int_t^{t+\omega} G_i(t,\tau)\Bigg[\int_{-\infty}^{\tau} D_i(\tau,u)\, f_i(x_i(u))\triangle u\end{aligned}$$

$$- a_i(\tau)g_i^\sigma(\tau, x_i(\tau - h(\tau)))\Big] \Delta\tau + \sum_{j:t_j\in[t,t+\omega)} G_i(t,t_j)I_j^{(i)}(x_i(t_j)),$$

for $i = 1, 2, \ldots, n$. Next, we prove the converse. Let

$$\begin{aligned} x_i(t) = & g_i(t, x_i(t - h(t))) + \int_t^{t+\omega} G_i(t,s)\left[\int_{-\infty}^s D_i(s,u)\, f_i(x_i(u))\triangle u\right.\\ & \left. - a_i(s)g_i^\sigma(s, x_i(s-h(s)))\right]\Delta s + \sum_{j:t_j\in[t,t+\omega)} G_i(t,t_j)I_j^{(i)}(x_i(t_j)), \end{aligned}$$

where

$$G_i(t,s) = (1 - e_{\ominus a_i}(\omega, 0))^{-1} e_{\ominus a_i}(t+\omega, s), \quad i = 1, 2, \ldots, n.$$

Then if $t \neq t_i$, $i \in \mathbb{Z}^+$, we have

$$\begin{aligned} & x_i^\Delta(t) \\ = & g_i^\Delta(t, x_i(t-h(t))) \\ & + \int_t^{t+\omega}\left\{G_i(t,s)\left[\int_{-\infty}^s D_i(s,u)\, f_i(x_i(u))\triangle u - a_i(s)g_i^\sigma(s, x_i(s-h(s)))\right]\right\}^\Delta \Delta s \\ & + G_i(t,t+\omega)\left[\int_{-\infty}^{t+\omega} D_i(t+\omega,u)\, f_i(x_i(u))\triangle u\right.\\ & \qquad \left. - a_i(t+\omega)g_i^\sigma(t+\omega, x_i(t+\omega-h(t+\omega)))\right] \\ & - G_i(t,t)\left[\int_{-\infty}^t D_i(t,u)\, f_i(x_i(u))\triangle u - a_i(t)g_i^\sigma(t, x_i(t-h(t)))\right] \\ = & g_i^\Delta(t, x_i(t-h(t))) + \int_{-\infty}^t D_i(t,u)\, f_i(x_i(u))\triangle u - a_i(t)g_i^\sigma(t, x_i(t-h(t))) \\ & + \int_t^{t+\omega}\left\{G_i(t,s)\left[\int_{-\infty}^s D_i(s,u)\, f_i(x_i(u))\triangle u - a_i(s)g_i^\sigma(s, x_i(s-h(s)))\right]\right\}^\Delta \Delta s \\ = & g_i^\Delta(t, x_i(t-h(t))) + \int_{-\infty}^t D_i(t,u)\, f_i(x_i(u))\triangle u - a_i(t)x_i^\sigma(t) \\ = & -a_i(t)x_i^\sigma(t) + g_i^\Delta(t, x_i(t-h(t))) + \int_{-\infty}^t D_i(t,u)\, f_i(x_i(u))\triangle u, \quad i = 1, 2, \ldots, n. \end{aligned}$$

If $t = t_i$, $i \in \mathbb{Z}^+$, we obtain

$$\begin{aligned} x_i(t_i^+) - x_i(t_i^-) = & \sum_{j:t_j\in[t_i^+,t_i^++\omega)} G_i(t_i,t_j)I_j^{(i)}(x_i(t_j)) - \sum_{j:t_j\in[t_i^-,t_i^-+\omega)} G_i(t_i,t_j)I_j^{(i)}(x_i(t_j)) \\ = & G_i(t_i, t_i+\omega)I_i^{(i)}(x_i(t_i+\omega)) - G_i(t_i,t_i)I_i^{(i)}(x_i(t_i)) \\ = & I_i^{(i)}(x_i(t_i)),\ i = 1, 2, \ldots, n. \end{aligned}$$

So we know that, x is also an ω-periodic solution of (1.1). This completes the proof. $\square$

Throughout this paper, we make the following assumptions.

