

A GENERALIZATION OF HERMITE-HADAMARD'S INEQUALITY

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ABSTRACT. In literature the Hermite-Hadamard inequality was eligible for many reasons, one of the most surprising and interesting that the Hermite-Hadamard inequality combine the midpoint and trapezoid formulae in an inequality. In this work, a Hermite-Hadamard like inequality that combines the composite trapezoid and composite midpoint formulae is proved. So that, the classical Hermite-Hadamard inequality becomes a special case of the presented result. Some Ostrowski's type inequalities for convex functions are proved as well.

1. INTRODUCTION

Let $f : [a, b] \rightarrow \mathbb{R}$, be a twice differentiable mapping such that $f''(x)$ exists on (a, b) and $\|f''\|_\infty = \sup_{x \in (a,b)} |f''(x)| < \infty$. Then the midpoint inequality is known as:

$$\left| \int_a^b f(x) dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^3}{24} \|f''\|_\infty,$$

and, the trapezoid inequality

$$\left| \int_a^b f(x) dx - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty,$$

also hold. Therefore, the integral $\int_a^b f(x) dx$ can be approximated in terms of the midpoint and the trapezoidal rules, respectively such as:

$$\int_a^b f(x) dx \cong (b-a) f\left(\frac{a+b}{2}\right),$$

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and

$$\int_a^b f(x) dx \cong (b-a) \frac{f(a) + f(b)}{2},$$

which are combined in a useful and famous relationship, known as the Hermite-Hadamard's inequality. That is,

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},$$

which hold for all convex functions f defined on a real interval $[a, b]$.

The real beginning was (almost) in the last twenty five years, where, in 1992 Dragomir [9] published his article about (1.1). The main result in [9] was

Theorem 1.1. *Let $f : [a, b]$ is convex function one can define the following mapping on $[0, 1]$ such as:*

$$H(t) = \frac{1}{(b-a)} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

then:

- (a) H is convex and monotonic non-decreasing on $[0, 1]$;
- (b) one has the bounds for H

$$\sup_{t \in [0,1]} H(t) = \frac{1}{(b-a)} \int_a^b f(x) dx = H(1),$$

and

$$\inf_{t \in [0,1]} H(t) = f\left(\frac{a+b}{2}\right) = H(0).$$

Few years after 1992, many authors have took (a real) attention to the Hermite-Hadamard inequality and sequence of several works under various assumptions for the function involved such as bounded variation, convex, differentiable functions whose n -derivative(s) belong to $L_p[a, b]$; ($1 \leq p \leq \infty$), Lipschitz, monotonic, etc., have been published. For a comprehensive list of results and excellent bibliography we recommend the interested to refer to [3, 4, 13].

In 1997, Yang and Hong [15], continued on Dragomir result (Theorem 1.1) and they proved the following theorem.

Theorem 1.2. *Suppose that $f : [a, b] \rightarrow \mathbb{R}$, is convex and the mapping $F : [0, 1] \rightarrow \mathbb{R}$ is defined by*

$$F(t) = \frac{1}{b-a} \int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}u\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}u\right) \right] du,$$

then:

- (a) the mapping F is convex and monotonic nondecreasing on $[0, 1]$;
- (b) we have the bounds

$$\inf_{t \in [0,1]} F(t, s) = \frac{1}{(b-a)} \int_a^b f(x) dx = F(0),$$

$$\sup_{t \in [0,1]} F(t) = \frac{f(a) + f(b)}{2} = F(1).$$

For other closely related results see [1, 5–8, 10–12].

In terms of composite numerical integration, we recall the composite midpoint rule [2, p. 202]

$$\int_a^b f(x) dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{(b-a)}{6} h^2 f''(\mu),$$

for some $\mu \in (a, b)$, where $f \in C^2[a, b]$, n is even, $h = \frac{b-a}{n+2}$ and $x_j = a + (j + 1)h$, for each $j = -1, 0, \dots, n + 1$; and, the composite trapezoid rule [2, p. 203]

$$\int_a^b f(x) dx = \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{(b-a)}{12} h^2 f''(\mu),$$

for some $\mu \in (a, b)$, where $f \in C^2[a, b]$, $h = \frac{b-a}{n}$ and $x_j = a + jh$, for each $j = 0, 1, \dots, n$.

