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SOME B-DIFFERENCE SEQUENCE SPACES DERIVED BY GENERALIZED MEANS AND COMPACT OPERATORS

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Abstract. This paper presents new sequence spaces $X(r,s,t,p;B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ defined by using generalized means and difference operator. It is shown that these spaces are complete paranormed spaces and the spaces $X(r,s,t,p;B)$ for $X \in \{c(p), c_0(p), l(p)\}$ have Schauder basis. Furthermore, the α-, β-, γ- duals of these sequence spaces are computed and also obtained necessary and sufficient conditions for some matrix transformations from $X(r,s,t,p;B)$ to X. Finally, some classes of compact operators on the space $l_p(r,s,t;B)$ are characterized by using the Hausdorff measure of noncompactness.

1. Introduction

The study of sequence spaces has importance in the several branches of analysis, namely, the structural theory of topological vector spaces, summability theory, Schauder basis theory. Besides this, the theory of sequence spaces is a powerful tool for obtaining some topological and geometrical results using Schauder basis.

Let w be the space of all real or complex sequences $x = (x_n)$, $n \in \mathbb{N}_0$. For an infinite matrix A and a sequence space λ, the matrix domain of A, denoted by λ_A, is defined as $\lambda_A = \{x \in w : Ax \in \lambda\}$ [35]. Some basic methods that are used to determine the topologies, matrix transformations and inclusion relations on sequence spaces can also be applied to study the matrix domain λ_A. In recent times, there is an approach of forming a new sequence space by using a matrix domain of a suitable matrix and characterize the matrix mappings between these sequence spaces.

Key words and phrases. Difference operator, generalized means, α-, β-, γ- duals, matrix transformation, compact operator, Hausdorff measure of noncompactness.

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Let (p_k) be a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1, H\}$. The linear spaces $c(p), c_0(p)$, $l_\infty(p)$ and $l(p)$ are studied by Maddox [21], where

$$\begin{aligned}
c(p) &= \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C}\right\},\\
c_0(p) &= \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\},\\
l_\infty(p) &= \left\{x = (x_k) \in w : \sup_{k\in\mathbb{N}_0} |x_k|^{p_k} < \infty\right\}, \text{ and}\\
l(p) &= \left\{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty\right\}.
\end{aligned}$$

The linear space $c_0(p)$ is a complete linear metric space with respect to the paranorm $g(x) = \sup_{k\in\mathbb{N}_0} |x_k|^{\frac{p_k}{M}}$ and the spaces $c(p)$, $l_\infty(p)$ are complete linear metric spaces under the paranorm $g(x)$ if and only if $\inf p_k > 0$ for all k. The space $l(p)$ is a complete linear metric space with the paranorm $\tilde{g}(x) = \left(\sum_{k\in\mathbb{N}_0} |x_k|^{p_k}\right)^{\frac{1}{M}}$.

Recently, several authors introduced sequence spaces by using matrix domain. For example, Başar et al. [6] studied the space $bs(p) = [l_\infty(p)]_S$, where S is the summation matrix. Altay and Başar [2] studied the sequence spaces $r^t(p)$ and $r^t_\infty(p)$, which consist of all sequences whose Riesz transforms are in the spaces $l(p)$ and $l_\infty(p)$ respectively, i.e., $r^t(p) = [l(p)]_{R^t}$ and $r^t_\infty(p) = [l_\infty(p)]_{R^t}$. Altay and Başar also studied the sequence spaces $r^t_c(p) = [c(p)]_{R^t}$ and $r^t_0(p) = [c_0(p)]_{R^t}$ in [3].

Kizmaz [20] first introduced and studied the difference sequence spaces. Later on, many authors including Ahmad and Mursaleen [1], Çolak and Et [11], Başar et al. [7] studied sequence spaces defined by using difference operator. Using Euler mean of order α, $0 < \alpha < 1$ and difference operator, Karakaya and Polat [19] introduced the paranormed sequence spaces $e^\alpha_0(p;\Delta), e^\alpha_c(p;\Delta)$ and $e^\alpha_\infty(p;\Delta)$. Mursaleen and Noman [30] introduced a sequence space of generalized means, which includes most of the earlier known sequence spaces. But till 2011, there was no such literature available in which a sequence space is generated by combining both the weighted mean and the difference operator. This was first initiated by Polat et al. [32]. Later on, *Demiriz* and Çakan [13] introduced the paranormed difference sequence spaces $\lambda(r', s', p; \Delta)$ for $\lambda \in \{l_\infty(p), c(p), c_0(p), l(p)\}$.

Recently, Başarır and Kara [9] introduced and studied the B-difference sequence space $l(r', s', p; B)$ defined as

$$l(r', s', p; B) = \{x \in w : [G(r', s') \cdot B]x \in l(p)\},$$

where $r' = (r'_n), s' = (s'_n)$ are non zero sequences and the matrices $G(r', s') = (g_{nk})$, $B = B(u, v) = (b_{nk}), u, v \neq 0$ are defined by

$$g_{nk} = \begin{cases} r'_n s'_k, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases} \quad \text{and} \quad b_{nk} = \begin{cases} 0, & \text{if } 0 \leq k < n-1, \\ v, & \text{if } k = n-1, \\ u, & \text{if } k = n, \\ 0, & \text{if } k > n. \end{cases}$$

By using matrix domain, one can write $l(r', s', p; B) = [l(p)]_{G(r',s')\cdot B}$. Recent developments in this direction can be obtained in the work of Başar ([4], [5]), Yeşilkayagil and Başar [36], Çapan et al. [10], Uçar and Başar [34] and Tuğ and Başar [33]. Quite recently, authors have introduced and studied new sequence spaces in this direction, see [25–27].

The aim of this present paper is to introduce and study sequence spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, which are more general and more comprehensive than the previous classes of sequences. We have shown that these sequence spaces are complete paranormed sequence spaces under some suitable paranorm. Some topological results and the α-, β-, γ- duals of these spaces are obtained. A characterization of some classes of matrix transformations between these new sequence spaces is established. Furthermore, a characterization of some classes of compact operators on the space $l_p(r, s, t; B)$ using the Hausdorff measure of noncompactness is given.

2. Preliminaries

Let l_∞, c and c_0 be the spaces of all bounded, convergent and null sequences $x = (x_n)$ respectively, with the norm $\|x\|_\infty = \sup_n |x_n|$. Let bs and cs be the spaces of all bounded and convergent series respectively. We denote by $e = (1, 1, \cdots)$ and e_n for the sequence whose n-th term is 1 and others are zero and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers. A sequence (b_n) in a paranormed space (X, g) is called a Schauder basis for X if for every $x \in X$ there is a unique sequence of scalars (μ_n) such that

$$g\left(x - \sum_{n=0}^{k} \mu_n b_n\right) \to 0, \quad \text{as } k \to \infty,$$

i.e., $x = \sum_{n=0}^{\infty} \mu_n b_n$ [35].

For any subsets U and V of w, the multiplier space $M(U, V)$ of U and V is defined as

$$M(U, V) = \{a = (a_n) \in w : au = (a_n u_n) \in V, \text{ for all } u \in U\}.$$

In particular,

$$U^\alpha = M(U, l_1), \; U^\beta = M(U, cs) \text{ and } U^\gamma = M(U, bs)$$

are called the α-, β- and γ- duals of U respectively [35].

Let $A = (a_{nk})_{n,k}$ be an infinite matrix with real or complex entries a_{nk}. We write A_n as the sequence of the n-th row of A, i.e., $A_n = (a_{nk})_k$ for every n. For

$x = (x_n) \in w$, the A-transform of x is defined as the sequence $Ax = ((Ax)_n)$, where $A_n(x) = (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$, provided the series on the right side converges for each n. For any two sequence spaces U and V, we denote by (U, V), the class of all infinite matrices A that map from U into V. Therefore $A \in (U, V)$ if and only if $Ax = ((Ax)_n) \in V$ for all $x \in U$. In other words, $A \in (U, V)$ if and only if $A_n \in U^\beta$ for all n [35].

3. Sequence Spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$

In this section, we first begin with the notion of generalized means given by Mursaleen et al. [30].

