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APPLICATION OF SYMMETRIC BILATERAL BAILEY TRANSFORMATION

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Abstract. In the present paper, making use of the symmetric bilateral Bailey transformation, we have developed certain relations for basic hypergeometric series and derived their special cases respectively.

1. Introduction

W. N. Bailey [27], in 1947 introduced the following transformation formula. If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

where α_r, δ_r, u_r and v_r are functions of r only, such that the series for γ_n exists.

This transformation leads to various results which play an important role in number theory and transformation theory of ordinary and basic hypergeometric series both.

Making use of the transformation, Bailey [27, 28] developed technique to obtain transformations for both ordinary and basic hypergeometric series and successfully used this transformations to obtain a number of identities of Rogers-Ramanujan type. In 1951 and 1952, L. J. Slater [15, 16] derived one hundred and thirty identities of Rogers-Ramanujan type with the help of Bailey transformation formula.

Key words and phrases. Basic hypergeometric series, Bilateral basic hypergeometric series, Symmetric bilateral Bailey transformation.

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After Bailey and Slater, a number of mathematicians working in the field of ordinary and basic hypergeometric series namely R. P. Agarwal [20], G. E. Andrews [7,8,10], D. M. Bressoud [5], D. M. Bressoud et al. [6], R. Y. Denis et al. [21], C. M. Joshi and Y. Vyas [4], A. Schilling and S. Ole Warnaar [1], S. P. Singh [23], U. B. Singh [24], P. Srivastava [19], A. Verma and V. K. Jain [2,3], S. Ole Warnaar [22] have used Bailey transformation to develop a number of results for both ordinary and basic hypergeometric series, i.e. for ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ etc. and ${}_2\phi_1$, ${}_3\phi_2$, ${}_4\phi_3$, ${}_5\phi_4$ etc. with single and multiple bases.

Before the year 1960, few applications of basic hypergeometric series were known. In 1961, N. G. Van Kampen [18] has used basic hypergeometric series to fluctuations in an electric circuit consisting of a condenser and a diode. In 1970, M. Baker and D. D. Coon [17] have used basic hypergeometric series on particle physics. In 1974, G. E. Andrews [9] wrote a paper to uncover applications of basic hypergeometric series in several area of pure and applied mathematics. In [9], G. E. Andrews surveys recent applications of basic hypergeometric series to partitions, number theory, finite vector spaces, combinatorial identities and physics. In the year 1999, V. K. Tuan [25], in 2000, I. Ali and V. K. Tuan [14], and in 2001, V. K. Tuan and M. Z. Nashed [26] have successfully utilized basic hypegeometric series to analytic continuation for functions of complex variable.

In 2007, G. E. Andrews and S. Ole Warnaar [11] have introduced Symmetric and Asymmetric bilateral Bailey transformation formulae and established certain results by making use of the formulae. In the present paper we have developed certain new relations for basic hypergeometric series, using Symmetric bilateral Bailey transformation formula due to G. E. Andrews and S. Ole Warnaar [11].

2. Definitions and Notations

We shall use the following q-symbols, for more details see [12]. For $|q| < 1$

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), & n = 1,2,\dots, \end{cases}$$

$$(a;q)_\infty = \prod_{s=0}^{\infty}(1-aq^s),$$

$$(a;q)_{-n} = \frac{(-q/a)^n}{(q/a;q)_n}\, q^{n(n-1)/2},$$

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_m;q)_n,$$

$$(a_1,a_2,\dots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\dots(a_m;q)_\infty.$$

A generalized basic hypergeometric series with base q is defined as

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,b_2,\dots,b_{r-1}\end{matrix} ;q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\dots(a_r;q)_n\, z^n}{(b_1;q)_n\dots(b_{r-1};q)_n(q;q)_n}. \tag{2.1}$$

if $|q| < 1$, series (2.1) converges absolutely for $|z| < 1$.

A generalized bilateral basic hypergeometric series with base q is defined as

$$ {}_r\psi_r\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n \ldots (a_r;q)_n}{(b_1;q)_n \ldots (b_r;q)_n} z^n, \tag{2.2}$$

if $|q| < 1$, series (2.2) converges for $\left|\frac{b_1 \ldots b_r}{a_1 \ldots a_r}\right| < |z| < 1$.

A truncated basic hypergeometric series with base q is defined as:

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_{r-1} \end{matrix} ; q, z\right]_k = \sum_{n=0}^{k} \frac{(a_1;q)_n \ldots (a_r;q)_n \; z^n}{(b_1;q)_n \ldots (b_{r-1};q)_n (q;q)_n}.$$

The above truncated series is converted into the generalized basic hypergeometric series (2.1), when $|z| < 1$, $|q| < 1$, provided that $k \to \infty$.

A truncated bilateral basic hypergeometric series with base q is defined as:

$$ {}_r\psi_r\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, z\right]_k = \sum_{n=-k}^{k} \frac{(a_1;q)_n \ldots (a_r;q)_n}{(b_1;q)_n \ldots (b_r;q)_n} z^n. \tag{2.3}$$

The above truncated series (2.3) is converted into the generalized bilateral basic hypergeometric series (2.2), when $\left|\frac{b_1 \ldots b_r}{a_1 \ldots a_r}\right| < |z| < 1$, $|q| < 1$, provided that $k \to \infty$.

Symmetric Bilateral Bailey Transformation due to G. E. Andrews and S. Ole Warnaar [11] is defined as follows.

If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{2.4}$$

and

$$\gamma_n = \sum_{r=|n|}^{\infty} \delta_r u_{r-n} v_{r+n}. \tag{2.5}$$

then

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.6}$$

where α_r, δ_r, u_r and v_r are any functions of r only, such that the series for γ_n exists.

To establish the main results following known summation formulae have been used.