The main purpose of this work, is to combine the composite trapezoid and composite midpoint formulae in an inequality that is similar to the classical Hermite-Hadamard inequality (1.1) for convex functions defined on a real interval $[a, b]$. In this way, we establish a conventional generalization of (1.1) which is in turn most useful and has a very constructional form.

2. A GENERALIZATION OF HERMITE-HADAMARD'S INEQUALITY

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$, then the double inequality*

$$(2.1) \quad h \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) \leq \int_a^b f(t) dt \leq \frac{h}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right],$$

holds, where $x_k = a + k\frac{b-a}{n}$, $k = 0, 1, 2, \dots, n$; with $h = \frac{b-a}{n}$, $n \in \mathbb{N}$. The constant '1' in the left-hand side and ' $\frac{1}{2}$ ' in the right-hand side are the best possible for all $n \in \mathbb{N}$. If f is concave then the inequality is reversed.

Nota bene: after revision of this paper, the anonymous referee informed us that the inequality (2.1) was proved in more general case in [14] (see also [13, p. 22–23]). We appreciate this remark from the reviewer.

Proof. Since f is convex on $[a, b]$, then f so is on each subinterval $[x_{j-1}, x_j]$, $j = 1, \dots, n$, then for all $t \in [0, 1]$, we have

$$(2.2) \quad f(tx_{j-1} + (1-t)x_j) \leq tf(x_{j-1}) + (1-t)f(x_j).$$

Integrating (2.2) with respect to t on $[0, 1]$ we get

$$(2.3) \quad \int_0^1 f(tx_{j-1} + (1-t)x_j) dt \leq \frac{f(x_{j-1}) + f(x_j)}{2}.$$

Substituting $u = tx_{j-1} + (1-t)x_j$, in the left hand side of (2.3), we get

$$\int_{x_{j-1}}^{x_j} f(u) du \leq \frac{x_j - x_{j-1}}{2} (f(x_{j-1}) + f(x_j)).$$

Taking the sum over j from 1 to n , we get

$$(2.4) \quad \begin{aligned} & \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(u) du \\ &= \int_a^b f(u) du \\ &\leq \sum_{j=1}^n \frac{x_j - x_{j-1}}{2} (f(x_{j-1}) + f(x_j)) \\ &\leq \frac{1}{2} \max_j \{x_j - x_{j-1}\} \cdot \sum_{j=1}^n (f(x_{j-1}) + f(x_j)) \\ &= \frac{h}{2} \left[f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{f(x_{j-1}) + f(x_j)\} + f(x_{n-1}) + f(x_n) \right] \\ &= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right]. \end{aligned}$$

On the other hand, again since f is convex on I_x , then for $t \in [0, 1]$, we have

$$(2.5) \quad \begin{aligned} f\left(\frac{x_{j-1} + x_j}{2}\right) &= f\left(\frac{tx_j + (1-t)x_{j-1}}{2} + \frac{(1-t)x_j + tx_{j-1}}{2}\right) \\ &\leq \frac{1}{2} [f(tx_j + (1-t)x_{j-1}) + f((1-t)x_j + tx_{j-1})]. \end{aligned}$$

Integrating inequality (2.5) with respect to t on $[0, 1]$ we get

$$\begin{aligned}
 (2.6) \quad & f\left(\frac{x_{j-1} + x_j}{2}\right) \\
 & \leq \frac{1}{2} \int_0^1 [f(tx_j + (1-t)x_{j-1}) + f((1-t)x_j + tx_{j-1})] dt \\
 & = \frac{1}{2} \int_0^1 f(tx_j + (1-t)x_{j-1}) dt + \frac{1}{2} \int_0^1 f((1-t)x_j + tx_{j-1}) dt.
 \end{aligned}$$

By putting $1-t = s$ in the second integral on the right-hand side of (2.6), we have

$$\begin{aligned}
 (2.7) \quad & f\left(\frac{x_{j-1} + x_j}{2}\right) \\
 & \leq \frac{1}{2} \int_0^1 f(tx_j + (1-t)x_{j-1}) dt + \frac{1}{2} \int_0^1 f((1-t)x_j + tx_{j-1}) dt \\
 & = \int_0^1 f(tx_j + (1-t)x_{j-1}) dt.
 \end{aligned}$$

Substituting $u = tx_j + (1-t)x_{j-1}$, in the left hand side of (2.7), and then taking the sum over j from 1 to n , we get

$$(2.8) \quad h \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) \leq \int_a^b f(t) dt.$$

From (2.4) and (2.8), we get the desired inequality (2.1).

