# SOME REVERSES OF THE CAUCHY-SCHWARZ AND TRIANGLE INEQUALITIES IN 2-INNER PRODUCT SPACES

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ABSTRACT. In this paper, we give some reverses of the Cauchy-Schwarz inequality and triangle inequality in 2-inner product spaces. Applications for determinantal integral inequalities are also provided.

## 1. Introduction

The Cauchy-Schwarz inequality plays an important role in the theory of inner product spaces (see, for instance, [22, 23]), which is one of the classical inequalities. It is well known that, in a semi-inner product space $(\mathscr{X}, \langle \cdot, \cdot \rangle)$, the Cauchy-Schwarz inequality has the form

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle ,$$

for all $x, y \in \mathscr{X}$.

In recent years, many authors have studied some related topics such as the reverse of the Cauchy-Schwarz inequality, the triangle and Bessel inequality as well as Grüss inequality (see [7, 10, 11, 18]). The probably first reverse of the Cauchy-Schwarz inequality for positive real numbers was obtained by Pólya and Szegö in 1925 (see [20, p. 57 and 213–214] and [21, p. 71–72 and 253–255]). Since then, there exist a lot of generalizations of the reverse of the Cauchy-Schwarz inequality. For example, in 2007, Dragomir [6, Chapter 2] contributed much to the reverses of the Cauchy-Schwarz inequality and also similar results for integrals, isotonic functionals as well as generalizations in the setting of inner product spaces are well-studied and understood

(see the book [5]). Some other interesting inequalities for the reverse of the Cauchy-Schwarz inequality can be found in [8, 9, 12].

In this paper, we continue and complement this research by proving some new reverses of the Cauchy-Schwarz inequality in framework of 2-inner product spaces. Furthermore, as applications, some reverse results for the generalized triangle inequality, i.e., upper bounds for the quantity

$$(0 \leq) \sum_{i=1}^{n} \|x_i, z\| - \left\| \sum_{i=1}^{n} x_i, z \right\|$$

under various assumptions for the vectors $z, x_i \in \mathscr{X}$, $i \in \{1, \cdots, n\}$, are established and also, some applications for the generalized triangle inequality are given.

## 2. Preliminaries

The concept of 2-normed spaces was introduced by Gähler [16] in 1963. After that, in 1973 and 1977, Diminnie, Gähler and White introduced the concept of 2-inner product spaces [3, 4]. For more details on 2-inner product spaces, see [2, 15, 17, 19]. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let $\mathscr{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a $\mathbb{K}$-valued function defined on $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ satisfying the following conditions:

(2I-1) $\langle x, x | z \rangle \geq 0$, and $\langle x, x | z \rangle = 0$ if and only if $x$ and $z$ are linearly dependent;
(2I-2) $\langle x, x | z \rangle = \overline{\langle z, z | x \rangle}$;
(2I-3) $\langle y, x | z \rangle = \overline{\langle x, y | z \rangle}$;
(2I-4) $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$, for any $\alpha \in \mathbb{K}$;
(2I-5) $\langle x + x', y | z \rangle = \langle x, y | z \rangle + \langle x', y | z \rangle$.

$\langle \cdot, \cdot | \cdot \rangle$ is called a *2-inner product* on $\mathscr{X}$ and $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner products $\langle \cdot, \cdot | \cdot \rangle$ can be immediately obtained as follows:

(P-1) If $\mathbb{K} = \mathbb{R}$, then (2I-3) reduces to $\langle y, x | z \rangle = \langle x, y | z \rangle$;
(P-2) $\langle 0, y | z \rangle = \langle x, 0 | z \rangle = \langle x, y | 0 \rangle = 0$;
(P-3) $\langle x, y | \alpha z \rangle = |\alpha|^2 \langle x, y | z \rangle$, for all $x, y, z \in \mathscr{X}$ and $\alpha \in \mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x, y | z \rangle|^2 \leq \langle x, x | z \rangle \langle y, y | z \rangle.$$

In any given 2-inner product space $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$, we can define a function $\|\cdot, \cdot\|$ on $\mathscr{X} \times \mathscr{X}$ by

(2.1) $$\|x, z\| = \sqrt{\langle x, x | z \rangle},$$

for all $x, z \in \mathscr{X}$. It is easy to see that this function satisfies the following conditions:

(2N-1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if $x$ and $z$ are linearly dependent;

(2N-2) $\|x, z\| = \|z, x\|$;

(2N-3) $\|\alpha x, z\| = |\alpha| \|x, z\|$ for any scalar $\alpha \in \mathbb{K}$;

(2N-4) $\|x + x', z\| \leq \|x, z\| + \|x', z\|$.