We denote the sets $\mathcal{U}$ and $\mathcal{U}_0$ as

$$\mathcal{U} = \Big\{u = (u_n)_{n=0}^{\infty} \in w : u_n \neq 0 \text{ for all } n\Big\} \text{ and } \mathcal{U}_0 = \Big\{u = (u_n)_{n=0}^{\infty} \in w : u_0 \neq 0\Big\}.$$

Let $r = (r_n), t = (t_n) \in \mathcal{U}$ and $s = (s_n) \in \mathcal{U}_0$. The sequence $y = (y_n)$ of generalized means of a sequence $x = (x_n)$ is defined by

$$y_n = \frac{1}{r_n}\sum_{k=0}^{n} s_{n-k}t_kx_k, \quad n \in \mathbb{N}_0.$$

The infinite matrix $A(r, s, t)$ of generalized means is defined by

$$(A(r, s, t))_{nk} = \begin{cases} \frac{s_{n-k}t_k}{r_n}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Since $A(r, s, t)$ is a triangle, it has a unique inverse and the inverse is also a triangle [18]. Take $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}}\begin{vmatrix} s_1 & s_0 & 0 & 0\cdots & 0 \\ s_2 & s_1 & s_0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4}\cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3}\cdots & s_1 \end{vmatrix}, \quad \text{for } n = 1, 2, 3, \ldots$$

Then the inverse of $A(r, s, t)$ is the triangle $\tilde{B} = (\tilde{b}_{nk})_{n,k}$, which is defined as

$$\tilde{b}_{nk} = \begin{cases} (-1)^{n-k}\ \frac{D_{n-k}^{(s)}}{t_n}r_k, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Throughout this paper, we consider $p = (p_k)$ is a bounded sequence of strictly positive real numbers such that $H = \sup_k p_k$ and $M = \max\{1, H\}$.

For $X(r, s, t, p; B)$ we now introduce the sequence spaces $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, combining both the generalized means and the matrix $B(u, v)$ as

$$X(r, s, t, p; B) = \{x \in w : y = A(r, s, t; B)x \in X\},$$

where $y = (y_k)$ is $A(r, s, t; B)$-transform of a sequence $x = (x_k)$, i.e.,

$$y_n = \frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u x_n\right), \quad n \in \mathbb{N}_0, \tag{3.1}$$

where we mean $\sum_n^m = 0$ for $n > m$. By using matrix domain, we can write $X(r, s, t, p; B) = X_{A(r,s,t,p;B)} = \{x \in w : A(r, s, t; B)x \in X\}$, where $A(r, s, t; B) = A(r, s, t) \cdot B$, product of two triangles $A(r, s, t)$ and $B(u, v)$. For $X = l_p$, $p \geq 1$, we write $X(r, s, t, p; B)$ as $l_p(r, s, t; B)$.

These sequence spaces include many known sequence spaces studied by several authors. For example,

(i) if $r_n = 1/r'_n$, $t_n = s'_n$, $s_n = 1$ for all n, then the sequence spaces $l(r, s, t, p; B)$ reduce to $l(r', s', p; B)$ studied by Başarır and Kara [9];

(ii) if $r_n = 1/r'_n$, $t_n = s'_n$, $s_n = 1$ for all n, $u = 1$ and $v = -1$ then the sequence spaces $X(r, s, t, p; B)$ reduce to $X(r', s', p; \Delta)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ studied by *Demiriz* and Çakan [13].

(iii) if $r_n = 1/n!$, $t_n = \alpha^n/n!$, $s_n = \frac{(1-\alpha)^n}{n!}$, where $0 < \alpha < 1$, $u = 1$, $v = -1$, then the sequence spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p)\}$ reduce to $e^\alpha_\infty(p; \Delta)$, $e^\alpha_c(p; \Delta)$, $e^\alpha_0(p; \Delta)$ respectively studied by Karakaya and Polat [19];

(iv) if $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers tending to infinity such that $r_n = \lambda_n$, $t_n = \lambda_n - \lambda_{n-1}$, $s_n = 1$ for all n, then $l_p(r, s, t; B)$ reduces to the sequence space $l^\lambda_p(B)$, where the sequence space l^λ_p is introduced and studied by Mursaleen and Noman [31];

(v) if $r_n = n + 1$, $t_n = 1 + \alpha^n$, where $0 < \alpha < 1$, $s_n = 1$ for all n, $u = 1$, $v = -1$, then the sequence space $l_p(r, s, t; B)$ reduces to the sequence space $a^\alpha_p(\Delta)$ studied by *Demiriz* and Çakan [12].

4. Main Results

Throughout the paper, we denote the sequence spaces $X(r, s, t, p; B)$ as $l(r, s, t, p; B)$, $c_0(r, s, t, p; B)$, $c(r, s, t, p; B)$ and $l_\infty(r, s, t, p; B)$ for $X = l(p), c_0(p), c(p)$ and $l_\infty(p)$ respectively. Now, we begin with some topological results of the new sequence spaces.

Theorem 4.1.
(a) *The sequence space $l(r, s, t, p; B)$ is a complete linear metric space paranormed by $\tilde{h}$ defined as*

$$\tilde{h}(x) = \left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u x_n\right)\right|^{p_n}\right)^{\frac{1}{M}}.$$

(b) *The sequence space $c_0(r, s, t, p; B)$ is a complete linear metric space paranormed by h defined as*

$$h(x) = \sup_{n\in\mathbb{N}_0}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u x_n\right)\right|^{\frac{p_n}{M}}.$$

The sequence spaces $X(r,s,t,p;B)$ for $X \in \{l_\infty(p),\ c(p)\}$ are also complete linear metric spaces under the paranorm h if and only if $\inf p_k > 0$ *for all k.*
(c) *The sequence space $\ell_p(r,s,t;B)$, $1 \leq p < \infty$ is a BK space with the norm given by*

$$\|x\|_{l_p(r,s,t;B)} = \|y\|_{l_p},$$

where $y = (y_k)$ is defined in (3.1).

Proof. We prove only the part (a) of this theorem. In a similar way, we can prove the other parts.

Let $x,\ y \in l(r,s,t,p;B)$. Using Minkowski's inequality

$$\begin{aligned} &\left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)(x_k+y_k) + s_0 t_n u(x_n+y_n)\right)\right|^{p_n}\right)^{\frac{1}{M}} \\ \leq &\left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u x_n\right)\right|^{p_n}\right)^{\frac{1}{M}} \\ &+\left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)y_k + s_0 t_n u y_n\right)\right|^{p_n}\right)^{\frac{1}{M}}. \end{aligned} \tag{4.1}$$

So, we have $x + y \in l(r,s,t,p;B)$. Let α be any scalar. Since $|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}$, we have $\tilde{h}(\alpha x) \leq \max\{1,|\alpha|\}\tilde{h}(x)$. Hence $\alpha x \in l(r,s,t,p;B)$. It is trivial to show that $\tilde{h}(\theta) = 0$, $\tilde{h}(-x) = \tilde{h}(x)$ for all $x \in l(r,s,t,p;B)$ and subadditivity of $\tilde{h}$, i.e., $\tilde{h}(x+y) \leq \tilde{h}(x) + \tilde{h}(y)$ follows from (4.1).

To show that the scalar multiplication is continuous, let (x^m) be a sequence in $l(r,s,t,p;B)$, where $x^m = (x_k^m) = (x_0^m, x_1^m, x_2^m, \ldots) \in l(r,s,t,p;B)$ for each $m \in \mathbb{N}_0$ such that $\tilde{h}(x^m - x) \to 0$ as $m \to \infty$ and (α_m) be a sequence of scalars such that $\alpha_m \to \alpha$ as $m \to \infty$. Then $\tilde{h}(x^m)$ is bounded that follows from the following inequality

$$\tilde{h}(x^m) \leq \tilde{h}(x) + \tilde{h}(x - x^m).$$

Now consider

$$\begin{aligned} &\tilde{h}(\alpha_m x^m - \alpha x) \\ &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)(\alpha_m x_k^m - \alpha x_k)\right.\right.\right. \\ &\qquad \left.\left.\left. + s_0 t_n u(\alpha_m x_n^m - \alpha x_n)\right)\right|^{p_n}\right)^{\frac{1}{M}} \\ &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)\left((\alpha_m - \alpha)(x_k^m - x_k) + \alpha(x_k^m - x_k)\right)\right.\right.\right. \end{aligned}$$

$$+ (\alpha_m - \alpha)x_k\Bigg) + s_0 t_n u\left((\alpha_m - \alpha)(x_n^m - x_n) + \alpha(x_n^m - x_n) + (\alpha_m - \alpha)\right)\Bigg|^{p_n}\Bigg)^{\frac{1}{M}}$$

$$\leq \max\{1, |\alpha_m - \alpha|\}\tilde{h}(x^m - x) + |\alpha|\tilde{h}(x^m - x) + \left(\sum_{n=0}^{\infty}|(\alpha_m - \alpha)y_n|^{p_n}\right)^{\frac{1}{M}},$$

where $y = (y_n)$ is defined in (3.1). By hypothesis $\alpha_m \to \alpha$ as $m \to \infty$, so there is a natural number m_0 such that $|\alpha_m - \alpha| < 1$ for $m \geq m_0$ and hence $|\alpha_m - \alpha|^{p_n} < 1$ for $m \geq m_0$. Since $\sum_{n=0}^{\infty}|y_n|^{p_n} < \infty$, so for given $\epsilon > 0$ there exists $n_0 \geq m_0$ such that $\sum_{n=n_0+1}^{\infty}|y_n|^{p_n} < \epsilon/2$. Then

$$\sum_{n=0}^{\infty}|(\alpha_m - \alpha)y_n|^{p_n} \leq \sum_{n=0}^{n_0}|(\alpha_m - \alpha)y_n|^{p_n} + \sum_{n=n_0+1}^{\infty}|y_n|^{p_n}.$$

Hence $(\sum_{n=0}^{\infty}|(\alpha_m - \alpha)y_n|^{p_n})^{\frac{1}{M}} \to 0$ as $m \to \infty$. Therefore we have $\tilde{h}(\alpha_m x^m - \alpha x) \to 0$ as $m \to \infty$. This shows that the scalar multiplication is continuous. Hence $\tilde{h}$ is a paranorm on the space $l(r, s, t, p; B)$.

