

SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION
 RELATED TO THE OPERATOR \oplus_B^k

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ABSTRACT. In this paper, we consider the equation

$$\oplus_B^k u(x) = \sum_{r=0}^m c_r \oplus_B^r \delta,$$

where \oplus_B^k is the operator iterated k -time and is defined by

$$\oplus_B^k = \left[(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^4 - (B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}})^4 \right]^k,$$

where $p+q=n$, $x = (x_1, \dots, x_n) \in \mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}$, $v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \dots, n$, c_r is a constant, k is a nonnegative integer, δ is the Dirac-delta distribution, $\oplus_B^0 \delta = \delta$ and n is the dimension of \mathbb{R}_n^+ . It is shown that, depending on the relationship between k and m , the solution to this equation can be ordinary functions, tempered distributions, or singular distributions.

1. INTRODUCTION

Bupasiri [5] has first introduced the elementary solution of the n -dimensional \oplus_B^k operator and showed that the solution of the convolution form $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{r*k}(x))^{*-1}$ is a unique elementary solution of the $\oplus_B^k u(x) = \delta$.

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Yildirim, Sarikaya and Ozturk [3] studied the Bessel diamond operator, iterated k -times,

$$\begin{aligned}
 (1.1) \quad \diamond_B^k &= \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right] \\
 &= \left[\sum_{i=1}^p B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right]^k \left[\sum_{i=1}^p B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right]^k.
 \end{aligned}$$

Yildirim, Sarikaya and Ozturk [3] showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator \diamond_B^k , where $*$ indicates convolution, and $R_{2k}(x), S_{2k}(x)$ are defined by (1.4) and (1.5) respectively, that is,

$$\diamond_B^k ((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta(x).$$

We consider the equation

$$\oplus_B^k u(x) = \sum_{r=0}^m c_r \oplus_B^r \delta,$$

where \oplus_B^k is the operator iterated k -time and is defined by

$$\begin{aligned}
 (1.2) \quad \oplus_B^k &= \left[\left(\sum_{i=1}^p B_{x_i} \right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^4 \right]^k \\
 &= \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k \\
 &= \diamond_B^k \odot_B^k,
 \end{aligned}$$

where

$$\begin{aligned}
 \odot_B^k &= \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k \\
 &= \left[\left(\frac{\Delta_B + \square_B}{2} \right)^2 + \left(\frac{\Delta_B - \square_B}{2} \right)^2 \right]^k \\
 &= \left(\frac{\Delta_B^2 + \square_B^2}{2} \right)^k.
 \end{aligned}$$

The purpose of this article, is finding the solution to the equation

$$(1.3) \quad \oplus_B^k u(x) = \sum_{r=0}^m c_r \oplus_B^r \delta$$

by using convolutions of the generalized function. It is also shown that the type of solution to (1.3) depends on the relationship between k and m , according to the following cases:

- (1) If $m < k$ and $m = 0$, then (1.3) has the solution

$$u(x) = c_0 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{**k}(x))^{*-1} \right),$$

which is an elementary solution of the \oplus_B^k operator in Theorem 2.2, is an ordinary function when $6k \geq n$, and is a tempered distribution when $6k < n$.

- (2) If $m < k$ then the solution of (1.3) is

$$u(x) = \sum_{r=1}^m c_r c_0 \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1} \right),$$

which is an ordinary function when $6k - 6r \geq n$ and is tempered distribution when $6k - 6r < n$.

- (3) If $m \geq k$ and $k \leq m \leq M$, then (1.3) has the solution

$$u(x) = \sum_{r=k}^M c_r \oplus_B^{r-k} \delta,$$

which is only a singular distribution. Before going that point, the following definitions and some concepts are needed.

Lemma 1.1. *Given the equation $\square_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \dots, x_n), x_1 > 0, \dots, x_n > 0\}$, where \square_B^k is the Bessel-ultra hyperbolic operator iterated k -times. Then $u(x) = R_{2k}(x)$ is an elementary solution of the operator \square_B^k , where*

$$\begin{aligned} \square_B^k &= \left[\sum_{i=1}^p B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right]^k, \\ (1.4) \quad R_{2k}(x) &= \frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)} \\ &= \frac{(x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)}, \end{aligned}$$

for

$$V = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2,$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-2k}{2}\right)}.$$