(i) q-Gauss sum [12, II.8, p. 236]

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} ; q, c/ab\right] = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}. \tag{2.7}$$

(ii) Ramanujan's sum [12, II.29, p. 239]

$$ {}_1\psi_1 \left[\begin{matrix} a \\ \\ b \end{matrix} ; q, z \right] = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}. \tag{2.8} $$

(iii) Bailey's sum [12, II.31, p. 239]

$$ {}_3\psi_3 \left[\begin{matrix} b, c, d \\ \\ q/b, q/c, q/d \end{matrix} ; q, \frac{q}{bcd} \right] = \frac{(q, q/bc, q/bd, q/cd; q)_\infty}{(q/b, q/c, q/d, q/bcd; q)_\infty}. \tag{2.9} $$

(iv) Basic bilateral analogue of Dixon's sum [12, II.32, p. 239]

$$ {}_4\psi_4 \left[\begin{matrix} -qa^{\frac{1}{2}}, b, c, d \\ \\ -a^{\frac{1}{2}}, aq/b, aq/c, aq/d \end{matrix} ; q, \frac{qa^{\frac{3}{2}}}{bcd} \right] \tag{2.10} $$
$$ = \frac{(aq, aq/bc, aq/bd, aq/cd, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c, qa^{\frac{1}{2}}/d, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, qa^{\frac{3}{2}}/bcd; q)_\infty}. $$

(v) Bailey's sum [12, II.33, p. 239]

$$ {}_6\psi_6 \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e \\ \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e \end{matrix} ; q, \frac{qa^2}{bcde} \right] \tag{2.11} $$
$$ = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}. $$

(vi) Result due to H. S. Shukla [13]

$$ {}_8\psi_8 \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, aq^2/f \\ \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f, f/q \end{matrix} ; q, \frac{a^2}{bcde} \right] \tag{2.12} $$
$$ = \left(1 - \frac{(1-bc/a)(1-bd/a)(1-be/a)}{(1-bq/f)(1-bf/aq)(1-bcde/a^2)}\right) \frac{(1-f/bq)(1-bf/aq)}{(1-f/aq)(1-f/q)} $$
$$ \times \frac{(q, aq, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2q/bcde; q)_\infty}. $$

where $|a^2/bcde| < 1$.

3. Main Results

The following results have been established.

(i)

$$
{}_5\phi_4\left[\begin{matrix} q, \frac{b}{a}, az, \frac{q}{az}, a \\ \frac{q}{a}, z, \frac{b}{az}, c \end{matrix}; q, \frac{c}{ab}\right] \tag{3.1}
$$

$$
= \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}\left\{ {}_2\phi_1\left[\begin{matrix} q, a \\ b \end{matrix}; q, z\right] + {}_2\phi_1\left[\begin{matrix} q, \frac{q}{b} \\ \frac{q}{a} \end{matrix}; q, \frac{b}{az}\right] - 1\right\}
$$

$$
+ {}_2\phi_1\left[\begin{matrix} a, a \\ c \end{matrix}; q, \frac{cz}{ab}\right] + {}_3\phi_2\left[\begin{matrix} \frac{q}{b}, a, b \\ \frac{q}{a}, c \end{matrix}; q, \frac{c}{a^2 z}\right]
$$

$$
- {}_4\phi_3\left[\begin{matrix} q, a, \frac{c}{a}, \frac{c}{b} \\ b, c, \frac{c}{ab} \end{matrix}; q, z\right] - {}_4\phi_3\left[\begin{matrix} q, \frac{q}{b}, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{a}, c, \frac{c}{ab} \end{matrix}; q, \frac{b}{az}\right].
$$

(ii)

$$
{}_6\phi_5\left[\begin{matrix} q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, a, b \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}, c \end{matrix}; q, \frac{c}{ab}\right] \tag{3.2}
$$

$$
= {}_4\phi_3\left[\begin{matrix} a, b, b, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix}; q, \frac{q}{ab^2 d}\right] + {}_4\phi_3\left[\begin{matrix} a, b, b, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix}; q, \frac{q^2}{ab^2 d}\right]
$$

$$
+ \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}\left\{ {}_4\phi_3\left[\begin{matrix} q, b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix}; q, \frac{q}{bcd}\right] + {}_4\phi_3\left[\begin{matrix} q, b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix}; q, \frac{q^2}{bcd}\right] - 1\right\}
$$

$$
- {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix}; q, \frac{q}{bcd}\right] - {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix}; q, \frac{q^2}{bcd}\right].
$$

(iii)

$$(3.3)\quad {}_{11}\phi_{10}\left[\begin{matrix} aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}, \frac{qa^{\frac{1}{2}}}{b}, \frac{qa^{\frac{1}{2}}}{c}, \frac{qa^{\frac{1}{2}}}{d}, q, \frac{q}{a}, a, b \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, \frac{qa^{\frac{3}{2}}}{bcd}, c \end{matrix}; q, \frac{c}{ab}\right]$$

$$= {}_5\phi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, b, d, a, b \\ -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix}; q, \frac{qa^{\frac{1}{2}}}{b^2 d}\right] + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}$$

$$\times \left\{ {}_5\phi_4\left[\begin{matrix} q, -qa^{\frac{1}{2}}, b, c, d \\ -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] + {}_5\phi_4\left[\begin{matrix} q, -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a} \\ -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] - 1 \right\}$$

$$- {}_6\phi_5\left[\begin{matrix} q, -qa^{\frac{1}{2}}, b, d, \frac{c}{a}, \frac{c}{b} \\ -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{c}{ab} \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right]$$

$$+ {}_6\phi_5\left[\begin{matrix} -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, a, b \\ -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, c \end{matrix}; q, \frac{qa^{\frac{1}{2}}}{b^2 d}\right] - {}_7\phi_6\left[\begin{matrix} q, -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{c}{a}, \frac{c}{b} \\ -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, c, \frac{c}{ab} \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right].$$