Any function $\|\cdot, \cdot\|$ defined on $\mathscr{X} \times \mathscr{X}$ and satisfying the above conditions is called a *2-norm* on $\mathscr{X}$ and $(\mathscr{X}, \|\cdot, \cdot\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ is given, we consider it as a linear 2-norm space $(\mathscr{X}, \|\cdot, \cdot\|)$ with the 2-norm defined by (2.1).

## 3. Some Reverses of the Cauchy-Schwarz Inequality

First, we have the following.

**Theorem 3.1.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$ and $x, y, z \in \mathscr{X}$, $r_1, r_2 > 0$ are such that*

$$(3.1) \qquad r_1 \leq |\|x, z\| - \|y, z\|| \leq \|x - y, z\| \leq r_2,$$

*then*

$$(3.2) \qquad \|x, z\| \|y, z\| - |\langle x, y | z \rangle| \leq \|x, z\| \|y, z\| - |\operatorname{Re} \langle x, y | z \rangle|$$
$$\leq \|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle$$
$$\leq \frac{1}{2} \left( r_2^2 - r_1^2 \right).$$

*Proof.* Taking the square in the third inequality in (3.1), we have

$$\|x, z\|^2 - 2 \operatorname{Re} \langle x, y | z \rangle + \|y, z\|^2 \leq r_2^2,$$

which is equivalent to

$$(3.3) \qquad 2 \left( \|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \right) + \left( \|x, z\| - \|y, z\| \right)^2 \leq r_2^2.$$

Using the first inequality in (3.1), we have

$$(3.4) \qquad r_1^2 \leq \left( \|x, z\| - \|y, z\| \right)^2.$$

Therefore, from (3.3) and (3.4), we have (3.2). This completes the proof. $\qquad \square$

**Corollary 3.1.** *With all the assumptions of Theorem* 3.1*, the following holds:*

$$(3.5) \qquad \|x, z\| + \|y, z\| - \|x + y, z\| \leq \sqrt{r_2^2 - r_1^2}.$$

*Proof.* It follows from (3.2) that

$$\left( \|x, z\| + \|y, z\| \right)^2 - \|x + y, z\|^2 = 2 \left( \|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \right)$$
$$\leq r_2^2 - r_1^2$$

gives

$$(3.6) \qquad \left( \|x, z\| + \|y, z\| \right)^2 \leq \|x + y, z\|^2 + r_2^2 - r_1^2.$$

Taking the square root in (3.6) and taking into account that

$$\sqrt{\alpha + \beta} \le \sqrt{\alpha} + \sqrt{\beta}$$

for all $\alpha, \beta \ge 0$, we have the desired inequality (3.5). This completes the proof. $\quad\square$

**Theorem 3.2.** *Let* $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ *be a 2-inner product space over the real or complex number field* $\mathbb{K}$. *For any* $x, y, z \in \mathscr{X}$,

$$\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \le r \tag{3.7}$$

*and*

$$\|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \le \frac{1}{2} r^2 \, \|x, z\| \, \|y, z\|$$

*are equivalent.*

*Proof.* It is obvious by taking the square in (3.7) and performing the required calculations. $\quad\square$

*Remark* 3.1. Since

$$\left\| \|y, z\| \, x - \|x, z\| \, y, z \right\| = \left\| \|y, z\| \, (x - y) + (\|y, z\| - \|x, z\|) y, z \right\|$$

$$\le \|y, z\| \, \|x - y, z\| + \left| \|y, z\| - \|x, z\| \right| \|y, z\|$$

$$\le 2 \|y, z\| \, \|x - y, z\| ,$$

the sufficient condition for (3.7) to hold is

$$\|x - y, z\| \le \frac{r}{2} \|x, z\| .$$

**Theorem 3.3.** *Let* $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ *be a 2-inner product space over the real or complex number field* $\mathbb{K}$. *Then, for any* $x, y, z \in \mathscr{X}$ *and* $p \ge 1$,