Now we prove the completeness of the space $l(r, s, t, p; B)$ with respect to the paranorm $\tilde{h}$. Let (x^m) be a Cauchy sequence in $l(r, s, t, p; B)$. So for every $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that

$$\tilde{h}(x^m - x^l) < \frac{\epsilon}{2}, \quad \text{for all } m, l \geq n_0.$$

Then by definition for each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}&\left|(A(r,s,t;B)x^m)_n - (A(r,s,t;B)x^l)_n\right| \\ &\leq \left(\sum_{n=0}^{\infty}\left|(A(r,s,t;B)x^m)_n - (A(r,s,t;B)x^l)_n\right|^{p_n}\right)^{\frac{1}{M}} < \frac{\epsilon}{2},\end{aligned} \tag{4.2}$$

for all $m, l \geq n_0$, which implies that the sequence $((A(r, s, t; B)x^m)_n)$ is a Cauchy sequence of scalars for each fixed $n \in \mathbb{N}_0$ and hence converges for each n. We write

$$\lim_{m\to\infty}(A(r,s,t;B)x^m)_n = (A(r,s,t;B)x)_n, \quad n \in \mathbb{N}_0.$$

Now taking $l \to \infty$ in (4.2), we obtain

$$\left(\sum_{n=0}^{\infty}|(A(r,s,t;B)x^m)_n - (A(r,s,t;B)x)_n|^{p_n}\right)^{\frac{1}{M}} < \epsilon$$

for all $m \geq n_0$ and each fixed $n \in \mathbb{N}_0$. Thus (x^m) converges to x in $l(r, s, t, p; B)$ with respect to $\tilde{h}$.

To show $x \in l(r, s, t, p; B)$, we take

$$\left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)x_k + s_0 t_n u x_n\right)\right|^{p_n}\right)^{\frac{1}{M}}$$

$$= \left(\sum_{n=0}^{\infty}\left|\frac{1}{r_n}\left(\sum_{k=0}^{n-1}(s_{n-k}t_k u + s_{n-k-1}t_{k+1}v)(x_k - x_k^m + x_k^m)\right.\right.\right.$$
$$\left.\left.\left. + s_0 t_n u(x_n - x_n^m + x_n^m)\right)\right|^{p_n}\right)^{\frac{1}{M}}$$
$$\leq \tilde{h}(x - x^m) + \tilde{h}(x^m),$$

which is finite for all $m \geq n_0$. Therefore $x \in l(r, s, t, p; B)$. This completes the proof. □

Theorem 4.2. *The sequence spaces $X(r, s, t, p; B)$ for $X \in \{l_\infty(p),\ c(p), c_0(p), l(p)\}$ are linearly isomorphic to the spaces $X \in \{l_\infty(p),\ c(p), c_0(p), l(p)\}$ respectively, i.e., $l_\infty(r, s, t, p; B) \cong l_\infty(p)$, $c(r, s, t, p; B) \cong c(p)$, $c_0(r, s, t, p; B) \cong c_0(p)$ and $l(r, s, t, p; B) \cong l(p)$.*

Proof. The proof is omitted as it is routine. □

Theorem 4.3. *Let $\nu_k = (A(r, s, t; B)x)_k$, $k \in \mathbb{N}_0$. For each $j \in \mathbb{N}_0$, define the sequence $b^{(j)} = (b_n^{(j)})_{n\in\mathbb{N}_0}$ of the elements of the space $c_0(r, s, t, p; B)$ as*

$$b_n^{(j)} = \begin{cases} \sum_{k=j}^{n}(-1)^{k-j}\dfrac{(-v)^{n-k}}{u^{n-k+1}}\dfrac{D_{k-j}^{(s)}}{t_k}r_j, & \text{if } 0 \leq j \leq n, \\ 0, & \text{if } j > n, \end{cases}$$

and

$$b_n^{(-1)} = \sum_{j=0}^{n}\sum_{k=j}^{n}(-1)^{k-j}\frac{(-v)^{n-k}}{u^{n-k+1}}\frac{D_{k-j}^{(s)}}{t_k}r_j.$$

Then the followings are true.
(i) *The sequence $(b^{(j)})_{j=0}^{\infty}$ is a basis for the space $X(r, s, t, p; B)$ for $X \in \{c_0(p), l(p)\}$ and any $x \in X(r, s, t, p; B)$ has a unique representation of the form*

$$x = \sum_{j=0}^{\infty}\nu_j b^{(j)}.$$

(ii) *The set $(b^{(j)})_{j=-1}^{\infty}$ is a basis for the space $c(r, s, t, p; B)$ and any $x \in c(r, s, t, p; B)$ has a unique representation of the form*

$$x = \ell b^{(-1)} + \sum_{j=0}^{\infty}(\nu_j - \ell)b^{(j)},$$

where $\ell = \lim_{n\to\infty}(A(r, s, t; B)x)_n$.

Remark 4.1. In particular, if we choose $r_n = 1/r'_n$, $t_n = s'_n$, $s_n = 1$ for all n, then the sequence space $l(r, s, t, p; B)$ reduces to $l(r', s', p; B)$ [9]. With this choice of s_n, we have $D_0^{(s)} = D_1^{(s)} = 1$ and $D_n^{(s)} = 0$ for $n \geq 2$. Thus the sequences $b^{(j)} = (b_n^{(j)})_{n \in \mathbb{N}_0}$ for $j = 0, 1, \ldots$ reduce to

$$b_n^{(j)} = \begin{cases} \frac{(-1)^{n-k}}{r'_j} \left(\frac{v^{n-j}}{u^{n-j+1}} \frac{1}{s'_j} + \frac{v^{n-j-1}}{u^{n-j}} \frac{1}{s'_{j+1}} \right), & \text{if } 0 \leq j < n, \\ \frac{1}{u} \frac{1}{r'_n s'_n}, & \text{if } j = n, \\ 0, & \text{if } j > n. \end{cases}$$

The sequence $(b^{(j)})$ is a Schauder basis for the space $l(r', s', p; B)$ studied in [9].

4.1. **The α-, β-, γ-duals of $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$.** K. G. Grosse-Erdmann [16] characterized the matrix transformations between the sequence spaces of Maddox, namely, $l_\infty(p), c(p), c_0(p)$ and $l(p)$. Let L denote a natural number, F be a nonempty finite subset of $\mathbb{N}$ and $A = (a_{nk})_{n,k}$ be an infinite matrix. We consider $p'_k = \frac{p_k}{p_k - 1}$ for $1 < p_k < \infty$. To compute the α-, β-, γ-duals of $X(r, s, t, p; B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$ and to characterize the classes of some matrix mappings between these spaces, we list the following conditions [16]:

$$\sup_F \sup_k \Big| \sum_{n \in F} a_{nk} \Big|^{p_k} < \infty, \tag{4.3}$$

$$\sup_F \sum_k \Big| \sum_{n \in F} a_{nk} L^{-1} \Big|^{p'_k} < \infty, \quad \text{for some } L, \tag{4.4}$$

$$\lim_n a_{nk} = 0, \quad \text{for every } k, \tag{4.5}$$

$$\sup_n \sup_k |a_{nk} L|^{p_k} < \infty, \quad \text{for all } L, \tag{4.6}$$

$$\sup_n \sum_k |a_{nk} L|^{p_k{}'} < \infty, \quad \text{for all } L, \tag{4.7}$$

$$\sup_n \sup_k |a_{nk}|^{p_k} < \infty, \tag{4.8}$$

$$\text{there exists } (\alpha_k) \lim_{n \to \infty} a_{nk} = \alpha_k, \quad \text{for all } k, \tag{4.9}$$

$$\text{there exists } (\alpha_k) \sup_n \sup_k \Big(|a_{nk} - \alpha_k| L \Big)^{p_k} < \infty, \quad \text{for all } L, \tag{4.10}$$

$$\text{there exists } (\alpha_k) \sup_n \sum_k \Big(|a_{nk} - \alpha_k| L \Big)^{p'_k} < \infty, \quad \text{for all } L, \tag{4.11}$$

