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STEINER HARARY INDEX

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ABSTRACT. The *Harary index* $H(G)$ of a connected graphs G is defined as $H(G) = \sum_{u,v\in V(G)} \frac{1}{d_G(u,v)}$ where $d_G(u,v)$ is the distance between vertices u and v of G. The Steiner distance in a graph, introduced by Chartrand et al. in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph G of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d_G(S)$ of the vertices of S is the minimum size of all connected subgraphs whose vertex set contain S. Recently, Furtula, Gutman, and Katanić introduced the concept of Steiner Harary index and give its chemical applications. The *k-center Steiner Harary index* $SH_k(G)$ of G is defined by $SH_k(G) = \sum_{S\subseteq V(G),\ |S|=k} \frac{1}{d_G(S)}$. Expressions for SH_k for some special graphs are obtained. We also give sharp upper and lower bounds of SH_k of a connected graph, and establish some of its properties in the case of trees.

1. INTRODUCTION

All graphs in this paper are undirected, finite, and simple. We refer to [3] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$, and $e(G)$ denote the set of vertices, the set of edges, and the size of G, respectively. Distance is one of the basic concepts of graph theory [4]. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u,v) = d_G(u,v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $\varepsilon(v)$ of v is defined by $\varepsilon(v) = \max\{d(u,v) \mid u \in V(G)\}$. Furthermore, the *radius* $\mathrm{rad}(G)$ and *diameter* $\mathrm{diam}(G)$ of G are defined by $\mathrm{rad}(G) = \min\{\varepsilon(v) \mid v \in V(G)\}$ and $\mathrm{diam}(G) = \max\{\varepsilon(v) \mid v \in V(G)\}$. These latter two concepts are related by the inequalities $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$. Goddard and Oellermann gave a survey paper on this subject [13].

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The *Wiener index* $W(G)$ of G is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

The first investigations of this distance–based graph invariant were done by Harold Wiener in 1947, who realized that there exist correlations between the boiling points of paraffins and their molecular structure, see [26–28]. Mathematicians study the Wiener index since the 1970s [10].

The Wiener index obtained wide attention and numerous results have been worked out, see the surveys [9, 15, 16, 30]. The *Harary index* $H(G)$ of G is defined by

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

For more details on the Harary index, we refer to [2, 17, 20, 29].

The Steiner distance of a graph, introduced by Chartrand et al. in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G(V,E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T(V',E')$ of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the *Steiner distance* $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size of connected subgraphs whose vertex set contain S. Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d_G(S)$, then H is a tree. Clearly, $d_G(S) = \min\{|E(T)|,\ S \subseteq V(T)\}$, where T is subtree of G. Furthermore, if $S = \{u,v\}$, then $d_G(S) = d_G(u,v)$ is nothing new, but the classical distance between u and v.

Let n and k be integers such that $2 \leq k \leq n$. The *Steiner k-eccentricity* $\varepsilon_k(v)$ of a vertex v of G is defined by $\varepsilon_k(v) = \max\{d_G(S) \mid S \subseteq V(G),\ |S| = k, \text{ and } v \in S\}$. The *Steiner k-radius* of G is $s\,\mathrm{rad}_k(G) = \min\{\varepsilon_k(v) \mid v \in V(G)\}$, while the *Steiner k-diameter* of G is $s\,\mathrm{diam}_k(G) = \max\{\varepsilon_k(v) \mid v \in V(G)\}$. Note that for every connected graph G, $\varepsilon_2(v) = \varepsilon(v)$ for all vertices v of G, $s\,\mathrm{rad}_2(G) = \mathrm{rad}(G)$ and $s\,\mathrm{diam}_2(G) = \mathrm{diam}(G)$. For more details on Steiner distance, we refer to [1, 5, 6, 8, 13, 25].

The following observation is easily seen.