To prove the sharpness let (2.1) hold with another constants $C_1, C_2 > 0$, which gives

$$\begin{aligned}
 (2.9) \quad C_1 \cdot h \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) & \leq \int_a^b f(t) dt \\
 & \leq C_2 \cdot h \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right].
 \end{aligned}$$

Let $f : [a, b] \rightarrow \mathbb{R}$ be the identity map $f(x) = x$, then the right-hand side of (2.9) reduces to

$$\begin{aligned} \frac{b^2 - a^2}{2} &\leq C_2 \cdot h \left[a + 2 \sum_{k=1}^{n-1} x_k + b \right] \\ &= C_2 \cdot h \left[a + 2 \sum_{k=1}^{n-1} \left(a + k \frac{b-a}{n} \right) + b \right] \\ &= C_2 \cdot h \left[a + 2 \sum_{k=1}^{n-1} a + 2 \frac{b-a}{n} \sum_{k=1}^{n-1} k + b \right] \\ &= C_2 \cdot \frac{b-a}{n} \left[a + 2(n-1)a + 2 \frac{b-a}{n} \cdot \frac{n(n-1)}{2} + b \right] \\ &= C_2 \cdot (b-a)(a+b). \end{aligned}$$

It follows that $\frac{1}{2} \leq C_2$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.1).

For the left-hand side, we have

$$\begin{aligned} \frac{b^2 - a^2}{2} &\geq C_1 \cdot h \sum_{k=1}^n \frac{x_{k-1} + x_k}{2} \\ &= C_1 \cdot h \sum_{k=1}^n \left\{ a + (2k-1) \frac{b-a}{2n} \right\}. \\ &= C_1 \cdot \frac{b-a}{n} \cdot \left[na + \frac{b-a}{2n} \left(2 \cdot \frac{n(n+1)}{2} - n \right) \right] \\ &= C_1 \cdot \frac{b^2 - a^2}{2}, \end{aligned}$$

which means that $1 \geq C_1$, and thus 1 is the best possible constant in the left-hand side of (2.1). Thus the proof of Theorem 2.1 is completely finished. \square

Remark 2.1. In Theorem 2.1, if we take $n = 1$, then we refer to the original Hermite-Hadamard inequality (1.1).

In viewing of (2.1), next we give direct sharp refinements of Hermite-Hadamard's type inequalities for convex functions defined on a real interval $[a, b]$, according to the number of division 'n' (in our case $n = 1, 2, 3, 4$) in Theorem 2.1.

Corollary 2.1. *In Theorem 2.1, we have*

(a) if $n = 1$, then

$$(b-a) f\left(\frac{a+b}{2}\right) \leq \int_a^b f(t) dt \leq (b-a) \frac{f(a) + f(b)}{2};$$

(b) if $n = 2$, then

$$\begin{aligned} & \frac{(b-a)}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ & \leq \int_a^b f(t) dt \\ & \leq \frac{(b-a)}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right]; \end{aligned}$$

(c) if $n = 3$, then

$$\begin{aligned} & \frac{(b-a)}{3} \left[f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+b}{2}\right) + f\left(\frac{a+5b}{6}\right) \right] \\ & \leq \int_a^b f(t) dt \\ & \leq \frac{(b-a)}{6} \left[f(a) + 2f\left(\frac{2a+b}{3}\right) + 2f\left(\frac{a+2b}{3}\right) + f(b) \right]; \end{aligned}$$

(d) if $n = 4$, then

$$\begin{aligned} & \frac{(b-a)}{4} \left[f\left(\frac{7a+b}{8}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] \\ & \leq \int_a^b f(t) dt \\ & \leq \frac{(b-a)}{12} \left[f(a) + 2f\left(\frac{3a+b}{4}\right) + 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right) + f(b) \right]. \end{aligned}$$

Now, let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Define the mappings $H_j, F_j : [0, 1] \rightarrow \mathbb{R}$, given by

$$(2.10) \quad H_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} f\left(tu + (1-t)\frac{x_{j-1} + x_j}{2}\right) du, \quad u \in [x_{j-1}, x_j],$$

and

$$(2.11) \quad F_j(t) = \frac{1}{h} \int_{x_{j-1}}^{x_j} \left[f\left(\frac{1+t}{2}x_{j-1} + \frac{1-t}{2}u\right) + f\left(\frac{1+t}{2}x_j + \frac{1-t}{2}u\right) \right] du,$$

where $u \in [x_{j-1}, x_j]$.