$$\sup_n \sup_k |a_{nk} L^{-1}|^{p_k} < \infty, \quad \text{for some } L, \tag{4.12}$$

$$\sup_F \sum_n \Big| \sum_{k \in F} a_{nk} L^{\frac{-1}{p_k}} \Big| < \infty, \quad \text{for some } L, \tag{4.13}$$

$$\sum_n \Big| \sum_k a_{nk} \Big| < \infty, \tag{4.14}$$

$$\sup_F \sum_n \Big| \sum_{k \in F} a_{nk} L^{\frac{1}{p_k}} \Big| < \infty, \quad \text{for all } L, \tag{4.15}$$

$$\sup_n \sum_k |a_{nk}| L^{-\frac{1}{p_k}} < \infty, \quad \text{for some } L, \tag{4.16}$$

$$\sup_n \Big| \sum_k a_{nk} \Big| < \infty, \tag{4.17}$$

$$\sup_n \sum_k |a_{nk}| L^{\frac{1}{p_k}} < \infty, \quad \text{for all } L, \tag{4.18}$$

$$\text{there exists } (\alpha_k) \sup_n \sum_k |a_{nk} - \alpha_k| L^{-\frac{1}{p_k}} < \infty, \quad \text{for some } L, \tag{4.19}$$

$$\text{there exists } \alpha \lim_n \Big| \sum_k a_{nk} - \alpha \Big| = 0, \tag{4.20}$$

$$\sup_n \sum_k |a_{nk}| L^{\frac{1}{p_k}} < \infty, \quad \text{for all } L, \tag{4.21}$$

$$\text{there exists } (\alpha_k) \lim_n \sum_k |a_{nk} - \alpha_k| L^{\frac{1}{p_k}} = 0, \quad \text{for all } L, \tag{4.22}$$

$$\sup_n \sum_k |a_{nk} L^{-1}|^{p'_k} < \infty, \quad \text{for some } L. \tag{4.23}$$

Lemma 4.1. [16]
(a) *Let* $1 < p_k \leq H < \infty$. *Then we have:*
 (i) $A \in (l(p), l_1)$ *if and only if* (4.4) *holds.*
 (ii) $A \in (l(p), c_0)$ *if and only if* (4.5) *and* (4.7) *hold.*
 (iii) $A \in (l(p), c)$ *if and only if* (4.9), (4.11) *and* (4.23) *hold.*
 (iv) $A \in (l(p), l_\infty)$ *if and only if* (4.23) *holds.*
(b) *Let* $0 < p_k \leq 1$. *Then we have:*
 (i) $A \in (l(p), l_1)$ *if and only if* (4.3) *holds.*
 (ii) $A \in (l(p), c_0)$ *if and only if* (4.5) *and* (4.6) *hold.*
 (iii) $A \in (l(p), c)$ *if and only if* (4.8), (4.9) *and* (4.10) *hold.*
 (iv) $A \in (l(p), l_\infty)$ *if and only if* (4.12) *holds.*

Lemma 4.2. [16] *Let* $0 < p_k \leq H < \infty$. *Then we have:*
(i) $A \in (c_0(p), l_1)$ *if and only if* (4.13) *holds,*
(ii) $A \in (c(p), l_1)$ *if and only if* (4.13) *and* (4.14) *hold,*
(iii) $A \in (l_\infty(p), l_1)$ *if and only if* (4.15) *holds.*

Lemma 4.3. [16] *Let* $0 < p_k \leq H < \infty$. *Then we have:*
(i) $A \in (c_0(p), l_\infty)$ *if and only if* (4.16) *holds,*
(ii) $A \in (c(p), l_\infty)$ *if and only if* (4.16) *and* (4.17) *hold,*
(iii) $A \in (l_\infty(p), l_\infty)$ *if and only if* (4.18) *holds.*

Lemma 4.4. [16] *For* $0 < p_k \leq H < \infty$, *we have:*
(i) $A \in (c_0(p), c)$ *if and only if* (4.9), (4.16) *and* (4.19) *hold,*
(ii) $A \in (c(p), c)$ *if and only if* (4.9), (4.16), (4.19), *and* (4.20) *hold,*

(iii) $A \in (l_\infty(p), c)$ *if and only if* (4.21) *and* (4.22) *hold.*

We now define the following sets to obtain the α-dual of the spaces $X(r,s,t,p;B)$:

$$H_1(p) = \bigcup_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_F \sum_n \left| \sum_{k\in F} \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n L^{\frac{-1}{p_k}} \right| < \infty \right\},$$

$$H_2(p) = \left\{ a=(a_n)\in w : \sum_n \left| \sum_k \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n \right| < \infty \right\},$$

$$H_3(p) = \bigcap_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_F \sum_n \left| \sum_{k\in F} \left(\sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n \right) L^{\frac{1}{p_k}} \right| < \infty \right\},$$

$$H_4(p) = \left\{ a=(a_n)\in w : \sup_F \sup_k \left| \sum_{n\in F} \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n \right|^{p_k} < \infty \right\},$$

$$H_5(p) = \bigcup_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_F \sum_k \left| \sum_{n\in F} \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n L^{-1} \right|^{p'_k} < \infty \right\}.$$

Theorem 4.4.
(a) *If* $p_k > 1$, *then* $[l(r,s,t,p;B)]^\alpha = H_5(p)$ *and* $[l(r,s,t,p;B)]^\alpha = H_4(p)$ *for* $0 < p_k \le 1$.
(b) *For* $0 < p_k \le H < \infty$, *we have:*
(i) $[c_0(r,s,t,p;B)]^\alpha = H_1(p)$,
(ii) $[c(r,s,t,p;B)]^\alpha = H_1(p) \cap H_2(p)$,
(iii) $[l_\infty(r,s,t,p;B)]^\alpha = H_3(p)$.

Proof.
(a) Let $p_k > 1$ for all k, $a=(a_n)\in w$ and $x \in l(r,s,t,p;B)$. Then for each n, we have

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n y_k = (Cy)_n,$$

where the matrix $C=(c_{nk})_{n,k}$ is defined as

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{j-k} \frac{(-v)^{n-j}}{u^{n-j+1}} \frac{D_{j-k}^{(s)}}{t_j} r_k a_n, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n, \end{cases}$$

and x_n is connected with y_n by the relation (3.1). Thus for each $x \in l(r,s,t,p;B)$, $(a_n x_n)_n \in l_1$ if and only if $Cy \in l_1$ where $y \in l(p)$. Therefore $a=(a_n) \in [l(r,s,t,p;B)]^\alpha$ if and only if $C \in (l(p), l_1)$. By using Lemma 4.1(a), we have

$$[l(r,s,t,p;B)]^\alpha = H_5(p).$$

If $0 < p_k \le 1$ $\forall k$, then using Lemma 4.1(b), we have $[l(r,s,t,p;B)]^\alpha = H_4(p)$.

(b) In a similar way, using Lemma 4.2, it can be derived that $[c_0(r,s,t,p;B)]^\alpha = H_1(p)$, $[c(r,s,t,p;B)]^\alpha = H_1(p) \cap H_2(p)$ and $[l_\infty(r,s,t,p;B)]^\alpha = H_3(p)$. □

To find the γ-dual of the spaces $X(r,s,t,p;B)$ for $X \in \{l_\infty(p), c(p), c_0(p), l(p)\}$, we consider the following sets:

$$\Gamma_1(p) = \bigcup_{L\in\mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \sum_k |e_{nk}| L^{-\frac{1}{p_k}} < \infty \right\},$$

$$\Gamma_2(p) = \left\{ a = (a_k) \in w : \sup_n \left| \sum_k e_{nk} \right| < \infty \right\},$$

$$\Gamma_3(p) = \bigcap_{L\in\mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \sum_k |e_{nk}| L^{\frac{1}{p_k}} < \infty \right\},$$

$$\Gamma_4(p) = \bigcup_{L\in\mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \sup_k |e_{nk} L^{-1}|^{p_k} < \infty \right\},$$

$$\Gamma_5(p) = \bigcup_{L\in\mathbb{N}} \left\{ a = (a_k) \in w : \sup_n \sum_k |e_{nk} L^{-1}|^{p'_k} < \infty \right\},$$

where the matrix $E = (e_{nk})$ is defined as

$$e_{nk} = \begin{cases} r_k \left[\dfrac{1}{u} \dfrac{a_k}{s_0 t_k} + \displaystyle\sum_{j=k}^{k+1} (-1)^{j-k} \dfrac{D^{(s)}_{j-k}}{t_j} \left(\sum_{l=k+1}^{n} \dfrac{(-v)^{l-j}}{u^{l-j+1}} a_l \right) \right. & \\ \left. + \displaystyle\sum_{j=k+2}^{n} (-1)^{j-k} \dfrac{D^{(s)}_{j-k}}{t_j} \left(\sum_{l=j}^{n} \dfrac{(-v)^{l-j}}{u^{l-j+1}} a_l \right) \right], & 0 \le k \le n, \\ 0, & k > n. \end{cases} \tag{4.24}$$