(iv)

$$(3.4)\quad {}_{11}\phi_{10}\left[\begin{matrix} aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, \frac{q}{a}, a, b \\ \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{qa^2}{bcde}, c \end{matrix}; q, \frac{c}{ab}\right]$$

$$= {}_7\phi_6\left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, d, e, a, b \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e} \end{matrix}; q, \frac{qa}{b^2 de}\right]$$

$$+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}\,{}_7\phi_6\left[\begin{matrix}q,qa^{\frac12},-qa^{\frac12},b,c,d,e\\ a^{\frac12},-a^{\frac12},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e}\end{matrix};q,\frac{qa^2}{bcde}\right]$$

$$+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}\left\{{}_7\phi_6\left[\begin{matrix}q,qa^{\frac12},-qa^{\frac12},b,c,d,e\\ a^{\frac12},-a^{\frac12},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e}\end{matrix};q,\frac{qa^2}{bcde}\right]-1\right\}$$

$$-\,{}_8\phi_7\left[\begin{matrix}q,qa^{\frac12},-qa^{\frac12},b,d,e,\frac{c}{a},\frac{c}{b}\\ a^{\frac12},-a^{\frac12},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{c}{ab}\end{matrix};q,\frac{qa^2}{bcde}\right]$$

$$+\,{}_8\phi_7\left[\begin{matrix}qa^{-\frac12},-qa^{-\frac12},\frac{b}{a},\frac{c}{a},\frac{d}{a},\frac{e}{a},a,b\\ a^{-\frac12},-a^{-\frac12},\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},c\end{matrix};q,\frac{qa}{b^2de}\right]$$

$$-\,{}_9\phi_8\left[\begin{matrix}q,qa^{-\frac12},-qa^{-\frac12},\frac{b}{a},\frac{c}{a},\frac{d}{a},\frac{e}{a},\frac{c}{a},\frac{c}{b}\\ a^{-\frac12},-a^{-\frac12},\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},c,\frac{c}{ab}\end{matrix};q,\frac{qa^2}{bcde}\right].$$

(v)

$$(3.5)\quad {}_{11}\phi_{10}\left[\begin{matrix}q,aq,\frac{q}{a},\frac{aq}{bc},\frac{aq}{bd},\frac{aq}{be},\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{de},a,b\\ \frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{qa^2}{bcde},c\end{matrix};q,\frac{c}{ab}\right]$$

$$=\frac{\dfrac{\left(1-\frac{f}{aq}\right)\left(1-\frac{f}{q}\right)}{\left(1-\frac{f}{bq}\right)\left(1-\frac{bf}{aq}\right)}}{\left(1-\dfrac{\left(1-\frac{bc}{a}\right)\left(1-\frac{bd}{a}\right)\left(1-\frac{be}{a}\right)}{\left(1-\frac{bq}{f}\right)\left(1-\frac{bf}{aq}\right)\left(1-\frac{bcde}{a^2}\right)}\right)}$$

$$
\times \Bigg\{ {}_9\phi_8 \left[\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, d, e, f, \frac{aq^2}{f}, a, b \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{f}{q} \end{matrix} ; q, \frac{a}{b^2de} \right]
$$

$$
+ \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} {}_9\phi_8 \left[\begin{matrix} q, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, \frac{aq^2}{f} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{f}{q} \end{matrix} ; q, \frac{a^2}{bcde} \right]
$$

$$
+ \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} \left({}_9\phi_8 \left[\begin{matrix} q, qa^{-\frac{1}{2}}, -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{f}{a}, \frac{q^2}{f} \\ a^{-\frac{1}{2}}, -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{f}{aq} \end{matrix} ; q, \frac{a^2}{bcdef} \right] - 1 \right)
$$

$$
- {}_{10}\phi_9 \left[\begin{matrix} q, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, d, e, f, \frac{aq^2}{f}, \frac{c}{a}, \frac{c}{b} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{f}{q}, \frac{c}{ab} \end{matrix} ; q, \frac{a^2}{bcde} \right]
$$

$$
+ {}_{10}\phi_9 \left[\begin{matrix} qa^{-\frac{1}{2}}, -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{f}{a}, \frac{q^2}{f}, a, b \\ a^{-\frac{1}{2}}, -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{f}{aq}, c \end{matrix} ; q, \frac{a}{b^2def} \right]
$$

$$
- {}_{11}\phi_{10} \left[\begin{matrix} q, qa^{-\frac{1}{2}}, -qa^{-\frac{1}{2}}, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \frac{f}{a}, \frac{q^2}{f}, \frac{c}{a}, \frac{c}{b} \\ a^{-\frac{1}{2}}, -a^{-\frac{1}{2}}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{f}{aq}, c, \frac{c}{ab} \end{matrix} ; q, \frac{a^2}{bcdef} \right] \Bigg\}.
$$

4. Proof of Main Results

In this section, we have given the proof of our main results given in §3.