(H1) The function $g = (g_1, g_2, \dots, g_n)$ satisfies a Lipschitz condition in x. That is, for $i \in \{1, 2, \dots, n\}$, there exists a positive constant L_i such that

$$|g_i(t, x) - g_i(t, y)| \le L_i \|x - y\|, \quad \text{for all } t \in \mathbb{T},\ x, y \in \mathbb{R}^n.$$

(H2) The function $f = (f_1, f_2, \dots, f_n)$ satisfies a Lipschitz condition in x. That is, for $i \in \{1, 2, \dots, n\}$, there exists a positive constants M_i such that

$$|f_i(x) - f_i(y)| \le M_i \|x - y\|, \quad \text{for all } t \in \mathbb{T},\ x, y \in \mathbb{R}^n.$$

(H3) For $j \in \mathbb{Z}$, $I_j = (I_j^{(1)}, I_j^{(2)}, \dots, I_j^{(n)})$ satisfies Lipschitz condition. That is, for $i \in \{1, 2, \dots, n\}$ there exists a positive constant $P_j^{(i)}$ such that

$$\left| I_j^{(i)}(x) - I_j^{(i)}(y) \right| \le P_j^{(i)} \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

(H4) There exists a positive constant N_i such that

$$\int_{-\infty}^{t} |D_i(t, u)| \triangle u \le N_i.$$

To apply Theorem 2.2 to (1.1), we define

$PC(\mathbb{T}) = \{x : \mathbb{T} \to \mathbb{R}^n : x|_{(t_j, t_{j+1})_\mathbb{T}} \in C(t_j, t_{j+1})_\mathbb{T},\ \exists x(t_j^-) = x(t_j), x(t_j^+),\ j \in \mathbb{Z}^+\}$.

Consider the Banach space

$$X = \{x \in PC(\mathbb{T}) : x(t + \omega) = x(t)\},$$

with the norm $\|x\| = \max_{t \in [0, \omega]_\mathbb{T}} |x(t)|_0$, where $|x(t)|_0 = \max_{1 \le i \le n} |x_i(t)|$.

Lemma 2.4 ([12]). *Let $x \in X$. Then there exists $\|x^\sigma\|$, and $\|x^\sigma\| = \|x\|$.*

Noticing that

$$G_i(t, s) \le (1 - e_{\ominus a_i}(\omega, 0))^{-1} := \eta_i,$$

for convenience, we introduce the notation

$$\bar{\eta} := \max_{1 \le i \le n} \eta_i, \quad \gamma := \max_{1 \le i \le n} \max_{t \in [0, \omega]_\mathbb{T}} |a_i(t)|, \quad L := \max_{1 \le i \le n} L_i, \quad M := \max_{1 \le i \le n} M_i,$$

$$N := \max_{1 \le i \le n} N_i, \quad P_j := \max_{1 \le i \le n} P_j^{(i)}, \quad P := \max_{1 \le j \le p} P_j.$$

Define the mapping $H : X \to X$ by

$$(2.5) \qquad (H\varphi)(t) = g(t, \varphi(t - h(t))) + \int_t^{t+\omega} G(t, s) \left[\int_{-\infty}^{s} D(s, u) f(\varphi(u)) \triangle u - A(s) g^\sigma(s, \varphi(s - h(s))) \right] \Delta s + \sum_{j : t_j \in [t, t+\omega)} G(t, t_j) I_j(x(t_j)).$$

To apply Theorem 2.2, we need to construct two mappings: one is a contraction and the other is continuous and compact. We express (2.5) as

$$(H\varphi)(t) = (\Phi\varphi)(t) + (\Psi\varphi)(t),$$

where

$$(\Phi\varphi)(t) = g(t, \varphi(t - h(t))), \tag{2.6}$$

and

$$\begin{aligned}(\Psi\varphi)(t) &\\ = \int_t^{t+\omega} G(t,s)\left[\int_{-\infty}^{s} D(s,u)\, f(\varphi(u))\triangle u - A(s)g^\sigma(s, \varphi(s-h(s)))\right]\Delta s &\\ + \sum_{j:t_j\in[t,t+\omega)} G(t,t_j)I_j(\varphi(t_j)).&\end{aligned} \tag{2.7}$$