Theorem 4.5.
(a) *If $p_k > 1$, then $[l(r,s,t,p;B)]^\gamma = \Gamma_5(p)$ and $[l(r,s,t,p;B)]^\gamma = \Gamma_4(p)$ for $0 < p_k \le 1$.*
(b) *For $0 < p_k \le H < \infty$, we have:*
 (i) *$[c_0(r,s,t,p;B)]^\gamma = \Gamma_1(p)$,*
 (ii) *$[c(r,s,t,p;B)]^\gamma = \Gamma_1(p) \cap \Gamma_2(p)$,*
 (iii) *$[l_\infty(r,s,t,p;B)]^\gamma = \Gamma_3(p)$.*

Proof.
(a) Let $p_k > 1$ for all k, $a = (a_n) \in w$ and $x \in l(r,s,t,p;B)$. Since the sequences x and y are connected with the relation (3.1), it is immediate that $x = B^{-1}\tilde{B}y$ which is equivalent to the following relation:

$$x_k = \sum_{l=0}^{k} \sum_{j=l}^{k} (-1)^{j-l} \frac{(-v)^{k-j}}{u^{k-j+1}} \frac{D^{(s)}_{j-l}}{t_j} r_l y_l, \tag{4.25}$$

for all $k \in \mathbb{N}_0$. Then, one can see by using (4.25) that

$$
\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=l}^{k}(-1)^{j-l}\frac{(-v)^{k-j}}{u^{k-j+1}}\frac{D_{j-l}^{(s)}}{t_j}r_l y_l a_k \\
&= \sum_{k=0}^{n-1}\sum_{l=0}^{k}\sum_{j=l}^{k}(-1)^{j-l}\frac{(-v)^{k-j}}{u^{k-j+1}}\frac{D_{j-l}^{(s)}}{t_j}a_k r_l y_l \\
&\quad + \sum_{l=0}^{n}\sum_{j=l}^{n}(-1)^{j-l}\frac{(-v)^{n-j}}{u^{n-j+1}}\frac{D_{j-l}^{(s)}}{t_j}a_n r_l y_l \\
&= \left[\frac{1}{u}\frac{D_0^{(s)}}{t_0}a_0 + \sum_{j=0}^{1}(-1)^{j}\frac{D_j^{(s)}}{t_j}\left(\sum_{l=1}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right. \\
&\quad \left. + \sum_{j=2}^{n}(-1)^{j}\frac{D_j^{(s)}}{t_j}\left(\sum_{l=j}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right] r_0 y_0 \\
&\quad + \left[\frac{1}{u}\frac{D_0^{(s)}}{t_1}a_1 + \sum_{j=1}^{2}(-1)^{j-1}\frac{D_{j-1}^{(s)}}{t_j}\left(\sum_{l=2}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right. \\
&\quad \left. + \sum_{j=3}^{n}(-1)^{j-1}\frac{D_{j-1}^{(s)}}{t_j}\left(\sum_{l=j}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right] r_1 y_1 + \ldots + \frac{1}{u}\frac{D_0^{(s)}}{t_n}a_n r_n y_n \\
&= \sum_{k=0}^{n}\left[\frac{1}{u}\frac{a_k}{s_0 t_k} + \sum_{j=k}^{k+1}(-1)^{j-k}\frac{D_{j-k}^{(s)}}{t_j}\left(\sum_{l=k+1}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right. \\
&\quad \left. + \sum_{j=k+2}^{n}(-1)^{j-k}\frac{D_{j-k}^{(s)}}{t_j}\left(\sum_{l=j}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_l\right)\right] r_k y_k \\
&= (Ey)_n,
\end{aligned}
$$

where E is the matrix defined in (4.24).

Thus $a \in \big[l(r,s,t,p;B)\big]^{\gamma}$ if and only if $ax = (a_k x_k) \in bs$ for $x \in l(r,s,t,p;B)$ if and only if $\left(\sum_{k=0}^{n} a_k x_k\right)_n \in l_\infty$, i.e., $Ey \in l_\infty$, for $y \in l(p)$. Hence, using Lemma 4.1(a), we have

$$\big[l(r,s,t,p;B)\big]^{\gamma} = \Gamma_5(p).$$

If $0 < p_k \leq 1$ for all k, then using Lemma 4.1(b), we have $[l(r,s,t,p;\Delta)]^{\gamma} = \Gamma_4(p)$.
(b) In a similar way, using Lemma 4.3, we can obtain $[c_0(r,s,t,p;B)]^{\gamma} = \Gamma_1(p)$, $[c(r,s,t,p;B)]^{\gamma} = \Gamma_1(p) \cap \Gamma_2(p)$ and $[l_\infty(r,s,t,p;B)]^{\gamma} = \Gamma_3(p)$. □

To obtain β-duals of $X(r,s,t,p;B)$, we define the following sets:

$$B_1 = \left\{ a=(a_n)\in w : \sum_{l=k+1}^{\infty} \frac{(-v)^{l-j}}{u^{l-j+1}} a_l \text{ exists for all } k \right\},$$

$$B_2 = \left\{ a=(a_n)\in w : \sum_{j=k+2}^{\infty} (-1)^{j-k} \frac{D_{j-k}^{(s)}}{t_j} \left(\sum_{l=j}^{\infty} \frac{(-v)^{l-j}}{u^{l-j+1}} a_l \right) \text{ exists for all } k \right\},$$

$$B_3 = \left\{ a=(a_n)\in w : \left(\frac{r_k a_k}{t_k}\right) \in l_\infty(p) \right\},$$

$$B_4 = \bigcup_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_n \sum_k \left| e_{nk} L^{-1} \right|^{p_k'} < \infty \right\},$$

$$B_5 = \left\{ a=(a_n)\in w : \sup_{n,k} |e_{nk}|^{p_k} < \infty \right\},$$

$$B_6 = \left\{ a=(a_n)\in w : \text{there exists } (\alpha_k) \ \lim_{n\to\infty} e_{nk} = \alpha_k \text{ for all } k \right\}$$

$$B_7 = \bigcap_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \text{there exists } (\alpha_k) \ \sup_{n,k} \left(|e_{nk}-\alpha_k| L\right)^{p_k} < \infty \right\},$$

$$B_8 = \bigcap_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \text{there exists } (\alpha_k) \ \sup_n \sum_k \left(|e_{nk}-\alpha_k| L\right)^{p_k'} < \infty \right\},$$

$$B_9 = \bigcup_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \text{there exists } (\alpha_k) \ \sup_n \sum_k |e_{nk}-\alpha_k| \, L^{\frac{-1}{p_k}} < \infty \right\},$$

$$B_{10} = \bigcup_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_n \sum_k |e_{nk}| L^{\frac{-1}{p_k}} < \infty \right\},$$

$$B_{11} = \left\{ a=(a_n)\in w : \text{there exists } \alpha \ \lim_n \left| \sum_k e_{nk} - \alpha \right| = 0 \right\},$$

$$B_{12} = \bigcap_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \sup_n \sum_k |e_{nk}| L^{\frac{1}{p_k}} < \infty \right\}$$

$$B_{13} = \bigcap_{L\in\mathbb{N}} \left\{ a=(a_n)\in w : \text{there exists } (\alpha_k) \ \lim_n \sum_k |e_{nk}-\alpha_k| L^{\frac{1}{p_k}} = 0 \right\}.$$

Theorem 4.6.
(a) *If $p_k > 1$ for all k, then $[l(r,s,t,p;B)]^\beta = B_1\cap B_2\cap B_3\cap B_4\cap B_6\cap B_8$ and if $0 < p_k \le 1$ for all k, then $[l(r,s,t,p;B)]^\beta = B_1\cap B_2\cap B_3\cap B_5\cap B_6\cap B_7$.*
(b) *Let $p_k > 0$ for all k. Then:*
(i) $[c_0(r,s,t,p;B)]^\beta = B_1\cap B_2\cap B_3\cap B_6\cap B_9\cap B_{10}$,
(ii) $[c(r,s,t,p;B)]^\beta = B_1\cap B_2\cap B_3\cap B_6\cap B_9\cap B_{10}\cap B_{11}$,

(iii) $[l_\infty(r,s,t,p;B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_{12} \cap B_{13}$.