Proposition 1.1. *Let k be an integer such that $2 \leq k \leq n$. If H is a spanning subgraph of G, then $s\,\mathrm{diam}_k(G) \leq s\,\mathrm{diam}_k(H)$.*

Li et al. [18] generalized the concept of Wiener index by Steiner distance. The *Steiner Wiener k-index* or *k-center Steiner Wiener index* $SW_k(G)$ of G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_G(S).$$

For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider SW_k for $2 \leq k \leq n-1$, but the above definition implies $SW_1(G) = 0$ and $SW_n(G) = n-1$. We refer to [18, 19, 21–24] for more details on Steiner Wiener index.

Furtula et al. [11] introduced the concept of Steiner Harary index. The *Steiner Harary k-index* or *k-center Steiner Harary index* $SH_k(G)$ of G is defined as

$$SH_k(G) = \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \frac{1}{d_G(S)}.$$

For $k = 2$, the above defined Steiner Harary index coincides with the ordinary Harary index. It is usual to consider SH_k for $2 \leq k \leq n-1$, but the above definition implies $SH_1(G) = 0$ and $SH_n(G) = \frac{1}{n-1}$.

In Section 2, we obtain the exact values of the Steiner Harary k-index of the path, complete graph, and complete bipartite graph. In Section 3, we obtain sharp lower and upper bounds for SH_k for connected graphs and for trees. In Section 4 we establish some relations for SH_k of trees. Our basic idea is from [18, 19].

2. Results for Some Special Graphs

Beginning this section, we note that the special case for $k = 2$ of all formulas derived here for the Steiner Harary index, thus pertaining to the ordinary Harary index, are well known and mentioned many times in the earlier literature.

Proposition 2.1. *Let K_n be the complete graph of order n, and let k be an integer such that $2 \leq k \leq n$. Then $SH_k(K_n) = \frac{1}{k-1}\binom{n}{k}$.*

Proof. For any $S \subseteq V(K_n)$ and $|S| = k$, without loss of generality, we let $S = \{u_1, u_2, \ldots, u_k\}$. Since K_n is the complete graph of order n, it follows that the tree T induced by the edges in $\{u_1u_2, u_1u_3, \ldots, u_1u_k\}$ is an S-Steiner tree, and hence $d_{K_n}(S) \leq k-1$. Since $|S| = k$, it follows that $d_{K_n}(S) \geq k-1$. Therefore, $d_{K_n}(S) = k-1$. From the arbitrariness of S and the symmetry of K_n, we have

$$SH_k(K_n) = \sum_{\substack{S\subseteq V(K_n)\\ |S|=k}} \frac{1}{d_{K_n}(S)} = \frac{1}{k-1}\binom{n}{k},$$

as desired. □

Proposition 2.2. *Let $K_{a,b}$ be the complete bipartite graph of order $a+b$ $(1 \leq a \leq b)$, and let k be an integer such that $2 \leq k \leq a+b$. Then*

$$SH_k(K_{a,b}) = \begin{cases} \frac{1}{k-1}\binom{a+b}{k} - \frac{1}{k(k-1)}\binom{a}{k} - \frac{1}{k(k-1)}\binom{b}{k}, & \text{if } 1 \leq k \leq a;\\ \frac{1}{k-1}\binom{a+b}{k} - \frac{1}{k(k-1)}\binom{b}{k}, & \text{if } a < k \leq b;\\ \frac{1}{k-1}\binom{a+b}{k}, & \text{if } b < k \leq a+b. \end{cases}$$

Proof. Let $G = K_{a,b}$, and let $U = \{u_1, u_2, \ldots, u_a\}$ and $W = \{w_1, w_2, \ldots, w_b\}$ be the two parts of $G = K_{a,b}$.