$$\begin{aligned}
0 &\le \|x, z\| \, \|y, z\| - \left| \langle x, y | z \rangle \right| \\
&\le \|x, z\| \, \|y, z\| - \left| \operatorname{Re} \langle x, y | z \rangle \right| \\
&\le \frac{1}{2} \begin{cases} \left( (\|x, z\| + \|y, z\|)^{2p} - \|x + y, z\|^{2p} \right)^{\frac{1}{p}}, \\ \left( \|x - y, z\|^{2p} - \left| \|x, z\| - \|y, z\| \right|^{2p} \right)^{\frac{1}{p}}. \end{cases}
\end{aligned} \tag{3.8}$$

*Proof.* Firstly, observe that

$$2 \left( \|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \right) = (\|x, z\| + \|y, z\|)^2 - \|x + y, z\|^2 .$$

Denoting

$$D := \|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle ,$$

we have

$$2D + \|x + y, z\|^2 = \left( \|x, z\| + \|y, z\| \right)^2 . \tag{3.9}$$

Taking in (3.9) the power $p \geq 1$ and using the elementary inequality

$$(a+b)^p \geq a^p + b^p, \quad a, b \geq 0,$$

we have

$$(\|x, z\| + \|y, z\|)^{2p} = \left(2D + \|x+y, z\|^2\right)^p \geq 2^p D^p + \|x+y, z\|^{2p}.$$

This implies that

(3.10) $$D^p \leq \frac{1}{2^p}\left((\|x, z\| + \|y, z\|)^{2p} - \|x+y, z\|^{2p}\right),$$

which is clearly equivalent to the first branch of the third inequality in (3.8).

With the above notation, we also have

(3.11) $$2D + (\|x, z\| - \|y, z\|)^2 = \|x-y, z\|^2.$$

Taking the power $p \geq 1$ in (3.11) and using the inequality (3.10), we have

$$\|x-y, z\|^{2p} \geq 2^p D^p + \left| \|x, z\| - \|y, z\| \right|^{2p}$$

and so we have the last part of (3.8). This completes the proof. $\square$

We state the following result that provides an invariant property for the constant in the Cauchy-Schwarz inequality.

**Theorem 3.4.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{K}$,*

$$\|x, z\|^2 \|y, z\|^2 - \left|\langle x, y|z\rangle\right|^2 = \|x - \lambda y, z\|^2 \|y, z\|^2 - \left|\langle x - \lambda y, y|z\rangle\right|^2.$$

*Proof.* By properties of 2-inner product, it follows that, for any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned}
&\|x - \lambda y, z\|^2 \|y, z\|^2 - \left|\langle x - \lambda y, y|z\rangle\right|^2 \\
&= \left(\|x, z\|^2 - 2\operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) + |\lambda|^2 \|y, z\|^2\right)\|y, z\|^2 \\
&\quad - \left|\langle x, y|z\rangle - \lambda\|y, z\|^2\right|^2 \\
&= \|x, z\|^2 \|y, z\|^2 - 2\|y, z\|^2 \operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) + |\lambda|^2\|y, z\|^4 \\
&\quad - \left|\langle y, x|z\rangle\right|^2 + 2\|y, z\|^2 \operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) - |\lambda|^2\|y, z\|^4 \\
&= \|y, z\|^2 \|x, z\|^2 - \left|\langle y, x|z\rangle\right|^2.
\end{aligned}$$

This completes the proof. $\square$

**Corollary 3.2.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{K}$,*

(3.12) $$\|x, z\|^2 \|y, z\|^2 - \left|\langle x, y|z\rangle\right|^2 \leq \|x - \lambda y, z\|^2 \|y, z\|^2.$$

*The equalities holds in (3.12) if and only if $\langle x, y|z\rangle = \lambda\|y, z\|^2$.*

For two parameters, we can get the following.