Lemma 2.5. *Suppose* (H1) *holds and* $L < 1$*, then* $\Phi : X \to X$*, as defined by* (2.6)*, is a contraction.*

Proof. Trivially, $\Phi : X \to X$. For $\varphi, \psi \in X$, we have

$$\begin{aligned}\|\Phi(\varphi) - \Phi(\psi)\| &= \max_{t\in[0,\omega]_{\mathbb{T}}} \max_{1\le i\le n} |g_i(t, \varphi_i(t - h(t))) - g_i(t, \psi_i(t - h(t)))| \\ &\le L\|\varphi - \psi\|.\end{aligned} \tag{2.8}$$

Hence Φ defines a contraction mapping with contraction constant L. □

Lemma 2.6. *Suppose* (H1)–(H4) *hold, then* $\Psi : X \to X$*, as defined by* (2.7)*, is continuous and compact.*

Proof. Evaluating (2.7) at $t + \omega$ gives

$$\begin{aligned}&(\Psi\varphi)(t+\omega)\\ &= \int_{t+\omega}^{t+2\omega} G(t+\omega,s)\left[\int_{-\infty}^{s} D(s,u)\, f(\varphi(u))\triangle u - A(s)g^\sigma(s,\varphi(s-h(s)))\right]\Delta s\\ &\quad + \sum_{j:t_j\in[t+\omega,t+2\omega)} G(t+\omega,t_j)I_j(\varphi(t_j)).\\ &= \int_{t}^{t+\omega} G(t+\omega,v+\omega)\left[\int_{-\infty}^{v+\omega} D(v+\omega,u)\, f(\varphi(u))\triangle u\right.\\ &\qquad\left. - A(v+\omega)g^\sigma(v+\omega,\varphi(v+\omega-h(v+\omega)))\right]\Delta v + \sum_{k:t_k\in[t,t+\omega)} G(t,t_k)I_j(\varphi(t_k))\\ &= \int_{t}^{t+\omega} G(t,v)\left[\int_{-\infty}^{v} D(v,u)\, f(\varphi(u))\triangle u\right.\\ &\qquad\left. - A(v)g^\sigma(v,\varphi(v-h(v)))\right]\Delta v + \sum_{k:t_k\in[t,t+\omega)} G(t,t_k)I_j(\varphi(t_k))\end{aligned}$$

$= (\Psi\varphi)(t).$

That is, $\Psi : X \to X$.

Now, we show that Ψ is continuous. Let $\varphi, \psi \in X$, given $\varepsilon > 0$, take

$$\delta = \frac{\varepsilon}{\overline{\eta}[\omega(MN + L\gamma) + P]},$$

such that for $\|\varphi - \psi\| \leq \delta$. By using the Lipschitz condition, we obtain

$$\begin{aligned}
&\|\Psi\varphi - \Psi\psi\| \\
\leq & \max_{t\in[0,\omega]_{\mathbb{T}}} \left| \int_t^{t+\omega} G(t,s) \left[\int_{-\infty}^{s} D\left(s,u\right) f(\varphi(u)) \triangle u - \int_{-\infty}^{s} D\left(s,u\right) f(\psi(u)) \triangle u \right] \Delta s \right|_0 \\
&+ \max_{t\in[0,\omega]_{\mathbb{T}}} \left| \int_t^{t+\omega} G(t,s) A(s) [g^\sigma(s, \varphi(s-h(s))) - g^\sigma(s, \psi(s-h(s)))] \Delta s \right|_0 \\
&+ \max_{t\in[0,\omega]_{\mathbb{T}}} \sum_{j:t_j\in[t,t+\omega)} |G(t,t_j)[I_j(\varphi(t_j)) - I_j(\psi(t_j))]|_0 \\
\leq & \overline{\eta} \int_0^{\omega} \int_{-\infty}^{s} |D\left(s,u\right) [f(\varphi(u)) - f(\psi(u))]\,|_0 \triangle u \Delta s \\
&+ \overline{\eta}\gamma \int_0^{\omega} |g^\sigma(s, \varphi(s-h(s))) - g^\sigma(s, \psi(s-h(s)))|_0 \Delta s \\
&+ \overline{\eta} \max_{1\leq j\leq p} |I_j(\varphi(t_j)) - I_j(\psi(t_j))|_0 \\
\leq & \overline{\eta}[\omega(MN + L\gamma) + P]\|\varphi - \psi\| < \varepsilon.
\end{aligned}$$