Applying Theorems 1.1 and 1.2, for $f : [x_{j-1}, x_j] \rightarrow \mathbb{R}$, $j = 1, 2, \dots, n$. Then the following statements hold:

- (a) $H_j(t)$ and $F_j(t)$ are convex for all $t \in [0, 1]$ and $u \in [x_{j-1}, x_j]$;
- (b) $H_j(t)$ and $F_j(t)$ are monotonic nondecreasing for all $t \in [0, 1]$ and $u \in [x_{j-1}, x_j]$.
- (c) we have the following bounds for $H_j(t)$

$$(2.12) \quad \frac{1}{h} \int_{x_{j-1}}^{x_j} f(u) du = H_j(1),$$

and

$$(2.13) \quad f\left(\frac{x_{k-1} + x_k}{2}\right) = H_j(0).$$

and the following bounds for $F_j(t)$

$$(2.14) \quad \frac{f(x_{j-1}) + f(x_j)}{2} = F_j(1),$$

and

$$(2.15) \quad \frac{1}{h} \int_{x_{j-1}}^{x_j} f(u) du = F_j(0).$$

Hence, we may establish two related mappings for the inequality (2.1).

Proposition 2.1. *Let f be as in Theorem 2.1, define the mappings $H, F : [0, 1] \rightarrow \mathbb{R}$, given by*

$$H(t) = \sum_{j=1}^n H_j(t) \quad \text{and} \quad F(t) = \sum_{j=1}^n F_j(t),$$

where $H_j(t)$ and $F_j(t)$ are defined in (2.10) and (2.11), respectively; then the following statements hold:

- (a) $H(t)$ and $F(t)$ are convex for all $t \in [0, 1]$ and $u \in [a, b]$;
- (b) $H(t)$ and $F(t)$ are monotonic nondecreasing for all $t \in [0, 1]$ and $u \in [a, b]$;
- (c) We have the following bounds for $H(t)$

$$\sup_{t \in [0,1]} H(t) = \frac{1}{h} \int_a^b f(u) du = H(1),$$

and

$$\inf_{t \in [0,1]} H(t) = \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) = H(0),$$

and the following bounds for $F(t)$

$$\sup_{t \in [0,1]} F(t) = \frac{1}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right] = F(1),$$

and

$$\inf_{t \in [0,1]} F(t) = \frac{1}{h} \int_a^b f(u) du = F(0).$$

Proof. Taking the sum over j from 1 to n , in (2.12)–(2.15) we get the required results, and we shall omit the details. \square

Remark 2.2. The inequality (2.1) may written in a convenient way as follows:

$$\sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right) - \sum_{k=1}^{n-1} f(x_k) \leq \frac{1}{h} \int_a^b f(t) dt - \sum_{k=1}^{n-1} f(x_k) \leq \frac{f(a) + f(b)}{2},$$

which is of Ostrowski's type.

Some sharps Ostrowski's type inequalities for convex functions defined on a real interval $[a, b]$, are proposed in the next theorems.

Theorem 2.2. *Let $f : [a, b] \rightarrow \mathbb{R}_+$ be a convex function on $[a, b]$, then the inequality*

$$(2.16) \quad \int_a^b f(x) dx - (b-a)f(y) \leq \frac{h}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right],$$

holds for all $y \in [a, b]$. where, $x_k = a + k \frac{b-a}{n}$, $k = 0, 1, 2, \dots, n$; with $h = \frac{b-a}{n}$, $n \in \mathbb{N}$. The constant $\frac{1}{2}$ in the right-hand side is the best possible, in the sense that it cannot be replaced by a smaller one for all $n \in \mathbb{N}$. If f is concave then the inequality is reversed.