**Theorem 3.5.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{K}$,*

$$(3.13) \qquad \left( \|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 \right) |\mu - \lambda|^2$$
$$= \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2 - \left| \langle x - \lambda y, x - \mu y | z \rangle \right|^2.$$

*Proof.* Denote $w := x - \lambda y$. Using some properties of a 2-inner product, we have

$$(3.14) \qquad \left| \langle x - \lambda y, x - \mu y | z \rangle \right|^2 = \left| \langle w, x - \mu y | z \rangle \right|^2$$
$$= \left| \langle w, x - \lambda y + (\lambda - \mu) y | z \rangle \right|^2$$
$$= \left| \langle w, w + (\lambda - \mu) y | z \rangle \right|^2$$
$$= \left| \|w, z\|^2 + \overline{(\lambda - \mu)} \langle w, y | z \rangle \right|^2$$
$$= \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re} (\lambda - \mu) \overline{\langle w, y | z \rangle}$$
$$\quad + |\lambda - \mu|^2 |\langle w, y | z \rangle|^2$$
$$= \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re} \left( (\lambda - \mu) \overline{\langle w, y | z \rangle} \right)$$
$$\quad + |\lambda - \mu|^2 \|w, z\|^2 \|y, z\|^2$$
$$\quad - |\lambda - \mu|^2 \left( \|w, z\|^2 \|y, z\|^2 - |\langle w, y | z \rangle|^2 \right).$$

Observe also that

$$(3.15) \qquad \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re} \left( (\lambda - \mu) \overline{\langle w, y | z \rangle} \right) + |\lambda - \mu|^2 \|w, z\|^2 \|y, z\|^2$$
$$= \|w, z\|^2 \left( \|w, z\|^2 + 2 \operatorname{Re} \left( \overline{(\lambda - \mu) \overline{\langle w, y | z \rangle}} \right) + |\lambda - \mu|^2 \|y, z\|^2 \right)$$
$$= \|w, z\|^2 \|w + (\lambda - \mu) y, z\|^2$$
$$= \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2.$$

Therefore, from (3.14) and (3.15), we have the desired result (3.13). This completes the proof. $\square$

**Corollary 3.3.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{K}$,*

$$\|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 \leq \frac{1}{|\mu - \lambda|^2} \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2.$$

As an application of Theorem 3.5, we have the following.

**Proposition 3.1.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Then, for all $x, y, z, e \in \mathscr{X}$ with $\|e, z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{K}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$,*

$$(3.16) \qquad |\langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle|$$

$$\leq \frac{1}{|\lambda - \mu| \, |\gamma - \eta|} \left( \|x - \lambda e, z\| \, \|x - \mu e, z\| \, \|y - \gamma e, z\| \, \|y - \eta e, z\| \right.$$

$$\left. - \left| \langle x - \lambda e, x - \mu e | z \rangle \right| \left| \langle y - \gamma e, y - \eta e | z \rangle \right| \right)$$

$$\leq \frac{1}{|\lambda - \mu| \, |\gamma - \eta|} \left( \|x - \lambda e, z\| \, \|x - \mu e, z\| \, \|y - \gamma e, z\| \, \|y - \eta e, z\| \right).$$

*Proof.* Applying the Cauchy-Schwarz inequality for the vectors $x - \langle x, e|z \rangle \, e$ and $y - \langle y, e|z \rangle \, e$ and taking into account that

$$\langle x - \langle x, e|z \rangle \, e, y - \langle y, e|z \rangle \, e|z \rangle = \langle x, y|z \rangle - \langle x, e|z \rangle \langle e, y|z \rangle,$$

and

$$\|x - \langle x, e|z \rangle \, e, z\|^2 = \|x, z\|^2 - |\langle x, e|z \rangle|^2,$$

and

$$\|y - \langle y, e|z \rangle \, e, z\|^2 = \|y, z\|^2 - |\langle y, e|z \rangle|^2,$$

we have

(3.17) $$\left| \langle x, y|z \rangle - \langle x, e|z \rangle \langle e, y|z \rangle \right|$$

$$\leq \left( \|x, z\|^2 - |\langle x, e|z \rangle|^2 \right)^{\frac{1}{2}} \left( \|y, z\|^2 - |\langle y, e|z \rangle|^2 \right)^{\frac{1}{2}},$$

for any $x, y, z, e \in \mathscr{X}$ with $\|e, z\| = 1$.

From (3.13), it follows that

(3.18) $$\left( \|x, z\|^2 - |\langle x, e|z \rangle|^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{|\mu - \lambda|} \left( \|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 - |\langle x - \lambda e, x - \mu e|z \rangle|^2 \right)^{\frac{1}{2}}$$

and

(3.19) $$\left( \|y, z\|^2 - |\langle y, e|z \rangle|^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{|\gamma - \eta|} \left( \|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 - |\langle y - \gamma e, y - \eta e|z \rangle|^2 \right)^{\frac{1}{2}},$$

for any $x, y, z, e \in \mathscr{X}$ with $\|e, z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{K}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$.