Proof of (i). Putting $u_r = v_r = 1$ and $\alpha_r = \dfrac{(a;q)_r\ z^r}{(b;q)_r}$ in (2.4), we get

$$\beta_n = \sum_{r=-n}^{n} \frac{(a;q)_r\ z^r}{(b;q)_r} = {}_1\psi_1 \left[\begin{matrix} a \\ b \end{matrix} ; q, z \right]_n$$

making use of the result (2.8), provided $n \to \infty$, we obtain β_n as

$$\beta_n = \frac{(q, b/a, az, q/az; q)_n}{(b, q/a, z, b/az; q)_n}.$$

Putting $u_r = v_r = 1$ in (2.5), we get

$$\begin{aligned} \gamma_n = \sum_{r=|n|}^{\infty} \delta_r &= \delta_{|n|} + \sum_{r=|n|+1}^{\infty} \delta_r \\ &= \delta_{|n|} + \sum_{r=0}^{|n|} \delta_r + \sum_{r=|n|+1}^{\infty} \delta_r - \sum_{r=0}^{|n|} \delta_r \\ &= \delta_{|n|} + \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{|n|} \delta_r . \end{aligned}$$

Putting

$$\delta_r = \frac{(a,b;q)_r \left(\dfrac{c}{ab}\right)^r}{(q,c;q)_r} \tag{4.1}$$

in above equation, we get

$$\begin{aligned} \gamma_n &= \frac{(a,b;q)_{|n|} \left(\dfrac{c}{ab}\right)^{|n|}}{(q,c;q)_{|n|}} + \sum_{r=0}^{\infty} \frac{(a,b;q)_r \left(\dfrac{c}{ab}\right)^r}{(q,c;q)_r} - \sum_{r=0}^{|n|} \frac{(a,b;q)_r \left(\dfrac{c}{ab}\right)^r}{(q,c;q)_r}, \\ &= \frac{(a,b;q)_{|n|} \left(\dfrac{c}{ab}\right)^{|n|}}{(q,c;q)_{|n|}} + {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, \frac{c}{ab} \right] - {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; q, \frac{c}{ab} \right]_{|n|} \end{aligned}$$

using equation (2.7), provided $n \to \infty$, we get γ_n as follows:

$$\gamma_n = \frac{(a,b;q)_{|n|} \left(\dfrac{c}{ab}\right)^{|n|}}{(q,c;q)_{|n|}} + \frac{\left(\dfrac{c}{a}, \dfrac{c}{b}; q\right)_\infty}{\left(c, \dfrac{c}{ab}; q\right)_\infty} - \frac{\left(\dfrac{c}{a}, \dfrac{c}{b}; q\right)_{|n|}}{\left(c, \dfrac{c}{ab}; q\right)_{|n|}}. \tag{4.2}$$

Putting the values of α_n, β_n, γ_n and δ_n in (2.6) and after simplification, we obtain result (3.1) as follows.

L.H.S. of the result (3.1) is obtained as follows

$$\sum_{n=0}^{\infty}\beta_n\delta_n=\sum_{n=0}^{\infty}\frac{(q,\frac{b}{a},az,\frac{q}{az};q)_n}{(b,\frac{q}{a},z,\frac{b}{az};q)_n}\ \frac{(a,b;q)_n\ \left(\frac{c}{ab}\right)^n}{(q,c;q)_n}$$

$$={}_5\phi_4\left[\begin{matrix}q,\frac{b}{a},az,\frac{q}{az},a\\ \\ \frac{q}{a},z,\frac{b}{az},c\end{matrix};q,\frac{c}{ab}\right].$$

R.H.S. of the result (3.1) is obtained as follows

$$\sum_{n=-\infty}^{\infty}\alpha_n\gamma_n=\sum_{n=-\infty}^{\infty}\frac{(a;q)_n\ z^n}{(b;q)_n}\left\{\frac{(a,b;q)_{|n|}\ \left(\frac{c}{ab}\right)^{|n|}}{(q,c;q)_{|n|}}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{|n|}}{\left(c,\frac{c}{ab};q\right)_{|n|}}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(a;q)_n\ z^n}{(b;q)_n}\left\{\frac{(a,b;q)_n\ \left(\frac{c}{ab}\right)^n}{(q,c;q)_n}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}\right\}$$

$$+\sum_{n=-\infty}^{-1}\frac{(a;q)_n\ z^n}{(b;q)_n}\left\{\frac{(a,b;q)_{-n}\ \left(\frac{c}{ab}\right)^{-n}}{(q,c;q)_{-n}}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{-n}}{\left(c,\frac{c}{ab};q\right)_{-n}}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(a;q)_n\ z^n}{(b;q)_n}\left\{\frac{(a,b;q)_n\ \left(\frac{c}{ab}\right)^n}{(q,c;q)_n}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}\right\}$$

$$+\sum_{n=1}^{\infty}\frac{(a;q)_{-n}\ z^{-n}}{(b;q)_{-n}}\left\{\frac{(a,b;q)_n\ \left(\frac{c}{ab}\right)^n}{(q,c;q)_n}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}\right\}.$$

For $n=0$, we get

$$\frac{(a;q)_{-n}\ z^{-n}}{(b;q)_{-n}}\left\{\frac{(a,b;q)_n\ \left(\frac{c}{ab}\right)^n}{(q,c;q)_n}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}\right\}$$

$$=1\times\left\{1+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}-1\right\}$$

$$= \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}.$$

Therefore, we can write

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \frac{(a;q)_n \; z^n}{(b;q)_n} \left\{ \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_n}{\left(c, \frac{c}{ab}; q\right)_n} \right\}$$
$$+ \sum_{n=0}^{\infty} \frac{(a;q)_{-n} \; z^{-n}}{(b;q)_{-n}} \left\{ \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_n}{\left(c, \frac{c}{ab}; q\right)_n} \right\}$$
$$- \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}.$$