Proof. Fix $y \in [x_{j-1}, x_j]$, $j = 1, \dots, n$. Since f is convex on $[a, b]$, then f so is on each subinterval $[x_{j-1}, x_j]$, in particular on $[x_{j-1}, y]$, then for all $t \in [0, 1]$, we have

$$(2.17) \quad f(tx_{j-1} + (1-t)y) \leq tf(x_{j-1}) + (1-t)f(y), \quad j = 1, \dots, n.$$

Integrating (2.17) with respect to t on $[0, 1]$ we get

$$(2.18) \quad \int_0^1 f(tx_{j-1} + (1-t)y) dt \leq \frac{f(x_{j-1}) + f(y)}{2}.$$

Substituting $u = tx_{j-1} + (1-t)y$, in the left hand side of (2.18), we get

$$(2.19) \quad \int_{x_{j-1}}^y f(u) du \leq \frac{y - x_{j-1}}{2} (f(x_{j-1}) + f(y)).$$

Now, we do similarly for the interval $[y, x_j]$, we therefore have

$$(2.20) \quad f(ty + (1-t)x_j) \leq tf(y) + (1-t)f(x_j), \quad j = 1, \dots, n.$$

Integrating (2.20) with respect to t on $[0, 1]$ we get

$$(2.21) \quad \int_0^1 f(ty + (1-t)x_j) dt \leq \frac{f(y) + f(x_j)}{2}.$$

Substituting $u = ty + (1-t)x_j$, in the left hand side of (2.21), we get

$$(2.22) \quad \int_y^{x_j} f(u) du \leq \frac{x_j - y}{2} (f(y) + f(x_j)).$$

Adding the inequalities (2.19) and (2.22), we get

$$\begin{aligned}
 (2.23) \quad & \int_{x_{j-1}}^y f(u) du + \int_y^{x_j} f(u) du \\
 &= \int_{x_{j-1}}^{x_j} f(u) du \\
 &\leq \frac{y - x_{j-1}}{2} (f(x_{j-1}) + f(y)) + \frac{x_j - y}{2} (f(y) + f(x_j)) \\
 &\leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + (x_j - x_{j-1}) f(y) \\
 &\leq \frac{x_j - x_{j-1}}{2} (f(x_{j-1}) + f(x_{j-1})) + hf(y).
 \end{aligned}$$

Taking the sum over j from 1 to n , we get

$$\begin{aligned}
 & \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(u) du \\
 &= \int_a^b f(u) du \\
 &= \sum_{j=1}^n \frac{x_j - x_{j-1}}{2} \{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^n hf(y) \\
 &\leq \frac{1}{2} \max_j \{x_j - x_{j-1}\} \cdot \sum_{j=1}^n (f(x_{j-1}) + f(x_j)) + (b - a) f(y) \\
 &= \frac{h}{2} \left[f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{f(x_{j-1}) + f(x_j)\} + f(x_{n-1}) + f(x_n) \right] + (b - a) f(y) \\
 &= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + (b - a) f(y),
 \end{aligned}$$

which gives that

$$\int_a^b f(u) du - (b - a) f(y) \leq \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],$$

for all $y \in [x_{j-1}, x_j] \subseteq [a, b]$ for all $j = 1, 2, \dots, n$, which gives the desired result (2.16).

To prove the sharpness let (2.16) hold with another constants $C > 0$, which gives

$$(2.24) \quad \int_a^b f(x) dx - (b - a) f(y) \leq C \cdot h \left[f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right].$$

Let $f : [0, 1] \rightarrow \mathbb{R}$ be the identity map $f(x) = x$, then the right-hand side of (2.24) reduces to

$$\begin{aligned} \frac{1}{2} - y &\leq C \cdot \frac{1}{n} \left[2 \sum_{k=1}^{n-1} x_k + 1 \right] \\ &= C \cdot \frac{1}{n} \left[2 \frac{1}{n} \cdot \frac{n(n-1)}{2} + 1 \right] \\ &= C. \end{aligned}$$

Choose $y = 0$, it follows that $\frac{1}{2} \leq C$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.16). □

Theorem 2.3. *Under the assumptions of Theorem 2.2, we have*

$$\begin{aligned} (2.25) \quad &\int_a^b f(x) dx - (b-a) f(y) \\ &\leq \left[\frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right], \end{aligned}$$

for all $y \in [a, b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible for all $n \in \mathbb{N}$. If f is concave then the inequality is reversed.

In particular, if $n = 1$ then

$$\int_a^b f(x) dx - (b-a) f(y) \leq \left[\frac{b-a}{2} + \left| y - \frac{a+b}{2} \right| \right] \cdot [f(a) + f(b)],$$

for all $y \in [a, b]$.