First, we consider the case $1 \leq k \leq a$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U = \varnothing$, or $S \cap W = \varnothing$, or $S \cap U \neq \varnothing$ and $S \cap W \neq \varnothing$. If $S \cap U = \varnothing$, then $S \subseteq W$. Without loss of generality, let $S = \{w_1, w_2, \ldots, w_k\}$. Then the tree T induced by the edges in $\{u_1w_1, u_1w_2, \ldots, u_1w_k\}$ is an S-Steiner tree, and hence $d_G(S) \leq k$. Since $G = K_{a,b}$ is a complete bipartite graph, it follows that any tree connecting S must use at least k edges, and hence $d_G(S) \geq k$. Therefore, $d_G(S) = k$. If $S \cap W = \varnothing$, then $S \subseteq U$. Without loss of generality, let $S = \{u_1, u_2, \ldots, u_k\}$. Then the tree T induced by the edges in $\{w_1u_1, w_1w_2, \ldots, w_1u_k\}$ is a Steiner tree connecting S, and hence $d_G(S) \leq k$. Since $G = K_{a,b}$ is a complete bipartite graph, it follows that any tree connecting S must use at least k edges, and hence $d_G(S) \geq k$. Therefore, $d_G(S) = k$. Suppose $S \cap U \neq \varnothing$ and $S \cap W \neq \varnothing$. Without loss of generality, let $S = \{u_1, u_2, \ldots, u_x, w_1, w_2, \ldots, w_{k-x}\}$. Then the tree T induced by the edges in $\{u_1w_1, w_1u_2, w_1u_3, \ldots, w_1u_x, u_1w_2, u_1w_3, \ldots, u_1w_{k-x}\}$ is an S-Steiner tree, and hence $d_G(S) \leq k-1$. Since $|S| = k$, it follows that any tree connecting S must use at least $k-1$ edges, and hence $d_G(S) = k-1$. Thus,

$$\begin{aligned}
SH_k(G) &= \sum_{\substack{S \subseteq V(G) \\ S \cap U = \varnothing}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U = \varnothing}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U \neq \varnothing, S \cap U \neq \varnothing}} \frac{1}{d_G(S)} \\
&= \frac{1}{k}\binom{a}{k} + \frac{1}{k}\binom{b}{k} + \frac{1}{k-1}\left[\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}\right] \\
&= \frac{1}{k}\binom{a}{k} + \frac{1}{k}\binom{b}{k} + \frac{1}{k-1}\left[\binom{a+b}{k} - \binom{b}{k} - \binom{a}{k}\right] \\
&= \frac{1}{k-1}\binom{a+b}{k} - \frac{1}{k(k-1)}\binom{a}{k} - \frac{1}{k(k-1)}\binom{b}{k}.
\end{aligned}$$

Next, we consider the case $a < k \leq b$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U = \varnothing$ or $S \cap U \neq \varnothing$. If $S \cap U = \varnothing$, then $S \subseteq W$ and $d_G(S) = k$. Suppose $S \cap U \neq \varnothing$. Then $d_G(S) = k-1$, and hence

$$\begin{aligned}
SH_k(G) &= \sum_{\substack{S \subseteq V(G) \\ S \cap U = \varnothing}} \frac{1}{d_G(S)} + \sum_{\substack{S \subseteq V(G) \\ S \cap U \neq \varnothing}} \frac{1}{d_G(S)} \\
&= \frac{1}{k}\binom{b}{k} + \frac{1}{k-1}\left[\sum_{x=1}^{a}\binom{a}{x}\binom{b}{k-x}\right] \\
&= \frac{1}{k}\binom{b}{k} + \frac{1}{k-1}\left[\sum_{x=1}^{\infty}\binom{a}{x}\binom{b}{k-x}\right] \\
&= \frac{1}{k}\binom{b}{k} + \frac{1}{k-1}\left[\binom{a+b}{k} - \binom{b}{k}\right] \\
&= \frac{1}{k-1}\binom{a+b}{k} - \frac{1}{k(k-1)}\binom{b}{k}.
\end{aligned}$$

In this end, we consider the remaining case $b < k \leq a+b$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap U \neq \varnothing$ and $S \cap U \neq \varnothing$. Then $d_G(S) = k-1$, and hence

$$SH_k(G) = \sum_{\substack{S\subseteq V(G)\\ S\cap U=\varnothing}} \frac{1}{d_G(S)} = \frac{1}{k-1}\binom{a+b}{k}.$$

The proof is now complete. □

From the above proposition, we can derive the following corollary.