Proof.
(a) Let $p_k > 1$ for all k. By Theorem 4.5, we have

$$\sum_{k=0}^{n} a_k x_k = Ey,$$

where the matrix E is defined in (4.24). Thus $a \in \left[l(r,s,t,p;B)\right]^\beta$ if and only if $ax = (a_k x_k) \in cs$, where $x \in l(r,s,t,p;B)$ if and only if $(Ey)_n \in c$, where $y \in l(p)$, i.e., $E \in (l(p), c)$. Hence by Lemma 4.1(a), we have

$$\sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \left|e_{nk}L^{-1}\right|^{p_k'} < \infty, \quad \text{for some } L \in \mathbb{N},$$
$$\text{there exists } (\alpha_k) \ \lim_{n\to\infty} e_{nk} = \alpha_k, \quad \text{for all } k,$$
$$\text{there exists } (\alpha_k) \ \sup_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} (|e_{nk} - \alpha_k|L)^{p_k'} < \infty, \quad \text{for all } L \in \mathbb{N}.$$

Therefore $[l(r,s,t,p;B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_6 \cap B_8$.
If $0 < p_k \leq 1$, for all k, then using Lemma 4.1(b), we have

$$\sup_{n,k\in\mathbb{N}_0} |e_{nk}|^{p_k} < \infty,$$
$$\text{there exists } (\alpha_k) \ \lim_{n\to\infty} e_{nk} = \alpha_k, \quad \text{for all } k,$$
$$\text{there exists } (\alpha_k) \ \sup_{n,k\in\mathbb{N}_0} (|e_{nk} - \alpha_k|L)^{p_k} < \infty, \quad \text{for all } L \in \mathbb{N}.$$

Thus $[l(r,s,t,p;B)]^\beta = B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7$.
(b) In a similar way, using Lemma 4.4, we can obtain the β-duals of $c_0(r,s,t,p;B)$, $c(r,s,t,p;B)$ and $l_\infty(r,s,t,p;B)$. ☐

4.2. **Matrix Mappings.**

Theorem 4.7. *Let $\tilde{E} = (\tilde{e}_{nk})$ be the matrix which is same as the matrix $E = (e_{nk})$ defined in* (4.24)*, where a_k and a_l are replaced by a_{nk} and a_{nl} respectively.*
(a) *Let $1 < p_k \leq H < \infty$, for $k \in \mathbb{N}_0$. Then $A \in (l(r,s,t,p;B), l_\infty)$ if and only if there exists $L \in \mathbb{N}$ such that*

$$\sup_n \sum_k |\tilde{e}_{nk}L^{-1}|^{p_k'} < \infty \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_6 \cap B_8.$$

(b) *Let $0 < p_k \leq 1$ for $k \in \mathbb{N}$. Then $A \in (l(r,s,t,p;B), l_\infty)$ if and only if there exists $L \in \mathbb{N}$ such that*

$$\sup_n \sup_k |\tilde{e}_{nk}L^{-1}|^{p_k} < \infty \quad \text{and} \quad (a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7.$$

Proof.
(a) Let $p_k > 1$, for all k. Since $(a_{nk})_k \in \big[l(r,s,t,p;B)\big]^\beta$ for each fixed n, Ax exists for all $x \in l(r,s,t,p;B)$. Now for each n, we have

$$\begin{aligned}\sum_{k=0}^{m} a_{nk}x_k &= \sum_{k=0}^{m} r_k \left[\frac{1}{u}\frac{a_{nk}}{s_0 t_k} + \sum_{j=k}^{k+1}(-1)^{j-k}\frac{D^{(s)}_{j-k}}{t_j}\left(\sum_{l=k+1}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_{nl}\right)\right.\\ &\qquad \left. + \sum_{j=k+2}^{n}(-1)^{j-k}\frac{D^{(s)}_{j-k}}{t_j}\left(\sum_{l=j}^{n}\frac{(-v)^{l-j}}{u^{l-j+1}}a_{nl}\right)\right] y_k\\ &= \sum_{k=0}^{m} \tilde{e}_{nk}y_k,\end{aligned}$$

Taking $m \to \infty$, we have

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty}\tilde{e}_{nk}y_k, \quad \text{for all } n \in \mathbb{N}_0.$$

We know that for any $T > 0$ and any complex numbers a, b

$$|ab| \le T(|aT^{-1}|^{p'} + |b|^p),$$

where $p > 1$ and $1/p + 1/p' = 1$. It is easy to compute that

$$\sup_n\left|\sum_k a_{nk}x_k\right| \le \sup_n \sum_k |\tilde{e}_{nk}|\,|y_k| \le T\left[\sup_n\sum_k|\tilde{e}_{nk}T^{-1}|^{p_k'} + \sum_k |y_k|^{p_k}\right] < \infty.$$

Conversely, assume that $A \in (l(r,s,t,p;B), l_\infty)$ and $1 < p_k \le H < \infty$ for all k. Then Ax exists for each $x \in l(r,s,t,p;B)$, which implies that $(a_{nk})_k \in [l(r,s,t,p;B)]^\beta$ for each n. Thus $(a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_6 \cap B_8$. Since $\sum_{k=0}^\infty a_{nk}x_k = \sum\limits_{k=0}^{\infty}\tilde{e}_{nk}y_k$, we have $\tilde{E} = (\tilde{e}_{nk}) \in (l(p), l_\infty)$. Now using Lemma 4.1(*a*), we have $\sup_n\sum_k|\tilde{e}_{nk}L^{-1}|^{p_k'} < \infty$ for some $L \in \mathbb{N}$. This completes the proof.
(b) We omit the proof of this part as it is similar to the previous part. □

Theorem 4.8.
(a) *Let* $1 < p_k \le H < \infty$, *for* $k \in \mathbb{N}$. *Then* $A \in (l(r,s,t,p;B), l_1)$ *iff there exists* $L \in \mathbb{N}$ *such that*

$$\sup_F \sum_k \Big|\sum_{n\in F}\tilde{e}_{nk}L^{-1}\Big|^{p_k'} < \infty, \quad \textit{for some } L \in \mathbb{N}$$

and

$$(a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_6 \cap B_8.$$

(b) *Let* $0 < p_k \le 1$, *for* $k \in \mathbb{N}$. *Then* $A \in (l(r,s,t,p;B), l_1)$ *if and only if*

$$\sup_F \sup_k \Big|\sum_{n\in F}\tilde{e}_{nk}\Big|^{p_k} < \infty$$

and

$$(a_{nk})_k \in B_1 \cap B_2 \cap B_3 \cap B_5 \cap B_6 \cap B_7.$$

Proof. We omit the proof as it follows in the same way. □

5. Compact Operators on the Space $l_p(r, s, t; B)$

In this section, we concentrate on $l_p(r, s, t; B)$, $p \geq 1$, which is a BK space and establish some identities or estimates for the Hausdorff measure of noncompactness of certain matrix operators on the space $l_p(r, s, t; B)$. Moreover, we characterize some classes of compact operators on this space.

The Hausdorff measure of noncompactness was first introduced and studied by Goldenstein, Gohberg and Markus in 1957 and later on stuided by Istrăţesku in 1972 [17]. It is quite natural to find necessary and sufficient conditions for a matrix mapping between BK spaces to define a compact operator as the matrix transformations between BK spaces are continuous. This can be achieved with the help of Hausdorff measure of noncompactness. Recently several authors, namely, Malkowsky and Rakočević [23], Djolović et al. [15], Djolović [14], Başar et al. [8], Mursaleen and Noman [28, 29], Başarır and Kara [9] have established some identities or estimates for the operator norms and the Hausdorff measure of noncompactness of matrix operators from an arbitrary BK space to arbitrary BK space. Let us recall some definitions and well-known results.

Let X, Y be two Banach spaces and S_X denote the unit sphere in X, i.e., $S_X = \{x \in X : \|x\| = 1\}$. We denote by $\mathcal{B}(X, Y)$, the set of all bounded (continuous) linear operators from X to Y which is a Banach space with the operator norm $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$. A linear operator $L : X \to Y$ is said to be compact if the domain of L is all of X and for every bounded sequence $(x_n) \in X$, the sequence $(L(x_n))$ has a subsequence which is convergent in Y and we denote by $\mathcal{C}(X, Y)$, the class of all compact operators in $\mathcal{B}(X, Y)$. An operator $L \in \mathcal{B}(X, Y)$ is said to be finite rank if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. If X is a BK space and $a = (a_k) \in w$, then we consider

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|, \tag{5.1}$$

provided the expression on the right side exists and is finite which is the case whenever $a \in X^\beta$ (see [29]).

Let (X, d) be a metric space and $\mathcal{M}_X$ be the class of all bounded subsets of X. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ denotes the open ball of radius $r > 0$ with centre at x. The Hausdorff measure of noncompactness of a set $Q \in \mathcal{M}_X$, denoted by $\chi(Q)$, is defined as

$$\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon, n \in \mathbb{N}_0 \right\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness. Let ϕ denote the set of all finite sequences, i.e., of sequences that terminate in zeros. Throughout we denote p' as the conjugate of p for $1 \leq p < \infty$, i.e., $p' = \frac{p}{p-1}$ for $p > 1$ and $p' = \infty$ for $p = 1$. The following known results are fundamental for our investigation.