Lemma 1.2. *Given the equation $\Delta_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where Δ_B^k is the Laplace Bessel operator iterated k -times. Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator Δ_B^k , where*

$$\begin{aligned}
 \Delta_B^k &= \left[\sum_{i=1}^p B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j} \right]^k, \\
 (1.5) \quad S_{2k}(x) &= \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \quad p+q=n, \\
 |x| &= (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}},
 \end{aligned}$$

and

$$w_n(2k) = \frac{\prod_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma(v_i + \frac{1}{2}) \Gamma(k)}{2^{n+2|v|-4k} \Gamma(\frac{n+2|v|-2k}{2})}.$$

Lemma 1.3. *The convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is an elementary solution for the operator \diamond_B^k iterated k -times and is defined by (1.1).*

Lemma 1.4. *$R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2|v|)$.*

We need to show that $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$ satisfy the Euler equation; that is,

$$\begin{aligned}
 (2k - n - 2|v|) R_{2k}(x) &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{2k}(x), \\
 (2k - n - 2|v|) S_{2k}(x) &= \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} S_{2k}(x).
 \end{aligned}$$

Lemma 1.5 (The B -convolution of tempered distribution). *$R_{2k}(x) * S_{2k}(x)$ exists and is a tempered distribution.*

Proof. For the proofs of Lemmas 1.1–1.5, see [3, p. 378–383]. □

Lemma 1.6 (The B -convolution of $R_{2k}(x)$ and $S_{2k}(x)$). *Let $R_{2k}(x)$ and $S_{2k}(x)$ defined by (1.4) and (1.5) respectively, then we obtain:*

- (1) $S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integers.
- (2) $R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integers.

Lemma 1.7. *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverse in the convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively. That is,*

$$\begin{aligned}
 R_{-2k}(x) * R_{2k}(x) &= R_{-2k+2k}(x) = R_0(x) = \delta(x), \\
 (-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) &= S_{-2k+2k}(x) = S_0(x) = \delta(x).
 \end{aligned}$$

Proof. For the proofs of Lemma 1.7 and Lemma 1.6, see [4]. □

Lemma 1.8. *Given the equation*

$$(1.6) \quad \oplus_B^k u(x) = \delta(x),$$

where \oplus_B^k is the operator iterated k -times defined by (1.2), $\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$(1.7) \quad u(x) = (R_{6k}(x) * (-1)^{3k} S_{6k}(x)) * (C^{*k}(x))^{*-1}$$

is a Green's function or an elementary solution for the operator \oplus_B^k iterated k -times where \oplus_B^k is defined by (1.2), and

$$(1.8) \quad C(x) = \frac{1}{2}R_4(x) + \frac{1}{2}(-1)^2 S_4(x),$$

where $C^{*k}(x)$ denotes the convolution of C with itself k times, $(C^{*k}(x))^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $u(x)$ is a tempered distribution.

Proof. For a proof of the above lemma, see [5]. □

2. MAIN RESULTS

Theorem 2.1. For $0 < r < k$,

$$\begin{aligned} & \oplus_B^k (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) \\ &= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}) \end{aligned}$$

and for $k \leq m$,

$$\oplus_B^m (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) = \oplus_B^{m-k} \delta.$$

Proof. For $0 < r < k$, from (1.6),

$$\oplus_B^k (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) = \delta.$$

Thus,

$$\oplus_B^{k-r} \oplus_B^r (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) = \delta$$

or

$$\oplus_B^{k-r} \delta * \oplus_B^r (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) = \delta.$$

Convoluting both sides by $((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1})$, we obtain

$$\begin{aligned} & \oplus_B^{k-r} (((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1})) \\ & * \oplus_B^r (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) \\ &= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}) * \delta. \end{aligned}$$

By Lemma 1.8,

$$\begin{aligned} & \delta * \oplus_B^r (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) \\ &= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}) * \delta. \end{aligned}$$

It follows that

$$\begin{aligned} & \oplus_B^r (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1})) \\ &= ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}), \end{aligned}$$

as required. For $k \leq m$

$$\begin{aligned} & \oplus_B^m (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}) \\ &= \oplus_B^{m-k} \oplus_B^k ((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}). \end{aligned}$$

It follows that

$$\oplus_B^m (c_0((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}) = \oplus_B^{m-k} \delta$$

by Lemma 1.8. This completes the proof. □

Theorem 2.2. *Consider the linear differential equation*

$$(2.1) \quad \oplus_B^k u(x) = \sum_{r=0}^m c_r \oplus_B^r \delta,$$

where $p + q = n$, n is odd with p odd and q even, or n is even with p odd and q odd, $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \dots, x_n), x_1 > 0, \dots, x_n > 0\}$, c_r is a constant, δ is the Dirac-delta distribution, and $\oplus_B^0 \delta = \delta$. Then the type of solution to (2.1) depends on the relationship between k and m , according to the following cases:

- (1) If $m < k$ and $m = 0$, then (2.1) has the solution

$$u(x) = c_0 ((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}),$$

which is an elementary solution of the \oplus_B^k operator in Theorem 2.1, is an ordinary function when $6k \geq n$, and is a temper distribution when $6k < n$.