Corollary 2.1. *Let S_n be the star of order n ($n \geq 3$), and let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(S_n) = \frac{kn-n+k}{k^2(k-1)}\binom{n-1}{k-1}.$$

Proof. From Proposition 2.2, we have that $SH_k(S_n) = SH_k(K_{1,n-1}) = \binom{n}{n}\frac{1}{n-1} = \frac{1}{n-1}$ for $k = n$ and $SH_k(S_n) = SH_k(K_{1,n-1}) = \frac{1}{k}\binom{n-1}{k} + \frac{1}{k-1}\binom{n-1}{k-1}$ for $2 \leq k \leq n-1$. We conclude that

$$SH_k(S_n) = \frac{1}{k}\binom{n-1}{k} + \frac{1}{k-1}\binom{n-1}{k-1} = \frac{kn-n+k}{k^2(k-1)}\binom{n-1}{k-1}.$$

□

Proposition 2.3. *Let P_n be the path of order n ($n \geq 3$), and let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(P_n) = n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1}.$$

Proof. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Choose $S \subseteq V(P_n)$ and $|S| = k$. Without loss of generality, let $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ where $i_1 \leq i_2 \leq \cdots \leq u_{i_k}$. Clearly, $k-1 \leq d(S) \leq n-1$. Observe that $d(S) = d_{P_n}(u_{i_1}, u_{i_k})$. Then $k-1 \leq d_{P_n}(u_{i_1}, u_{i_k}) \leq n-1$. Let $d_{P_n}(u_{i_1}, u_{i_k}) = t$. Thus, $k-1 \leq t \leq n-1$ and $1 \leq i_1 \leq n-t$. Therefore, we have $(n-t)$ ways to choose u_{i_1}. Once the vertex u_{i_1} is chosen, then $u_{i_k} = u_{i_1+t}$ is determined. Since $d_{P_n}(u_{i_1}, u_{i_k}) = t$, we have $\binom{t-1}{k-2}$ ways to choose $u_{i_2}, u_{i_3}, \ldots, u_{i_k-1}$. So there are $(n-t)\binom{t-1}{k-2}$ ways to determine S. Thus,

$$\begin{aligned} SH_k(P_n) &= \sum_{k-1\leq t\leq n-1} (n-t)\frac{1}{t}\binom{t-1}{k-2} \\ &= n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \sum_{k-1\leq t\leq n-1} \binom{t-1}{k-2} \\ &= n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1}, \end{aligned}$$

as desired. □

3. Lower and Upper Bounds for General Graphs

The following proposition is immediate.

Proposition 3.1. *Let G be a connected graph of order n, $e \in E(G)$, and let k be an integer such that $2 \leq k \leq n$. Furthermore, let F be the graph with vertex set $V(F) = V(G)$ and edge set $E(G) \setminus e$. Then*

$$SH_k(F) \leq SH_k(G).$$

This straightforwardly leads to the following theorem.

Proposition 3.2. *Let G be a connected graph of order n, and T a spanning tree of G. Let k be an integer such that $2 \leq k \leq n$. Then*

$$SH_k(T) \leq SH_k(G)$$

with equality if and only if G is a tree.

A lower bound for the Steiner Harary index of an arbitrary tree is given by the next theorem.

Theorem 3.1. *Let T be a tree of order n, and let k be an integer such that $2 \leq k \leq n$. Then*

$$n \sum_{k-1 \leq t \leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1} \leq SH_k(T) \leq \frac{kn-n+k}{k^2(k-1)}\binom{n-1}{k-1}.$$

Moreover, among all trees of order n, the star S_n maximizes the Steiner Harary k-index whereas the path P_n minimizes the Steiner Harary k-index.