Since,

$$\frac{(a;q)_{-n} \; z^{-n}}{(b;q)_{-n}} = \frac{\frac{\left(-\frac{q}{a}\right)^n}{\left(\frac{q}{a};q\right)_n} \; q^{\frac{n(n-1)}{2}}}{\frac{\left(-\frac{q}{b}\right)^n}{\left(\frac{q}{b};q\right)_n} \; q^{\frac{n(n-1)}{2}}} \times \frac{1}{z^n}$$
$$= \frac{\left(\frac{q}{b};q\right)_n}{\left(\frac{q}{a};q\right)_n} \times \left(\frac{b}{az}\right)^n.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \frac{(a;q)_n \; z^n}{(b;q)_n} \left\{ \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_n}{\left(c, \frac{c}{ab}; q\right)_n} \right\}$$
$$+ \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b};q\right)_n}{\left(\frac{q}{a};q\right)_n} \left(\frac{b}{az}\right)^n \left\{ \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_n}{\left(c, \frac{c}{ab}; q\right)_n} \right\}$$
$$- \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(a;q)_n \; z^n}{(b;q)_n} \; \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n \; z^n}{(b;q)_n} \\
&\quad - \sum_{n=0}^{\infty} \frac{(a;q)_n \; z^n}{(b;q)_n} \; \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} + \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b};q\right)_n \; \left(\frac{b}{az}\right)^n}{\left(\frac{q}{a};q\right)_n} \; \frac{(a,b;q)_n \; \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} \\
&\quad + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b};q\right)_n \; \left(\frac{b}{az}\right)^n}{\left(\frac{q}{a};q\right)_n} - \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b};q\right)_n \; \left(\frac{b}{az}\right)^n}{\left(\frac{q}{a};q\right)_n} \; \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \\
&\quad - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \\
&= {}_2\phi_1\left[\begin{matrix} a, a & \\ & ;q, \frac{cz}{ab} \\ c & \end{matrix}\right] + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \times \; {}_2\phi_1\left[\begin{matrix} q, a & \\ & ;q, z \\ b & \end{matrix}\right] \\
&\quad - {}_4\phi_3\left[\begin{matrix} q, a, \frac{c}{a}, \frac{c}{b} & \\ & ;q, z \\ b, c, \frac{c}{ab} & \end{matrix}\right] + {}_3\phi_2\left[\begin{matrix} \frac{q}{b}, a, b & \\ & ;q, \frac{c}{a^2 z} \\ \frac{q}{a}, c & \end{matrix}\right] \\
&\quad + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \; {}_2\phi_1\left[\begin{matrix} q, \frac{q}{b} & \\ & ;q, \frac{b}{az} \\ \frac{q}{a} & \end{matrix}\right] - {}_4\phi_3\left[\begin{matrix} q, \frac{q}{b}, \frac{c}{a}, \frac{c}{b} & \\ & ;q, \frac{b}{az} \\ \frac{q}{a}, c, \frac{c}{ab} & \end{matrix}\right] \\
&\quad - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \\
&= \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} \left\{ {}_2\phi_1\left[\begin{matrix} q, a & \\ & ;q, z \\ b & \end{matrix}\right] + {}_2\phi_1\left[\begin{matrix} q, \frac{q}{b} & \\ & ;q, \frac{b}{az} \\ \frac{q}{a} & \end{matrix}\right] - 1 \right\}
\end{aligned}
$$

$$+ {}_2\phi_1\left[\begin{matrix} a, a \\ c \end{matrix} ; q, \frac{cz}{ab}\right] + {}_3\phi_2\left[\begin{matrix} \frac{q}{b}, a, b \\ \frac{q}{a}, c \end{matrix} ; q, \frac{c}{a^2 z}\right]$$

$$- {}_4\phi_3\left[\begin{matrix} q, a, \frac{c}{a}, \frac{c}{b} \\ b, c, \frac{c}{ab} \end{matrix} ; q, z\right] - {}_4\phi_3\left[\begin{matrix} q, \frac{q}{b}, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{a}, c, \frac{c}{ab} \end{matrix} ; q, \frac{b}{az}\right].$$

From equation (2.6), we have

$$\sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=-\infty}^{\infty} \alpha_n \gamma_n.$$

Hence, the result (3.1) is given as follows

$${}_5\phi_4\left[\begin{matrix} q, \frac{b}{a}, az, \frac{q}{az}, a \\ \frac{q}{a}, z, \frac{b}{az}, c \end{matrix} ; q, \frac{c}{ab}\right]$$

$$= \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} \left\{ {}_2\phi_1\left[\begin{matrix} q, a \\ b \end{matrix} ; q, z\right] + {}_2\phi_1\left[\begin{matrix} q, \frac{q}{b} \\ \frac{q}{a} \end{matrix} ; q, \frac{b}{az}\right] - 1 \right\}$$

$$+ {}_2\phi_1\left[\begin{matrix} a, a \\ c \end{matrix} ; q, \frac{cz}{ab}\right] + {}_3\phi_2\left[\begin{matrix} \frac{q}{b}, a, b \\ \frac{q}{a}, c \end{matrix} ; q, \frac{c}{a^2 z}\right]$$

$$- {}_4\phi_3\left[\begin{matrix} q, a, \frac{c}{a}, \frac{c}{b} \\ b, c, \frac{c}{ab} \end{matrix} ; q, z\right] - {}_4\phi_3\left[\begin{matrix} q, \frac{q}{b}, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{a}, c, \frac{c}{ab} \end{matrix} ; q, \frac{b}{az}\right].$$ □

Proof of (ii). Putting $u_r = v_r = 1$ and $\alpha_r = \dfrac{(b, c, d; q)_r \; (q/bcd)^r}{(q/b, q/c, q/d; q)_r}$ in (2.4), we get

$$\beta_n = \sum_{r=-n}^{n} \frac{(b, c, d; q)_r \left(\frac{q}{bcd}\right)^r}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}; q\right)_r} = {}_3\psi_3\left[\begin{matrix} b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q}{bcd}\right]_n$$

using result (2.9), provided $n \to \infty$, we get β_n as follows:

$$\beta_n = \frac{\left(q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_n}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}; q\right)_n}.$$

Putting the values of α_n, β_n and values of δ_n, γ_n from (4.1) and (4.2) in (2.6) and after simplification, we obtain result (3.2) as follows.