This proves Ψ is continuous. Next, we need to show that Ψ is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset X$ and assume that the sequence is uniformly bounded. Let $\Theta > 0$ be such that $\|\varphi_n\| \leq \Theta$, for all $n \in N$. In view of (H1)–(H3), we arrive at

$$\begin{aligned}
\|g^\sigma(t,x)\| &\leq \|g^\sigma(t,x) - g^\sigma(t,0)\| + \|g^\sigma(t,0)\| \\
&= \max_{t\in[0,\omega]_{\mathbb{T}}} \max_{1\leq i\leq n} |g_i^\sigma(t,x) - g_i^\sigma(t,0)| + \alpha_g \\
&\leq L\|x\| + \alpha_g,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\|f(x)\| &\leq \|f(x) - f(0)\| + \|f(0)\| \\
&= \max_{1\leq i\leq n} |f_i(x) - f_i(0)| + \alpha_f \\
&\leq M\|x\| + \alpha_f,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\|I_j(x)\| &\leq \|I_j(x) - I_j(0)\| + \|I_j(0)\| \\
&= \max_{1\leq i\leq n} n|I_j^{(i)}(x) - I_j^{(i)}(0)| + \alpha_{I_j} \\
&\leq P_j\|x\| + \alpha_{I_j}, \text{ for } j \in \mathbb{Z}^+,
\end{aligned} \tag{2.11}$$

where $\alpha_g = \|g^\sigma(t,0)\|$, $\alpha_f = \|f(0)\|$ and $\alpha_{I_j} = \|I_j(0)\|$. Hence,

$$
\begin{aligned}
(2.12)\quad &\|\Psi\varphi_n\| \\
&\le \max_{t\in[0,\omega]_{\mathbb{T}}} \left| \int_t^{t+\omega} G(t,s) \left[\int_{-\infty}^{s} D(s,u)\, f(\varphi_n(u)) \triangle u \right.\right. \\
&\qquad \left.\left. - A(s) g^{\sigma}(s, \varphi_n(s-h(s))) \right] \Delta s \right|_0 + \max_{t\in[0,\omega]_{\mathbb{T}}} \sum_{j: t_j \in [t, t+\omega)} |G(t,t_j) I_j(\varphi_n(t_j))|_0 \\
&\le \overline{\eta} \int_0^{\omega} \int_{-\infty}^{s} |D(s,u)\, f(\varphi_n(u))|_0 \triangle u \Delta s + \overline{\eta}\gamma \int_0^{\omega} |g^{\sigma}(s, \varphi_n(s-h(s)))|_0 \Delta s \\
&\quad + \overline{\eta} \sum_{j=1}^{p} |I_j(\varphi_n(t_j))|_0 \\
&\le \overline{\eta}\omega N (M\|\varphi_n\| + \alpha_f) + \overline{\eta}\gamma\omega(L\|\varphi_n\| + \alpha_g) + \overline{\eta}(\max_{1\le j\le p}(P_j\|\varphi_n\| + \alpha_{I_j})) \\
&\le \overline{\eta}\omega\Theta(MN + \gamma L) + \overline{\eta}(\omega N\alpha_f + \gamma\omega\alpha_g + P\Theta + \alpha) := D,
\end{aligned}
$$

where $\alpha = \max_{1\le j\le p} \alpha_{I_j}$. Thus the sequence $\{\Psi\varphi_n\}$ is uniformly bounded. Now, it can be easily checked that

$$
\begin{aligned}
(\Psi\varphi_n)^{\Delta}(t) = &- A(t)(\Psi\varphi_n)^{\sigma}(t) + \int_{-\infty}^{t} D(t,u)\, f(\varphi_n(u)) \triangle u \\
&- A(t) g^{\sigma}(t, \varphi_n(t-h(t))).
\end{aligned}
$$