Proof. The validity of the second inequality is verified by induction on n. For $n = k$, we have $d_T(S) = k - 1 = n - 1$ for $S \subseteq V(T)$, where T is a tree of order n. Then $SH_k(T) = SH_n(T) = \frac{1}{n-1} = \frac{1}{d_T(S)} = \frac{n^2-n+n}{n^2(n-1)}\binom{n-1}{n-1}$, as desired. Assume now that the second inequality holds for all trees of order n. Let T be a tree on $n+1$ vertices and v its pendent vertex. Let u be the vertex adjacent to v in T. Furthermore, let T' be the subtree of T induced by $V(T) \setminus v$. Then the inequality holds for T'. By the induction hypothesis,

$$SH_k(T') \leq \frac{kn-n+k}{k^2(k-1)}\binom{n-1}{k-1}.$$

If $v \in S$ but $u \notin S$, then $d_T(S) \geq k$ for $S \subseteq V(T), |S| = k$. If both $v \in S$ and $u \in S$, then $d_T(S) \geq k-1$ for $S \subseteq V(T), |S| = k$. Therefore, we have

$$\begin{aligned} SH_k(T) &= SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v \in S, u \notin S}} \frac{1}{d_T(S)} + \sum_{\substack{S \subseteq V(T), |S|=k \\ u,v \in S}} \frac{1}{d_T(S)} \\ &\leq \frac{kn-n+k}{k^2(k-1)}\binom{n-1}{k-1} + \frac{1}{k}\binom{n-1}{k-1} + \frac{1}{k-1}\binom{n-1}{k-2} \end{aligned}$$

$$= \frac{(k-1)(n+1)+k}{k^2(k-1)}\binom{n}{k-1}.$$

The first inequality is also verified by induction on n. For $n=k$, we have $d_T(S) = k-1 = n-1$ for $S \subseteq V(T)$, where T is a tree of order n. Then $SH_k(T) = SH_n(T) = \frac{1}{d_T(S)} = \frac{1}{n-1} = n \cdot \frac{1}{n-1}\binom{n-2}{n-2} - \binom{n-1}{n-1}$, as desired. Assume now that the second inequality holds for all trees of order n. Let T be a tree on $n+1$ vertices and v its pendent vertex. Furthermore, let T' be the subtree of T induced by $V(T) \setminus v$. Then the inequality holds for T' and we obtain

$$SH_k(T) = SH_k(T') + \sum_{\substack{S \subseteq V(T), |S|=k \\ v \in S}} \frac{1}{d_T(S)}.$$

By the induction hypothesis,

$$SH_k(T') \geq n \sum_{k-1 \leq t \leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1}.$$

Let $V(T') = \{v_1, v_2, \ldots, v_n\}$. For the vertex $v \in V(T)$, we can find a subtree T_1 such that $v \in V(T_1)$ and $d(V(T_1)) = k-1$. Without loss of generality, let $V(T_1) = \{v, v_1, v_2, \ldots, v_{k-1}\}$. Clearly, there is only one subset of $V(T_1)$ containing v and hence

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_1, v_2, \ldots, v_{k-1} \in S}} \frac{1}{d_T(S)} = \frac{1}{d(V(T_1))} = \frac{1}{k-1}.$$

Pick up a vertex from $V(T) \setminus \{v, v_1, v_2, \ldots, v_{k-1}\} = \{v_k, v_{k+1}, \ldots, v_n\}$, say v_k, such that v_k is adjacent to one element of $\{v_1, v_2, \ldots, v_{k-1}\}$. Clearly, the tree T_2 induced by the edges in $E(T_1) \cup \{v_k v_j\}$ is of order $k+1$, where $1 \leq j \leq k-1$ and $v_k v_j \in E(T)$. Then $d_T(V(T_2)) = d_T(\{v, v_1, v_2, \ldots, v_{k-1}, v_k\}) = k$. It is clear that there are at most $\binom{k-1}{k-2}$ subsets of $V(T_2)$ containing both v and v_k. For each such subset $S \subseteq V(T_2)$ with $|S| = k$, $d_T(S) \leq d_T(V(T_2)) \leq k$. Thus, we have

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_k \in S}} \frac{1}{d_T(S)} \geq \binom{k-1}{k-2}\frac{1}{d_T(V(T_2))} \geq \frac{1}{k}\binom{k-1}{k-2}.$$