- (2) If $m < k$ then the solution of (2.1) is

$$u(x) = \sum_{r=1}^m c_r ((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1}),$$

which is an ordinary function when $6k - 6r \geq n$ and is tempered distribution when $6k - 6r < n$.

- (3) If $m \geq k$ and $k \leq m \leq M$, then (2.1) has the solution

$$u(x) = \sum_{r=k}^M c_r \oplus_B^{r-k} \delta,$$

which is only a singular distribution.

Proof. (1) For $m = 0$, we have $\oplus_B^k u(x) = c_0 \delta$, and by Theorem 2.1 we obtain

$$u(x) = ((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}).$$

Now, $(-1)^{3k} S_{6k}(x)$ and $R_{6k}(x)$ are the analytic function for $6k \geq n$ and also $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exists and is an analytic function by (1.7). It follows that $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ is an ordinary function for $6k \geq n$. By Lemma 1.5, $(-1)^{3k} S_{6k}(x)$, $R_{6k}(x)$ are tempered distributions with $6k < n$, we obtain $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ exists and is a tempered distribution.

(2) For the case $0 < m < k$, we have

$$\oplus_B^k u(x) = c_1 \oplus_B \delta + c_2 \oplus_B^2 \delta + \cdots + c_m \oplus_B^m \delta.$$

We convolved both sides of the above equation by $(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$ to obtain

$$\begin{aligned} & \oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) * u(x) \\ &= c_1 \oplus_B \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) \\ & \quad + c_2 \oplus_B^2 \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) \\ & \quad + \cdots + c_m \oplus_B^m \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right). \end{aligned}$$

By Theorem 2.1, we obtain

$$\begin{aligned} u(x) &= c_1 \left((-1)^{3(k-1)} S_{6(k-1)}(x) * R_{6(k-1)}(x) * (C^{*(k-1)}(x))^{*-1} \right) \\ & \quad + c_2 \left((-1)^{4(k-2)} S_{6(k-2)}(x) * R_{6(k-2)}(x) * (C^{*(k-2)}(x))^{*-1} \right) \\ & \quad + \cdots + c_m \left((-1)^{3(k-m)} S_{6(k-m)}(x) * R_{6(k-m)}(x) * (C^{*(k-m)}(x))^{*-1} \right), \end{aligned}$$

or

$$u(x) = \sum_{r=1}^m c_r \left((-1)^{3(k-r)} S_{6(k-r)}(x) * R_{6(k-r)}(x) * (C^{*(k-r)}(x))^{*-1} \right).$$

Similarly, as in case (1), $u(x)$ is an ordinary function for $6k - 6r \geq n$ and is a tempered distribution for and $6k - 6r < n$.

(3) For the case $m \geq k$ and $k \leq m \leq M$, we have

$$\oplus_B^k u(x) = c_k \oplus_B^k \delta + c_{k+1} \oplus_B^{k+1} \delta + \cdots + c_M \oplus_B^M \delta.$$

Convolved both sides of the above equation by

$$(-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1}$$

to obtain

$$\begin{aligned} & \oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) * u(x) \\ &= c_k \oplus_B^k \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) \\ & \quad + c_{k+1} \oplus_B^{k+1} \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right) \\ & \quad + \cdots + c_M \oplus_B^M \left((-1)^{3k} S_{6k}(x) * R_{6k}(x) * (C^{*k}(x))^{*-1} \right). \end{aligned}$$

By Theorem 2.1 again, we obtain

$$u(x) = c_k \delta + c_{k+1} \oplus_B \delta + c_{k+2} \oplus_B^2 \delta + \cdots + c_M \oplus_B^{M-k} \delta = \sum_{r=k}^M c_r \oplus_B^{r-k} \delta.$$

Since $\oplus_B^{r-k} \delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. This completes the proofs. \square

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