Consequently, it follows from (2.10), (2.11), (2.12) and Lemma 2.4 that

$$
\begin{aligned}
|(\Psi\varphi_n)^{\Delta}(t)|_0 &\le \|A\| \|(\Psi\varphi_n)^{\sigma}\| + \left\| \int_{-\infty}^{t} D(t,u)\, f(\varphi_n(u)) \triangle u \right\| \\
&\quad + |A(t) g^{\sigma}(t, \varphi_n(t-h(t)))|_0 \\
&\le \|A\| \|(\Psi\varphi_n)\| + N(M\|\varphi_n\| + \alpha_f) + \|A\|(L\|\varphi_n\| + \alpha_g) \\
&\le \|A\| (D + L\Theta + \alpha_g) + N(M\Theta + \alpha_f),
\end{aligned}
$$

for all n. That is, $\|(\Psi\varphi_n)^{\Delta}\| \le \|A\|(D + L\Theta + \alpha_g) + N(M\Theta + \alpha_f)$, thus the sequence $\{\Psi\varphi_n\}$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that Ψ is compact. □

3. Main Results

Our main results reads as follows.

Theorem 3.1. *Assume that* (H1)–(H4) *hold and* $L < 1$. *Suppose that there is a positive constant* G *such that all solutions* x *of* (1.1), $x \in X$, *satisfy* $\|x\| \le G$, *and the inequality*

$$
\frac{\gamma\omega\alpha_g + \omega N\alpha_f + \alpha}{1/\overline{\eta} - \omega(\gamma L + MN) - L/\overline{\eta} - P} \le G,
$$

holds. Then (1.1) *has an* ω*-periodic solution.*

Proof. Define $M = \{\varphi \in X : \|\varphi\| \le G\}$. Then Lemma 2.6 implies $\Psi : X \to X$ and Ψ is compact and continuous. Also, from Lemma 2.5, the mapping Φ is a contraction and $\Phi : X \to X$. We need to show that if $\varphi, \psi \in M$, then $\|\Phi\varphi + \Psi\psi\| \le G$. Let $\varphi, \psi \in M$ with $\|\varphi\|, \|\psi\| \le G$, from (2.9)–(2.11), we have

$$\begin{aligned}\|\Phi\varphi + \Psi\psi\| &\le \|\Phi\varphi\| + \|\Psi\psi\|\\ &\le LG + \overline{\eta}\omega G(\gamma L + MN) + \overline{\eta}(\gamma\omega\alpha_g + \omega N\alpha_f + GP + \alpha) \le G.\end{aligned}$$

Thus $\Phi\varphi + \Psi\psi \in M$. We see that all the conditions of Krasnoselskii theorem are satisfied on the set M. Hence there exists a fixed point z in M such that $z = \Phi z + \Psi z$. By Lemma 2.3, this fixed point is a solution of (1.1). □

Theorem 3.2. *Suppose that* (H1)–(H4) *hold. If*

$$\Upsilon := \overline{\eta}[\omega(\gamma L + MN) + P] < 1,$$

then (1.1) *has an unique ω-periodic solution.*

Proof. For $\varphi, \psi \in X$, we have

$$\begin{aligned}\|H\varphi - H\psi\| \le& \overline{\eta}\int_0^\omega \int_{-\infty}^s |D(s,u)\, f(\varphi(u)) - f(\psi(u))|_0 \triangle u \Delta s\\ &+ \overline{\eta}\gamma \int_0^\omega |g^\sigma(s, \varphi(s-h(s))) - g^\sigma(s, \psi(s-h(s)))|_0 \Delta s\\ &+ \overline{\eta}\sum_{j=1}^p |I_j(\varphi(t_j)) - I_j(\psi(t_j))|_0\\ \le& \overline{\eta}\omega MN\|\varphi - \psi\| + \overline{\eta}\gamma\omega L\|\varphi - \psi\| + \overline{\eta}P\|\varphi - \psi\|\\ <& \overline{\eta}[\omega(\gamma L + MN) + P]\|\varphi - \psi\|\\ =& \Upsilon\|\varphi - \psi\|.\end{aligned}$$

This completes the proof. □

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