Pick up a vertex from $V(T) \setminus \{v, v_1, v_2, \ldots, v_{k-1}, v_k\} = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$, say v_{k+1}, such that v_{k+1} is adjacent to one element of $\{v_1, v_2, \ldots, v_k\}$. Clearly, the tree T_3 induced by the edges in $E(T_2) \cup \{v_{k+1} v_j\}$ is of order $k+2$, where $1 \leq j \leq k$ and $v_{k+1} v_j \in E(T)$. Then $d_T(V(T_3)) = d_T(\{v, v_1, v_2, \ldots, v_{k+1}\}) = k+1$. Obviously, we can find $\binom{k}{k-2}$ subsets of $V(T_3)$ containing both v and v_{k+1}. Thus,

$$\sum_{\substack{S \subseteq V(T), |S|=k \\ v, v_{k+1} \in S}} \frac{1}{d_T(S)} \geq \binom{k}{k-2}\frac{1}{d_T(V(T_3))} \geq \frac{1}{k+1}\binom{k}{k-2}.$$

Continue the above procedure, we get

$$
\begin{aligned}
SH_k(T) =& SH_k(T') + \sum_{\substack{S\subseteq V(T),|S|=k\\ v\in S}} \frac{1}{d_T(S)}\\
\geq& SH_k(T') + \sum_{\substack{S\subseteq V(T),|S|=k\\ v,v_1,v_2,\ldots,v_{k-1}\in S}} \frac{1}{d_T(S)} + \sum_{\substack{S\subseteq V(T),|S|=k\\ v,v_k\in S}} \frac{1}{d_T(S)}\\
&+ \sum_{\substack{S\subseteq V(T),|S|=k\\ v,v_{k+1}\in S}} \frac{1}{d_T(S)} + \cdots + \sum_{\substack{S\subseteq V(T),|S|=k\\ v,v_n\in S}} \frac{1}{d_T(S)}\\
\geq& SH_k(T') + \frac{1}{k-1}\binom{k-2}{k-2} + \frac{1}{k}\binom{k}{k-2} + \cdots + \frac{1}{n}\binom{n-1}{k-2}\\
=& n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1} + \sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} + \frac{1}{n}\binom{n-1}{k-2}\\
=& (n+1)\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1} + \frac{1}{n}\binom{n-1}{k-2}\\
=& (n+1)\sum_{k-1\leq t\leq n} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-2} - \binom{n-1}{k-1}\\
=& (n+1)\sum_{k-1\leq t\leq n} \frac{1}{t}\binom{t-1}{k-2} - \binom{n}{k-1}.
\end{aligned}
$$

□

We recall that Theorem 3.1 provides a generalization of the much older results known for the Harary index, i.e., it yields this previous result by setting $k = 2$.

Theorem 3.2. *Let G be a connected graph of order n, and let k be an integer such that $2 \leq k \leq n$. Then*

$$
n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1} \leq SH_k(G) \leq \frac{1}{k-1}\binom{n}{k}.
$$

Moreover, the lower bound is sharp.

Proof. From Proposition 2.1, we have $SH_k(K_n) = \frac{1}{k-1}\binom{n}{k}$. According to Proposition 3.1, each subgraph G of K_n with $E(G) \subseteq E(K_n)$ has Steiner Harary index less than the Steiner Harary of K_n. Since each graph of order n is a subgraph of the complete graph, the inequality holds. From Proposition 3.2, we know that $SH_k(T) \leq SH_k(G)$, where T is a spanning tree of G. Combining this with Theorem 3.1, we have $SH_k(G) \geq SH_k(T) \geq SH_k(P_n) = n\sum_{k-1\leq t\leq n-1} \frac{1}{t}\binom{t-1}{k-2} - \binom{n-1}{k-1}$, as desired. □

4. On Steiner Harary Indices of Trees

For $k = n, n-1$, we have the following results for trees.

Theorem 4.1. *Let T be a tree of order n, possessing p pendent vertices. Then*

$$SH_{n-1}(T) = \frac{n^2 - 2n - p}{(n-1)(n-2)}, \tag{4.1}$$

irrespective of any other structural detail of T.