Now, if we multiply (3.18) with (3.19), then we have

(3.20) $$\left( \|x, z\|^2 - |\langle x, e|z \rangle|^2 \right)^{\frac{1}{2}} \left( \|y, z\|^2 - |\langle y, e|z \rangle|^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{|\mu - \lambda| \, |\gamma - \eta|}$$

$$\times \left( \|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e|z \rangle|^2 \right)^{\frac{1}{2}}$$

$$\times \left( \|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e|z \rangle|^2 \right)^{\frac{1}{2}}.$$

Further, if we use the elementary inequality

$$\left( a^2 - b^2 \right)^{\frac{1}{2}} \left( c^2 - d^2 \right)^{\frac{1}{2}} \leq ac - bd$$

for all $a \geq b \geq 0$ and $c \geq d \geq 0$, then we also have

$$(3.21) \qquad \left( \|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e | z \rangle|^2 \right)^{\frac{1}{2}}$$

$$\times \left( \|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e | z \rangle|^2 \right)^{\frac{1}{2}}$$

$$\leq \|x - \lambda e, z\| \, \|x - \mu e, z\| \, \|y - \gamma e, z\| \, \|y - \eta e, z\|$$

$$- |\langle x - \lambda e, x - \mu e | z \rangle| |\langle y - \gamma e, y - \eta e | z \rangle|.$$

Finally, using (3.17), (3.20) and (3.21), we have the second inequality (3.16). This completes the proof. $\qquad \square$

## 4. Some Reverses of the Triangle Inequality

In this section, we give some reverses of the triangle inequality.

**Theorem 4.1.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Let $z, x_i, \in \mathscr{X}$, $i \in \{1, \ldots, n\}$, and $r_{ij} > 0$ for each $1 \leq i \leq j \leq n$ be such that*

$$(4.1) \qquad 0 \leq \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \leq r_{ij}.$$

*Then the following quadratic reverse of the triangle inequality holds:*

$$(4.2) \qquad \left( \sum_{i=1}^{n} \|x_i, z\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i, z \right\|^2 + 2 \sum_{1 \leq i \leq j \leq n} r_{ij}.$$

*The case of the equality holds in (4.2) if and only if it holds in (4.1) for each $i, j$ with $1 \leq i \leq j \leq n$.*

*Proof.* Observe that

$$(4.3) \qquad \left( \sum_{i=1}^{n} \|x_i, z\| \right)^2 - \left\| \sum_{i=1}^{n} x_i, z \right\|^2 = \sum_{i,j=1}^{n} \|x_i, z\| \, \|x_j, z\| - \left\langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j | z \right\rangle$$

$$= \sum_{i,j=1}^{n} \|x_i, z\| \, \|x_j, z\| - \sum_{i,j=1}^{n} \operatorname{Re} \langle x_i, x_j | z \rangle$$

$$= \sum_{i,j=1}^{n} \left( \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \right)$$

$$= \sum_{1 \leq i \leq j \leq n} \left( \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \right)$$

$$+ \sum_{1 \leq j \leq i \leq n} \left( \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \right)$$

$$= 2 \sum_{1 \leq i \leq j \leq n} \left( \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \right).$$

Using the condition (4.1), we have

$$\sum_{1 \leq i \leq j \leq n} \left( \|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \right) \leq \sum_{1 \leq i \leq j \leq n} r_{ij}$$

and, by (4.3), we have the desired inequality (4.2).

The case of the equality is obvious by the identity (4.3) and we omit the details. This completes the proof. $\square$

*Remark* 4.1. From (4.2), one may deduce the coarser inequality that might be useful in some applications:

$$0 \leq \sum_{i=1}^{n} \|x_i, z\| - \left\| \sum_{i=1}^{n} x_i, z \right\| \leq \sqrt{2} \left( \sum_{1 \leq i \leq j \leq n} r_{ij} \right)^{\frac{1}{2}}.$$