Proof. Repeating the steps of the proof of Theorem 2.2, therefore by (2.23)

$$\begin{aligned} (2.26) \quad &\int_{x_{j-1}}^{x_j} f(u) du \leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + hf(y) \\ &\leq \max \left\{ \frac{y - x_{j-1}}{2}, \frac{x_j - y}{2} \right\} \cdot (f(x_{j-1}) + f(x_{j-1})) + hf(y) \\ &\leq \left[\frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot (f(x_{j-1}) + f(x_{j-1})) \\ &\quad + hf(y) \end{aligned}$$

Taking the sum over j from 1 to n , we get

$$\begin{aligned}
& \int_a^b f(u) du \\
&= \sum_{j=1}^n \left[\frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^n hf(y) \\
&\leq \max_{1 \leq j \leq n} \left[\frac{x_j - x_{j-1}}{2} + \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \sum_{j=1}^n (f(x_{j-1}) + f(x_j)) + (b-a)f(y) \\
&\leq \left[\frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \\
&\quad \times \left[f(x_0) + f(x_1) + \sum_{j=2}^{n-1} \{f(x_{j-1}) + f(x_j)\} + f(x_{n-1}) + f(x_n) \right] + (b-a)f(y) \\
&= \left[\frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + (b-a)f(y),
\end{aligned}$$

which gives that

$$\begin{aligned}
& \int_a^b f(u) du - (b-a)f(y) \\
&\leq \left[\frac{h}{2} + \max_{1 \leq j \leq n} \left| y - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] m
\end{aligned}$$

for all $y \in [x_{j-1}, x_j] \subseteq [a, b]$ for all $j = 1, 2, \dots, n$, which gives the desired result (2.25). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. \square

Corollary 2.2. *Let $\alpha_i \geq 0$, for all $i = 0, 1, 2, \dots, n$, be positive real numbers such that $\sum_{i=0}^n \alpha_i = 1$, then under the assumptions of Theorem 2.3, we have*

$$\begin{aligned}
& \int_a^b f(x) dx - (b-a)f\left(\frac{1}{n+1} \sum_{i=0}^n \alpha_i x_i\right) \\
&\leq \left[\frac{h}{2} + \max_{1 \leq j \leq n} \left| \frac{1}{n+1} \sum_{i=0}^n \alpha_i x_i - \frac{x_{j-1} + x_j}{2} \right| \right] \cdot \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],
\end{aligned}$$

for all $y \in [a, b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed.

Theorem 2.4. *Under the assumptions of Theorem 2.3, we have*

$$(2.27) \quad \frac{1}{b-a} \int_a^b f(u) du - \frac{1}{n} \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) - \frac{1}{n} \sum_{j=1}^{n-1} f(x_j) \leq \frac{f(a) + f(b)}{2n},$$

for all $j = 1, 2, \dots, n$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed.

Proof. Repeating the steps of the proof of Theorem 2.3, (2.26) if we choose $y = \frac{x_{j-1} + x_j}{2}$, then we get

$$\int_{x_{j-1}}^{x_j} f(u) du \leq \frac{x_j - x_{j-1}}{2} \cdot (f(x_{j-1}) + f(x_j)) + hf\left(\frac{x_{j-1} + x_j}{2}\right).$$

Taking the sum over j from 1 to n , we get

$$\begin{aligned} \int_a^b f(u) du &\leq \sum_{j=1}^n \frac{x_j - x_{j-1}}{2} \cdot \{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^n hf\left(\frac{x_{j-1} + x_j}{2}\right) \\ &\leq \frac{h}{2} \sum_{j=1}^n \{f(x_{j-1}) + f(x_j)\} + h \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right), \end{aligned}$$

which gives that

$$\int_a^b f(u) du - h \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \leq \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right],$$

for all $j = 1, 2, \dots, n$, which gives the desired result (2.27). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. \square