L.H.S. of the result (3.2) is obtained as

$$\begin{aligned}\sum_{n=0}^{\infty} \beta_n \delta_n &= \sum_{n=0}^{\infty} \frac{\left(q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}; q\right)_n}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}; q\right)_n} \frac{(a, b; q)_n \left(\frac{c}{ab}\right)^n}{(q, c; q)_n} \\ &= {}_6\phi_5 \left[\begin{matrix} q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, a, b \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}, c \end{matrix} ; q, \frac{c}{ab} \right].\end{aligned}$$

R.H.S. of the result (3.2) is obtained as

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = &\sum_{n=-\infty}^{\infty} \frac{(b, c, d; q)_n \left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}; q\right)_n} \\ &\times \left\{ \frac{(a, b; q)_{|n|} \left(\frac{c}{ab}\right)^{|n|}}{(q, c; q)_{|n|}} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_{|n|}}{\left(c, \frac{c}{ab}; q\right)_{|n|}} \right\} \\ = &\sum_{n=0}^{\infty} \frac{(b, c, d; q)_n \left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}; q\right)_n} \\ &\times \left\{ \frac{(a, b; q)_n \left(\frac{c}{ab}\right)^n}{(q, c; q)_n} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_n}{\left(c, \frac{c}{ab}; q\right)_n} \right\} \\ &+ \sum_{n=-\infty}^{-1} \frac{(b, c, d; q)_n \left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}; q\right)_n} \\ &\times \left\{ \frac{(a, b; q)_{-n} \left(\frac{c}{ab}\right)^{-n}}{(q, c; q)_{-n}} + \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} - \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_{-n}}{\left(c, \frac{c}{ab}; q\right)_{-n}} \right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(b,c,d;q)_n \left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n} \\
&\quad \times \left\{ \frac{(a,b;q)_n \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \right\} \\
&\quad + \sum_{n=1}^{\infty} \frac{(b,c,d;q)_{-n} \left(\frac{q}{bcd}\right)^{-n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{-n}} \\
&\quad \times \left\{ \frac{(a,b;q)_n \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \right\}.
\end{aligned}
$$

For $n=0$, we get

$$
\begin{aligned}
&\frac{(b,c,d;q)_{-n} \left(\frac{q}{bcd}\right)^{-n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{-n}} \left\{ \frac{(a,b;q)_n \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \right\} \\
&= 1 \times \left\{ 1 + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - 1 \right\} = \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}.
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n &= \sum_{n=0}^{\infty} \frac{(b,c,d;q)_n \left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n} \\
&\quad \times \left\{ \frac{(a,b;q)_n \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \right\} \\
&\quad + \sum_{n=0}^{\infty} \frac{(b,c,d;q)_{-n} \left(\frac{q}{bcd}\right)^{-n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{-n}} \\
&\quad \times \left\{ \frac{(a,b;q)_n \left(\frac{c}{ab}\right)^n}{(q,c;q)_n} + \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n} \right\} - \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}.
\end{aligned}
$$

Since,

$$\frac{(b,c,d;q)_{-n}\left(\frac{q}{bcd}\right)^{-n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{-n}}$$

$$=\frac{(b;q)_{-n}(c;q)_{-n}(d;q)_{-n}\left(\frac{q}{bcd}\right)^{-n}}{\left(\frac{q}{b};q\right)_{-n}\left(\frac{q}{c};q\right)_{-n}\left(\frac{q}{d};q\right)_{-n}}$$

$$=\frac{\frac{\left(-\frac{q}{b}\right)^{n}}{\left(\frac{q}{b};q\right)_{n}}q^{\frac{n(n-1)}{2}}\frac{\left(-\frac{q}{c}\right)^{n}}{\left(\frac{q}{c};q\right)_{n}}q^{\frac{n(n-1)}{2}}\frac{\left(-\frac{q}{d}\right)^{n}}{\left(\frac{q}{d};q\right)_{n}}q^{\frac{n(n-1)}{2}}}{\frac{(-b)^{n}}{(b;q)_{n}}q^{\frac{n(n-1)}{2}}\frac{(-c)^{n}}{(c;q)_{n}}q^{\frac{n(n-1)}{2}}\frac{(-d)^{n}}{(d;q)_{n}}q^{\frac{n(n-1)}{2}}}\left(\frac{bcd}{q}\right)^{n}$$

$$=\frac{(b;q)_{n}(c;q)_{n}(d;q)_{n}}{\left(\frac{q}{b};q\right)_{n}\left(\frac{q}{c};q\right)_{n}\left(\frac{q}{d};q\right)_{n}}\left(\frac{q^{2}}{bcd}\right)^{n}$$

$$=\frac{(b,c,d;q)_{n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{n}}\left(\frac{q^{2}}{bcd}\right)^{n}.$$

Therefore,

$$\sum_{n=-\infty}^{\infty}\alpha_n\gamma_n$$

$$=\sum_{n=0}^{\infty}\frac{(b,c,d;q)_{n}\left(\frac{q}{bcd}\right)^{n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{n}}\left\{\frac{(a,b;q)_{n}\left(\frac{c}{ab}\right)^{n}}{(q,c;q)_{n}}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{\infty}}{\left(c,\frac{c}{ab};q\right)_{\infty}}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{n}}{\left(c,\frac{c}{ab};q\right)_{n}}\right\}$$

$$+\sum_{n=0}^{\infty}\frac{(b,c,d;q)_{n}\left(\frac{q^{2}}{bcd}\right)^{n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{n}}\left\{\frac{(a,b;q)_{n}\left(\frac{c}{ab}\right)^{n}}{(q,c;q)_{n}}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{\infty}}{\left(c,\frac{c}{ab};q\right)_{\infty}}-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{n}}{\left(c,\frac{c}{ab};q\right)_{n}}\right\}$$

$$-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{\infty}}{\left(c,\frac{c}{ab};q\right)_{\infty}}$$