Lemma 5.1 ([29]). *Let $1 \leq p < \infty$ and $A \in (l_p, c)$. Then the following holds:*
(i) $\alpha_k = \lim_{n\to\infty} a_{nk}$ *exists for all* $k \in \mathbb{N}_0$,
(ii) $\alpha = (\alpha_k) \in l_{p'}$,
(iii) $\sup_n \|A_n - \alpha\|_{l_{p'}} < \infty$,
(iv) $\lim_{n\to\infty}(Ax)_n = \sum_{k=0}^{\infty} \alpha_k x_k$, *for all* $x = (x_k) \in l_p$.

Lemma 5.2 ([23], Theorem 1.29). *Let $1 \leq p < \infty$. Then we have $l_p^\beta = l_{p'}$ and $\|a\|_{l_p}^* = \|a\|_{l_{p'}}$ for all $a \in l_{p'}$.*

Lemma 5.3 ([29]). *Let $X \supset \phi$ and Y be BK spaces. Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, i.e., every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$.*

Lemma 5.4 ([14]). *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or l_∞. If $A \in (X, Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 5.5 ([23]). *Let $Q \in \mathcal{M}_X$, where $X = l_p$ for $1 \leq p < \infty$ or c_0 and $P_m : c_0 \to c_0$ $(m \in \mathbb{N}_0)$ is an operator defined by $P_m(x) = (x_0, x_1, \cdots, x_m, 0, 0, \cdots)$ for all $x = (x_k) \in X$. Then we have*

$$\chi(Q) = \lim_{m\to\infty} \left(\sup_{x\in Q} \|(I - P_m)(x)\| \right),$$

where I is the identity operator on X.

Let $z = (z_n) \in c$. Then z has a unique representation $z = \ell e + \sum_{n=0}^{\infty}(z_n - \ell)e_n$, where $\ell = \lim_{n\to\infty} z_n$. We now define the operator P_m $(m \in \mathbb{N}_0)$ from c onto the linear span of $\{e, e_0, e_1, \cdots, e_m\}$ as

$$P_m(z) = \ell e + \sum_{n=0}^{m}(z_n - \ell)e_n,$$

for all $z \in c$ and $\ell = \lim_{n\to\infty} z_n$.

Then the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

Lemma 5.6 ([23]). *Let $Q \in \mathcal{M}_c$ and $P_m : c \to c$ be the projector from c onto the linear span of $\{e, e_0, e_1, \ldots e_m\}$. Then we have*

$$\frac{1}{2} \lim_{m\to\infty} \left(\sup_{x\in Q} \|(I - P_m)(x)\|_{l_\infty} \right) \leq \chi(Q) \leq \lim_{m\to\infty} \left(\sup_{x\in Q} \|(I - P_m)(x)\|_{l_\infty} \right),$$

where I is the identity operator on c.

Lemma 5.7 ([23]). *Let X, Y be two Banach spaces and $L \in \mathcal{B}(X, Y)$. Then*

$$\|L\|_\chi = \chi(L(S_X))$$

and

$$L \in \mathcal{C}(X, Y), \quad \textit{if and only if } \|L\|_\chi = 0.$$

Let $\mathcal{F}_m = \{F \in \mathcal{F} : n > m,\ \forall n \in F\}$, $m \in \mathbb{N}$ and $\mathcal{F}$ is the collection of nonempty and finite subsets of $\mathbb{N}$.

Lemma 5.8 ([28]). *Let $X \supset \phi$ be a BK space.*
(a) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^*$$

and

$$L_A \textit{ is compact if and only if } \lim_{n\to\infty} \|A_n\|_X^* = 0.$$

(b) *If $A \in (X, l_\infty)$, then*

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \|A_n\|_X^*$$

and

$$L_A \textit{ is compact if } \lim_{n\to\infty} \|A_n\|_X^* = 0.$$

(c) *If $A \in (X, l_1)$, then*

$$\lim_{m\to\infty} \left(\sup_{F\in\mathcal{F}_m} \left\| \sum_{n\in F} A_n \right\|_X^* \right) \le \|L_A\|_\chi \le 4 \lim_{m\to\infty} \left(\sup_{F\in\mathcal{F}_m} \left\| \sum_{n\in F} A_n \right\|_X^* \right)$$

and

$$L_A \textit{ is compact if and only if } \lim_{m\to\infty} \left(\sup_{F\in\mathcal{F}_m} \left\| \sum_{n\in F} A_n \right\|_X^* \right) = 0.$$

We establish the following lemmas which are required to characterize the classes of compact operators with the help of Hausdorff measure of noncompactness.

Lemma 5.9. *If $a = (a_k) \in [l_p(r, s, t; B)]^\beta$, then $\tilde{a} = (\tilde{a}_k) \in l_{p'}$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k$$

holds for every $x = (x_k) \in l_p(r, s, t; B)$ and $y = (y_k) \in l_p$, where $y = [A(r, s, t) \cdot B]x$ and

$$\tilde{a}_k = r_k \left[\frac{a_k}{s_0 t_k} \frac{1}{u} + \sum_{i=k}^{k+1} (-1)^{i-k} \frac{D_{i-k}^{(s)}}{t_i} \left(\sum_{j=k+1}^{\infty} \frac{(-v)^{j-i}}{u^{j-i+1}} a_j \right) \right.$$

$$+\sum_{i=k+2}^{\infty}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\left(\sum_{j=i}^{\infty}\frac{(-v)^{j-i}}{u^{j-i+1}}a_j\right)\Bigg]. \tag{5.2}$$

Proof. Let $a=(a_k)\in[l_p(r,s,t;B)]^\beta$. Then by [14, Theorem 2.3, Remark 2.4], we have $R(a)=(R_k(a))\in l_p^\beta=l_{p'}$ and also

$$\sum_{k=0}^{\infty}a_kx_k=\sum_{k=0}^{\infty}R_k(a)T_k(x),\quad \text{for all } x\in l_p(r,s,t;B),$$

where

$$\begin{aligned}R_k(a)&=r_k\Bigg[\frac{a_k}{s_0t_k}\frac{1}{u}+\sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\Big(\sum_{j=k+1}^{\infty}\frac{(-v)^{j-i}}{u^{j-i+1}}a_j\Big)\\&\quad+\sum_{i=k+2}^{\infty}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\Big(\sum_{j=i}^{\infty}\frac{(-v)^{j-i}}{u^{j-i+1}}a_j\Big)\Bigg]\\&=\tilde{a}_k.\end{aligned}$$

and $y=T(x)=[A(r,s,t)\cdot B]x$. This completes the proof. □

Lemma 5.10. *Let $1\le p<\infty$. Then we have*

$$\|a\|^*_{l_p(r,s,t;B)}=\|\tilde{a}\|_{l_{p'}}=\begin{cases}\left(\sum_{k=0}^{\infty}|\tilde{a}_k|^{p'}\right)^{\frac{1}{p'}}, & 1<p<\infty,\\ \sup_k|\tilde{a}_k|, & p=1.\end{cases}$$

for all $a=(a_k)\in[l_p(r,s,t;B)]^\beta$, where $\tilde{a}=(\tilde{a}_k)$ is defined in (5.2).

Proof. Let $a=(a_k)\in[l_p(r,s,t;B)]^\beta$. Then from Lemma 5.9, we have $\tilde{a}=(\tilde{a}_k)\in l_{p'}$. Also $x\in S_{l_p(r,s,t;B)}$ if and only if $y=T(x)\in S_{l_p}$ as $\|x\|_{l_p(r,s,t;B)}=\|y\|_{l_p}$. From (5.1), we have

$$\|a\|^*_{l_p(r,s,t;B)}=\sup_{x\in S_{l_p(r,s,t;B)}}\left|\sum_{k=0}^{\infty}a_kx_k\right|=\sup_{y\in S_{l_p}}\left|\sum_{k=0}^{\infty}\tilde{a}_ky_k\right|=\|\tilde{a}\|^*_{l_p}.$$

Using Lemma 5.2, we have $\|a\|^*_{l_p(r,s,t;B)}=\|\tilde{a}\|^*_{l_p}=\|\tilde{a}\|_{l_{p'}}$, which is finite as $\tilde{a}\in l_{p'}$. This completes the proof. □

Lemma 5.11. *Let $1\le p<\infty$, Y be any sequence space and $A=(a_{nk})_{n,k}$ be an infinite matrix. If $A\in(l_p(r,s,t;B),Y)$ then $\tilde{A}\in(l_p,Y)$ such that $Ax=\tilde{A}y$ holds for all $x\in l_p(r,s,t;B)$ and $y\in l_p$, which are connected by the relation $y=[A(r,s,t)\cdot B]x$ and $\tilde{A}=(\tilde{a}_{nk})_{n,k}$ is given by*

$$\tilde{a}_{nk}=r_k\left[\frac{a_{nk}}{s_0t_k}\frac{1}{u}+\sum_{i=k}^{k+1}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\left(\sum_{j=k+1}^{\infty}\frac{(-v)^{j-i}}{u^{j-i+1}}a_{nj}\right)\right.$$

$$+\sum_{i=k+2}^{\infty}(-1)^{i-k}\frac{D_{i-k}^{(s)}}{t_i}\left(\sum_{j=i}^{\infty}\frac{(-v)^{j-i}}{u^{j-i+1}}a_{nj}\right)\Bigg],$$

provided the series on the right side converges for all n, k.