Theorem 2.5. Let $I \subset \mathbb{R}$ be an open interval and $a, b \in I$, $a < b$. Let $f : I \rightarrow \mathbb{R}_+$ be an increasing convex function on $[a, b]$, then the inequality

$$(2.28) \quad \int_a^b f(t) dt - \frac{(b-a)}{2} f(y) \geq \frac{h}{2} \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \geq 0,$$

is valid for all $y \in [a, b] \subset I$. The constant $\frac{1}{2}$ in the right-hand side is the best possible, in the sense that it cannot be replaced by a greater one. If f is concave then the inequality is reversed. In particular, if $n = 1$ then

$$\int_a^b f(t) dt - \frac{(b-a)}{2} f(y) \geq \frac{b-a}{2} f\left(\frac{a+b}{2}\right) \geq 0,$$

Proof. Let $y \in [x_{j-1}, x_j]$ be an arbitrary point such that $x_{j-1} < y - p \leq y \leq y + p < x_j$ for all $j = 1, 2, \dots, n$ with $p > 0$.

It is well known that f is convex on I iff

$$f(y) \leq \frac{1}{2p} \int_{y-p}^{y+p} f(t) dt,$$

for every subinterval $[y - p, y + p] \subset [a, b] \subset I$ for some $p > 0$. But since f increases on $[a, b]$, we also have

$$f(y) \leq \frac{1}{2p} \int_{y-p}^{y+p} f(t) dt \leq \frac{1}{2p} \int_{x_{j-1}}^{x_j} f(t) dt.$$

Choosing $p \geq \frac{h}{2}$, (this choice is available since it is true for every subinterval in I), therefore from the last inequality we get

$$f(y) \leq \frac{1}{h} \int_{y-p}^{y+p} f(t) dt \leq \frac{1}{h} \int_{x_{j-1}}^{x_j} f(t) dt.$$

Again by convexity we have

$$hf\left(\frac{x_{j-1} + x_j}{2}\right) \leq \int_{x_{j-1}}^{x_j} f(t) dt.$$

Adding the last two inequalities, we get

$$hf(y) + hf\left(\frac{x_{j-1} + x_j}{2}\right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) dt,$$

or we write

$$hf\left(\frac{x_{j-1} + x_j}{2}\right) \leq 2 \int_{x_{j-1}}^{x_j} f(t) dt - hf(y).$$

Taking the sum over j from 1 to n , we get

$$h \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \leq 2 \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(t) dt - \sum_{j=1}^n hf(y)$$

hence,

$$\int_a^b f(t) dt - \frac{(b-a)}{2} f(y) \geq \frac{h}{2} \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \geq 0,$$

holds by positivity of f and this proves our assertion.

To prove the sharpness let (2.28) holds with another constant $C > 0$, which gives

$$(2.29) \quad \int_a^b f(t) dt - \frac{(b-a)}{2} f(y) \geq C \cdot h \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \geq 0.$$

Let $f : [0, 1] \rightarrow \mathbb{R}_+$ be the identity map $f(x) = x$, then the right-hand side of (2.29) reduces to

$$\begin{aligned} \frac{1}{2} - \frac{1}{2}y &\geq C \cdot \frac{1}{n} \sum_{j=1}^n \frac{2j-1}{2n} \\ &= C \cdot \frac{1}{n} \left[\sum_{j=1}^n \frac{j}{n} - \sum_{j=1}^n \frac{1}{2n} \right] \\ &= C \cdot \frac{1}{n} \left[\frac{1}{n} \cdot \frac{n(n+1)}{2} - \frac{1}{2n} \cdot n \right] \\ &= \frac{1}{2}C. \end{aligned}$$

Choose $y = \frac{1}{2}$, it follows that $\frac{1}{4} \geq \frac{1}{2}C$ which means that $\frac{1}{2} \geq C$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.28). \square

Corollary 2.3. *Let $\alpha_i \geq 0$, for all $i = 0, 1, 2, \dots, n$, be positive real numbers such that $\sum_{i=0}^n \alpha_i = 1$, then under the assumptions of Theorem 2.5, we have*

$$\int_a^b f(t) dt - \frac{(b-a)}{2} f\left(\frac{1}{n+1} \sum_{i=0}^n \alpha_i x_i\right) \geq \frac{h}{2} \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right) \geq 0.$$

The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed. In particular case if $n = 1$, then

$$\int_a^b f(t) dt - \frac{(b-a)}{2} f\left(\frac{\alpha a + (1-\alpha)b}{2}\right) \geq \frac{b-a}{2} f\left(\frac{a+b}{2}\right) \geq 0,$$

for all $\alpha \in [0, 1]$.

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