Proof. Since $k = n-1$, the respective subsets S contain all except one vertices of T. If the vertex missing from S is pendent, then the vertices contained in S form a tree of order $n-1$. Therefore $d_G(S) = n-2$. There are p such subsets, contributing to SH_{n-1} by $p \times \frac{1}{n-2}$.

If the vertex of T, not present in S, is non-pendent, then the vertices contained in S cannot form a tree, and the respective Steiner tree must contain all the n vertices of T. Therefore, $d_G(S) = n-1$. There are $n-p$ such subsets, contributing to SH_{n-1} by $(n-p) \times \frac{1}{n-1}$.

Thus, $SH_{n-1}(T) = \frac{p}{n-2} + \frac{n-p}{n-1}$, which straightforwardly leads to (4.1). □

Theorem 4.2. *Let T be a tree of order n, possessing p pendent vertices. Let q be the number of vertices of degree 2 in T such that each of them is adjacent to a pendant vertex. Then*

$$SH_{n-2}(T) = \left[\binom{p}{2} + q\right]\frac{1}{n-3} + \binom{n-p}{2}\frac{1}{n-1} + \frac{pn - p^2 - q}{n-2}. \tag{4.2}$$

Proof. For any $S \subseteq V(G)$ and $|S| = n-2$, we let $\bar{S} = \{u, v\}$. If $d_T(u) = d_T(v) = 1$, then $d_T(S) = n-3$ and this case contributes SH_{n-2} by

$$\sum_{\substack{u,v\in\bar{S}\\ d_T(u)=d_T(v)=1}} \frac{1}{d_T(S)} = \binom{p}{2}\frac{1}{n-3}.$$

If $d_T(u) \geq 2$ and $d_T(v) \geq 2$, then $d_T(S) = n-1$ and this case contributes SH_{n-2} by

$$\sum_{\substack{u,v\in\bar{S}\\ d_T(u)\geq 2,\ d_T(v)\geq 2}} \frac{1}{d_T(S)} = \binom{n-p}{2}\frac{1}{n-1}.$$

Suppose that $d_T(u) = 1$ and $d_T(v) \geq 2$. If $d_T(u) = 1$, $d_T(v) = 2$ and $uv \in E(T)$, then $d_T(S) = n-3$. If $d_T(u) = 1$, $d_T(v) \geq 3$ and $uv \in E(T)$, then $d_T(S) = n-2$. If $d_T(u) = 1$, $d_T(v) \geq 2$ and $uv \notin E(T)$, then $d_T(S) = n-2$. Therefore, this case contributes SH_{n-2} by

$$\begin{aligned} SH_{n-2}(T) &= \sum_{\substack{u,v\in\bar{S}\\ d_T(u)=1,\ d_T(v)\geq 2}} \frac{1}{d_T(S)} \\ &= \sum_{\substack{u,v\in\bar{S},uv\in E(T)\\ d_T(u)=1,\ d_T(v)=2}} \frac{1}{d_T(S)} + \sum_{\substack{u,v\in\bar{S},uv\in E(T)\\ d_T(u)=1,\ d_T(v)\geq 3}} \frac{1}{d_T(S)} + \sum_{\substack{u,v\in\bar{S},uv\notin E(T)\\ d_T(u)=1,\ d_T(v)\geq 2}} \frac{1}{d_T(S)} \end{aligned}$$

$$= \frac{q}{n-3} + \frac{p-q}{n-2} + \frac{p(n-p-1)}{n-2}.$$

From the above argument, we have

$$\begin{aligned} SH_{n-2}(T) &= \binom{p}{2}\frac{1}{n-3} + \binom{n-p}{2}\frac{1}{n-1} + \frac{q}{n-3} + \frac{p-q}{n-2} + \frac{p(n-p-1)}{n-2} \\ &= \left[\binom{p}{2} + q\right]\frac{1}{n-3} + \binom{n-p}{2}\frac{1}{n-1} + \frac{pn-p^2-q}{n-2}. \end{aligned}$$ □

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