Proof. We assume that $A \in (l_p(r,s,t;B), Y)$. Then $A_n \in [l_p(r,s,t;B)]^\beta$ for all n. Thus it follows from Lemma 5.9, we have $\tilde{A}_n \in l_p^\beta = l_{p'}$ for all n and $Ax = \tilde{A}y$ holds for every $x \in l_p(r,s,t;B)$ and $y \in l_p$, which are connected by the relation $y = [A(r,s,t)\cdot B]x$. Hence $\tilde{A}y \in Y$. Since $x = B^{-1}\cdot A(r,s,t)^{-1}y$, for every $y \in l_p$, we get some $x \in l_p(r,s,t;B)$ and hence $\tilde{A} \in (l_p, Y)$. This completes the proof. □

Theorem 5.1. *Let $1 < p < \infty$.*
(a) *If $A \in (l_p(r,s,t;B), c_0)$ then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}}$$

and L_A is compact if and only if $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 0$.
(b) *If $A \in (l_p(r,s,t;B), l_\infty)$ then*

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}|$$

and L_A is compact if $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 0$.

Proof. Let $1 < p < \infty$ and $A \in (l_p(r,s,t;B), c_0)$, then $A_n \in [l_p(r,s,t;B)]^\beta$ for all n and hence $\tilde{A}_n \in l_{p'}$ by Lemma 5.9. Again using Lemma 5.10, we have

$$\|A_n\|^*_{l_p(r,s,t;B)} = \|\tilde{A}_n\|_{l_{p'}} = \left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}}.$$

Now by Lemma 5.8, we have

$$\|L_A\|_\chi = \limsup_{n\to\infty}\|A_n\|^*_{l_p(r,s,t;B)} = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}}.$$

Using Lemma 5.7, we have L is compact if and only if $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|^{p'}\right)^{\frac{1}{p'}} = 0$. Similarly, we can prove the part (b). □

Theorem 5.2. *If $A \in (l_p(r,s,t;B), c)$ then*

$$\frac{1}{2}\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k| \le \|L_A\|_\chi \le \limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|, \tag{5.3}$$

where $\tilde{\alpha}_k = \lim\limits_{n\to\infty}\tilde{a}_{nk}$ for all k.

Proof. Let $A \in (l_p(r,s,t;B), c)$. Then by using Lemmas 5.11 and 5.1, we can deduce that the expression in (5.3) exists. We write $S = S_{l_p(r,s,t;B)}$ in short. Then by Lemma 5.7, we have $\|L_A\|_\chi = \chi(AS)$. Since $l_p(r,s,t;B)$ and c are BK spaces, A induces continuous map L_A from $l_p(r,s,t;B)$ to c by Lemma 5.3. Thus AS is bounded in c, i.e., $AS \in \mathcal{M}_c$. Let $P_m : c \to c$, $(m \in \mathbb{N}_0)$ be an operator from c onto the span of $\{e, e_0, e_1, \cdots, e_m\}$ defined by

$$P_m(z) = \ell e + \sum_{k=0}^{m}(z_k - \ell)e_k,$$

where $\ell = \lim_{k\to\infty} z_k$. Thus for every m, we have

$$(I - P_m)(z) = \sum_{k=m+1}^{\infty}(z_k - \ell)e_k,$$

where I is the identity operator. Therefore $\|(I - P_m)(z)\|_\infty = \sup_{k=m+1} |z_k - \ell|$, for all $z = (z_k) \in c$. So by applying Lemma 5.6, we have

$$\frac{1}{2}\lim_{m\to\infty}\left(\sup_{x\in S}\|(I-P_m)(Ax)\|_{l_\infty}\right) \le \|L_A\|_\chi \le \lim_{m\to\infty}\left(\sup_{x\in S}\|(I-P_m)(Ax)\|_{l_\infty}\right). \tag{5.4}$$

Since $A \in (l_p(r,s,t;B), c)$, we have by Lemma 5.11, $\tilde{A} \in (l_p, c)$ and $Ax = \tilde{A}y$ for every $x \in l_p(r,s,t;B)$ and $y \in l_p$ which are connected by the relation $y = [A(r,s,t)\cdot B]x$. Again applying Lemma 5.1, we have $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ exists for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in X^\beta = l_{p'}$ and $\lim_{n\to\infty}(\tilde{A}y)_n = \sum_{k=0}^\infty \tilde{\alpha}_k y_k$. Since $\|(I - P_m)(z)\|_{l_\infty} = \sup_{k>m}|z_k - \ell|$, we have

$$\begin{aligned}\|(I-P_m)(Ax)\|_\infty &= \|(I-P_m)(\tilde{A}y)\|_\infty\\ &= \sup_{n>m}\left|(\tilde{A}y)_n - \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right| = \sup_{n>m}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\right|.\end{aligned}$$

Also we know that $x \in S = S_{l_p(r,s,t;B)}$ if and only if $y \in S_{l_p}$. From (5.1) and Lemma 5.2, we deduce that

$$\begin{aligned}\sup_{x\in S}\|(I-P_m)(Ax)\|_\infty &= \sup_{n>m}\left(\sup_{y\in S_{l_p}}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\right|\right)\\ &= \sup_{n>m}\|\tilde{A}_n - \tilde{\alpha}\|^*_{l_p} = \sup_{n>m}\|\tilde{A}_n - \tilde{\alpha}\|_{l_{p'}}\end{aligned}$$

Hence from (5.4), we have

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|^{p'}\right)^{\frac{1}{p'}} \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|^{p'}\right)^{\frac{1}{p'}}.$$

This completes the proof. □

Theorem 5.3. *Let $1 \le p < \infty$. If $A \in (l_1(r,s,t;B), l_p)$, then*

$$\|L_A\|_\chi = \lim_{m\to\infty}\left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p\right)^{\frac{1}{p}}\right).$$

Proof. We omit the proof as it follows easily from Lemma 5.5 and Lemma 5.7. □

Theorem 5.4. *Let $1 < p < \infty$. If $A \in (l_p(r,s,t;B), l_1)$, then*

$$\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right) \le \|L_A\|_\chi \le 4\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right) = 0.$$

Proof. Let $A \in (l_p(r,s,t;B), l_1)$. Then $A_n \in [(l_p(r,s,t;B)]^\beta$ for all n and hence $\tilde{A}_n \in l_{p'}$ by Lemma 5.9. Using Lemma 5.10, we have

$$\left\|\sum_{n\in F} A_n\right\|^*_{l_p(r,s,t;B)} = \left\|\sum_{n\in F}\tilde{A}_n\right\|_{l'_p} = \left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}.$$

Now using Lemma 5.8(c), we obtain

$$\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right) \le \|L_A\|_\chi \le 4\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right).$$

Thus L_A is compact if and only if

$$\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}\tilde{a}_{nk}\right|^{p'}\right)^{\frac{1}{p'}}\right) = 0.$$

This completes the proof. □

Theorem 5.5. *Let $1 < p < \infty$. If $A \in (l_p(r,s,t;B), bv)$, then*

$$\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk} - \tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)$$

$$\le \|L_A\|_\chi \le 4\lim_{m\to\infty}\left(\sup_{F\in\mathfrak{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk} - \tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)$$

and

$$L_A \textit{ is compact if and only if } \lim_{m\to\infty}\left(\sup_{F\in\mathcal{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk}-\tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)=0.$$

Proof. Using matrix domain, the sequence space bv can be written as $bv=(l_1)_\Delta$. Let $A\in(l_p(r,s,t;B),bv)$. Then for each $x\in l_p(r,s,t;B)$, we get $(\Delta A)x=\Delta(Ax)\in l_1$. So by Theorem 5.4, we have

$$\lim_{m\to\infty}\left(\sup_{F\in\mathcal{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk}-\tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)$$
$$\leq\|L_A\|_\chi\leq 4\lim_{m\to\infty}\left(\sup_{F\in\mathcal{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk}-\tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)$$

and L_A is compact if and only if

$$\lim_{m\to\infty}\left(\sup_{F\in\mathcal{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in F}(\tilde{a}_{nk}-\tilde{a}_{n-1,k})\right|^{p'}\right)^{\frac{1}{p'}}\right)=0.$$

This proves this theorem. □

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