$$=\sum_{n=0}^{\infty}\frac{(b,c,d;q)_{n}\left(\frac{q}{bcd}\right)^{n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{n}}\frac{(a,b;q)_{n}\left(\frac{c}{ab}\right)^{n}}{(q,c;q)_{n}}+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_{\infty}}{\left(c,\frac{c}{ab};q\right)_{\infty}}\sum_{n=0}^{\infty}\frac{(b,c,d;q)_{n}\left(\frac{q}{bcd}\right)^{n}}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_{n}}$$

$$-\sum_{n=0}^{\infty}\frac{(b,c,d;q)_n\left(\frac{q}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n}\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}+\sum_{n=0}^{\infty}\frac{(b,c,d;q)_n\left(\frac{q^2}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n}\frac{(a,b;q)_n\left(\frac{c}{ab}\right)^n}{(q,c;q)_n}$$

$$+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(b,c,d;q)_n\left(\frac{q^2}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n}-\sum_{n=0}^{\infty}\frac{(b,c,d;q)_n\left(\frac{q^2}{bcd}\right)^n}{\left(\frac{q}{b},\frac{q}{c},\frac{q}{d};q\right)_n}\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c,\frac{c}{ab};q\right)_n}$$

$$-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}$$

$$={}_4\phi_3\left[\begin{matrix}a,b,b,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q}{ab^2d}\right]+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}{}_4\phi_3\left[\begin{matrix}q,b,c,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q}{bcd}\right]$$

$$-{}_5\phi_4\left[\begin{matrix}q,b,d,\frac{c}{a},\frac{c}{b}\\ \frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{c}{ab}\end{matrix};q,\frac{q}{bcd}\right]+{}_4\phi_3\left[\begin{matrix}a,b,b,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q^2}{ab^2d}\right]$$

$$+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}{}_4\phi_3\left[\begin{matrix}q,b,c,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q^2}{bcd}\right]-{}_5\phi_4\left[\begin{matrix}q,b,d,\frac{c}{a},\frac{c}{b}\\ \frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{c}{ab}\end{matrix};q,\frac{q^2}{bcd}\right]$$

$$-\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}$$

$$={}_4\phi_3\left[\begin{matrix}a,b,b,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q}{ab^2d}\right]+{}_4\phi_3\left[\begin{matrix}a,b,b,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q^2}{ab^2d}\right]$$

$$+\frac{\left(\frac{c}{a},\frac{c}{b};q\right)_\infty}{\left(c,\frac{c}{ab};q\right)_\infty}\left\{{}_4\phi_3\left[\begin{matrix}q,b,c,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q}{bcd}\right]+{}_4\phi_3\left[\begin{matrix}q,b,c,d\\ \frac{q}{b},\frac{q}{c},\frac{q}{d}\end{matrix};q,\frac{q^2}{bcd}\right]-1\right\}$$

$$- {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix} ; q, \frac{q}{bcd}\right] - {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix} ; q, \frac{q^2}{bcd}\right].$$

From equation (2.6), we have

$$\sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=-\infty}^{\infty} \alpha_n \gamma_n.$$

Hence, the result (3.2) is given as follows

$${}_6\phi_5\left[\begin{matrix} q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, a, b \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}, c \end{matrix} ; q, \frac{c}{ab}\right]$$

$$= {}_4\phi_3\left[\begin{matrix} a, b, b, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q}{ab^2d}\right] + {}_4\phi_3\left[\begin{matrix} a, b, b, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q^2}{ab^2d}\right]$$

$$+ \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty}\left\{ {}_4\phi_3\left[\begin{matrix} q, b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q}{bcd}\right] + {}_4\phi_3\left[\begin{matrix} q, b, c, d \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q^2}{bcd}\right] - 1\right\}$$

$$- {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix} ; q, \frac{q}{bcd}\right] - {}_5\phi_4\left[\begin{matrix} q, b, d, \frac{c}{a}, \frac{c}{b} \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{c}{ab} \end{matrix} ; q, \frac{q^2}{bcd}\right]. \qquad \square$$

Following the similar procedure as in the proof of main results (i) and (ii), one can easily derive the main results (iii), (iv) and (v). which are given as follows.

Proof of (iii). Putting $u_r = v_r = 1$ and $\alpha_r = \frac{(-qa^{\frac{1}{2}}, b, c, d; q)_r \; (qa^{\frac{3}{2}}/bcd)^r}{(-a^{\frac{1}{2}}, aq/b, aq/c, aq/d; q)_r}$ in (2.4) and making use of the result (2.10), provided $n \to \infty$, we get β_n, which is as follows:

$$\beta_n = \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}, \frac{qa^{\frac{1}{2}}}{b}, \frac{qa^{\frac{1}{2}}}{c}, \frac{qa^{\frac{1}{2}}}{d}, q, \frac{q}{a}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, \frac{qa^{\frac{3}{2}}}{bcd}; q\right)_n}.$$

Putting the values of α_n, β_n and values of δ_n, γ_n from (4.1) and (4.2) in (2.6) and after simplification, we obtain result (3.3). □

Proof of (iv). Putting $u_r = v_r = 1$ and $\alpha_r = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_r \ (qa^2/bcde)^r}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_r}$ in (2.4) and making use of the result (2.11), provided $n \to \infty$, we get β_n, which is as follows:

$$\beta_n = \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, \frac{q}{a}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{qa^2}{bcde}; q\right)_n}.$$

Putting the values of α_n, β_n and values of δ_n, γ_n from (4.1) and (4.2) in (2.6) and after simplification, we obtain result (3.4). □