**Theorem 4.2.** *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Let $z, x_i, \in \mathscr{X}$, $i \in \{1, \ldots, n\}$, and $r > 0$ be such that*

$$\|x_i - x_j, z\| \leq r, \tag{4.4}$$

*for each $1 \leq i \leq j \leq n$. Then*

$$\left( \sum_{i=1}^{n} \|x_i, z\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i, z \right\|^2 + \frac{n(n-1)}{2} r^2. \tag{4.5}$$

*The case of the equality holds in (4.5) if and only if*

$$\|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle = \frac{1}{2} r^2,$$

*for each $1 \leq i \leq j \leq n$.*

*Proof.* The inequality (4.4) is obviously equivalent to

$$\|x_i, z\|^2 + \|x_j, z\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j | z \rangle + r^2,$$

for each $1 \leq i \leq j \leq n$. Since

$$2 \|x_i, z\| \, \|x_j, z\| \leq \|x_i, z\|^2 + \|x_j, z\|^2,$$

for each $1 \leq i \leq j \leq n$, we have

$$\|x_i, z\| \, \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \leq \frac{1}{2} r^2,$$

for each $1 \leq i \leq j \leq n$.

Applying Theorem 4.1 for $r_{ij} := \frac{1}{2} r^2$ and taking into account that

$$\sum_{1 \leq i \leq j \leq n} r_{ij} = \frac{n(n-1)}{4} r^2,$$

we deduce the desired inequality (4.5). The case of the equality is also obvious by the Theorem 4.1 and we omit the details. This completes the proof. $\square$

## 5. Applications for Determinantal Integral Inequalities

Let $(\Omega, \Sigma, \mu)$ be a measure space consisting of a set $\Omega$, a $\sigma$-algebra $\Sigma$ of subsets of $\Omega$ and a countably additive and positive measure $\mu$ on $\Sigma$ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_\rho^2(\Omega)$ the Hilbert space of all real-valued functions $f$ defined on $\Omega$ that are 2-$\rho$-integrable on $\Omega$, i.e.

$$\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty,$$

where $\rho : \Omega \to [0, \infty)$ is a measurable function on $\Omega$.

We can introduce the following 2-inner product on $L_\rho^2(\Omega)$ by the formula

$$\langle f, g | h \rangle_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) \, d\mu(t),$$

where, by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix},$$

generating the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f, h\|_\rho := \left( \frac{1}{2} \int_\Omega \int_\Omega \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) \, d\mu(t) \right)^{\frac{1}{2}}.$$

A simple calculation with integrals reveals that

$$(5.1) \qquad \langle f, g | h \rangle_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}$$

and

$$(5.2) \qquad \|f, h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{\frac{1}{2}},$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$ (see, e.g. [13] or [14]).

Using the representations (5.1), (5.2) and the Theorem 4.1, one may state interesting determinantal integral inequality, as follows.

**Proposition 5.1.** *Let $f_1, \ldots, f_n, g, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on $\Omega$ and $r_{ij} > 0$ such that*

$$\left| \begin{matrix} \int\limits_\Omega \rho(f_i - f_j)^2 d\mu & \int\limits_\Omega \rho(f_i - f_j) h d\mu \\ \int\limits_\Omega \rho(f_i - f_j) h d\mu & \int\limits_\Omega \rho h^2 d\mu \end{matrix} \right|^{\frac{1}{2}} \leq r,$$

*for each $1 \leq i \leq j \leq n$, then*

$$\left( \sum_{i=1}^n \left| \begin{matrix} \int\limits_\Omega \rho f_i{}^2 d\mu & \int\limits_\Omega \rho f_i h d\mu \\ \int\limits_\Omega \rho f_i h d\mu & \int\limits_\Omega \rho h^2 d\mu \end{matrix} \right|^{\frac{1}{2}} \right)^2 \leq \left| \begin{matrix} \sum_{i=1}^n \int\limits_\Omega \rho f_i{}^2 d\mu & \sum_{i=1}^n \int\limits_\Omega \rho f_i h d\mu \\ \sum_{i=1}^n \int\limits_\Omega \rho f_i h d\mu & \sum_{i=1}^n \int\limits_\Omega \rho h^2 d\mu \end{matrix} \right| + 2 \sum_{1 \leq i \leq j \leq n} r_{ij}.$$

*Proof.* The proof follows by Theorem 4.1, applied for the 2-inner product $\langle \cdot, \cdot | \cdot \rangle_\rho$ and we omit the details. $\square$

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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