Proof of (v). Putting $u_r = v_r = 1$ and

$$\alpha_r = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f, aq^2/f; q)_r \ (a^2/bcde)^r}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f, f/q; q)_r}$$

in (2.4) and making use of the result (2.12), provided $n \to \infty$, we get β_n, which is as follows

$$\beta_n = \left(1 - \frac{\left(1-\frac{bc}{a}\right)\left(1-\frac{bd}{a}\right)\left(1-\frac{be}{a}\right)}{\left(1-\frac{bq}{f}\right)\left(1-\frac{bf}{aq}\right)\left(1-\frac{bcde}{a^2}\right)}\right) \frac{\left(1-\frac{f}{bq}\right)\left(1-\frac{bf}{aq}\right)}{\left(1-\frac{f}{aq}\right)\left(1-\frac{f}{q}\right)}$$
$$\times \frac{\left(q, aq, \frac{q}{a}, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{qa^2}{bcde}; q\right)_n}.$$

Putting the values of α_n, β_n and values of δ_n, γ_n from (4.1) and (4.2) in (2.6) and after simplification, we obtain result (3.5). □

5. Special Cases

Certain interesting results have been developed from the main results as special cases, which are as follows:

(i) Putting $b = q$ in result (3.1), special case is as follows

$${}_3\phi_2 \left[\begin{matrix} q, az, a \\ \\ z, c \end{matrix} ; q, \frac{c}{aq} \right] = \frac{\left(\frac{c}{a}, \frac{c}{q}; q\right)_\infty}{\left(c, \frac{c}{aq}; q\right)_\infty} {}_1\phi_0 \left[\begin{matrix} a \\ \\ - \end{matrix} ; q, z \right]$$

$$+ {}_2\phi_1 \left[\begin{matrix} a, a \\ \\ c \end{matrix} ; q, \frac{cz}{aq} \right] - {}_3\phi_2 \left[\begin{matrix} a, \frac{c}{a}, \frac{c}{q} \\ \\ c, \frac{c}{aq} \end{matrix} ; q, z \right].$$

(ii) Putting $a = c$ in result (3.2), special case is as follows

$${}_5\phi_4 \left[\begin{matrix} q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}, b \\ \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd} \end{matrix} ; q, \frac{1}{b} \right]$$
$$= {}_4\phi_3 \left[\begin{matrix} b, b, c, d \\ \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q}{b^2cd} \right] + {}_4\phi_3 \left[\begin{matrix} b, b, c, d \\ \\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d} \end{matrix} ; q, \frac{q^2}{b^2cd} \right] - 2.$$

(iii) Putting $a = c$ in the result (3.3), special case is as follows

$${}_5\phi_4 \left[\begin{matrix} cq, \frac{cq}{bd}, \frac{qc^{\frac{1}{2}}}{b}, \frac{qc^{\frac{1}{2}}}{d}, b \\ \\ \frac{cq}{b}, \frac{cq}{d}, qc^{\frac{1}{2}}, \frac{qc^{\frac{1}{2}}}{bd} \end{matrix} ; q, \frac{1}{b} \right] = {}_5\phi_4 \left[\begin{matrix} -qc^{\frac{1}{2}}, b, b, c, d \\ \\ -c^{\frac{1}{2}}, \frac{cq}{b}, q, \frac{cq}{d} \end{matrix} ; q, \frac{qc^{\frac{1}{2}}}{b^2d} \right] - 1.$$

(iv) Putting $a = c$ in the result (3.4), special case is as follows

$${}_5\phi_4 \left[\begin{matrix} cq, \frac{cq}{bd}, \frac{cq}{be}, \frac{cq}{de}, b \\ \\ \frac{cq}{b}, \frac{cq}{d}, \frac{cq}{e}, \frac{cq}{bde} \end{matrix} ; q, \frac{1}{b} \right] = {}_7\phi_6 \left[\begin{matrix} qc^{\frac{1}{2}}, -qc^{\frac{1}{2}}, b, b, c, d, e \\ \\ c^{\frac{1}{2}}, -c^{\frac{1}{2}}, \frac{cq}{b}, q, \frac{cq}{d}, \frac{cq}{e} \end{matrix} ; q, \frac{cq}{b^2de} \right] - 1.$$

(v) Putting $a = c$ in the result (3.5), special case is as follows

$${}_5\phi_4 \left[\begin{matrix} cq, \frac{cq}{bd}, \frac{cq}{be}, \frac{cq}{de}, b \\ \\ \frac{cq}{b}, \frac{cq}{d}, \frac{cq}{e}, \frac{cq}{bde} \end{matrix} ; q, \frac{1}{b} \right]$$

$$
= \frac{\dfrac{\left(1-\dfrac{f}{cq}\right)\left(1-\dfrac{f}{q}\right)}{\left(1-\dfrac{f}{bq}\right)\left(1-\dfrac{bf}{cq}\right)}}{\left(1-\dfrac{(1-b)\left(1-\dfrac{bd}{c}\right)\left(1-\dfrac{be}{c}\right)}{\left(1-\dfrac{bq}{f}\right)\left(1-\dfrac{bf}{cq}\right)\left(1-\dfrac{bde}{c}\right)}\right)}
\times \left\{ {}_9\phi_8 \left[\begin{matrix} qc^{\frac{1}{2}}, -qc^{\frac{1}{2}}, b, b, c, d, e, f, \dfrac{cq^2}{f} \\ c^{\frac{1}{2}}, -c^{\frac{1}{2}}, \dfrac{cq}{b}, q, \dfrac{cq}{d}, \dfrac{cq}{e}, \dfrac{cq}{f}, \dfrac{f}{q} \end{matrix} ; q, \frac{c}{b^2de} \right] - 1 \right\}.
$$

Similarly, other special cases can be derived by taking appropriate selections in the main results.

6. Conclusions

Certain results in the form of two-term, three-term and four-term transformation formulae had been discussed by the authors [12]. Certain new results in terms of basic hypergeometric series have been established in this paper. Our work extends the list of transformation formulae given in [12], that will provide some scope for future research in this field.

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