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NORMALIZED LAPLACIAN SPECTRUM OF DIFFERENT TYPE OF CORONAS OF TWO REGULAR GRAPHS

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Abstract. In this paper, we determine the full normalized Laplacian spectrum of the corona, edge corona and neighborhood corona of a connected regular graph with an arbitrary regular graph in terms of the normalized Laplacian eigenvalues of the original graphs. Moreover, applying these results we find some non-regular normalized Laplacian co-spectral graphs.

1. Introduction

There are several kinds of spectrums associated with a graph, for example, adjacency spectrum, Laplacian spectrum, signless Laplacian spectrum, normalized Laplacian spectrum etc. Normalized Laplacian spectrum determines the bipartiteness from the largest eigenvalue and the number of connected components from the second smallest eigenvalue [5]. F Chung [5] introduced the *normalized Laplacian matrix* of a graph G, denoted by $\mathcal{L}(G)$, which is a square matrix with rows and columns are indexed by vertices of G, and for any two vertices u and v of G the $(u,v)^{th}$ entry of it is given by

$$\mathcal{L}(u,v) = \begin{cases} 1, & \text{if } u = v \text{ and } d_v \neq 0, \\ \frac{-1}{\sqrt{d_u d_v}}, & \text{if } u \text{ and } v \text{ are adjacent}, \\ 0, & \text{otherwise}. \end{cases}$$

where d_u and d_v are degree of u and v respectively. If $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix (that is $A(u,v) = 1$ if and only if vertex u is adjacent to vertex v and 0 otherwise) of G, then we can write $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ with the convention that $D(G)^{-1}(u,u) = 0$ if

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$d_u = 0$. We denote the characteristic polynomial $\det(\lambda I - \mathcal{L})$ of $\mathcal{L}(G)$ by $f_G(\lambda)$. The roots of $f_G(\lambda)$ are known as the *normalized Laplacian eigenvalues* of G. The multiset of the normalized Laplacian eigenvalues of G is called the *normalized Laplacian spectrum* of G. Since $\mathcal{L}(G)$ is a symmetric and positive semi-definite matrix, its eigenvalues, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are all real, non-negative and can be arranged in non-decreasing order $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. Two graphs G and H are called *cospectral* if $A(G)$ and $A(H)$ have the same spectrum. Analogously graphs G and H are called *normalized Laplacian cospectral* or simply *$\mathcal{L}$-cospectral* if the spectrum of $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are the same. In [5] Chung proved that all normalized Laplacian eigenvalues lie within the interval $[0, 2]$ and 0 is always a normalized Laplacian eigenvalue of any graph G. She also determined normalized Laplacian spectrum of different kind of graphs like complete graphs, bipartite graphs, hypercubes etc. In [1], Banerjee and Jost investigated how the normalized Laplacian spectrum is affected by operations like motif doubling, graph splitting or joining. In [3], Butler and Grout produced (exponentially) large families of non-bipartite, non-regular graphs which are mutually cospectral, and also gave an example of a graph which is cospectral with its complement but is not self-complementary. In [11], Li studied the effect on the second smallest normalized Laplacian eigenvalue by grafting some pendant paths. In this paper we are interested on finding normalized Laplacian spectrum of some coronas of graphs, which are defined below.

Definition 1.1. For $i = 1, 2$, let G_i be the graph with n_i vertices and m_i edges. Then
(i) The corona [8] of G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 for $i = 1, 2, \ldots, n_1$. The corona $G_1 \circ G_2$ has $n_1(n_2 + 1)$ vertices and $m_1 + n_1(m_2 + n_2)$ edges.
(ii) The edge corona [10] of G_1 and G_2, denoted by $G_1 \diamond G_2$, is the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_2 for $i = 1, 2, \ldots, m_1$. The edge corona $G_1 \diamond G_2$ has $n_1 + m_1 n_2$ vertices and $m_1 + 2m_1 n_2 + m_1 m_2$ edges.
(iii) The neighborhood corona [7] of G_1 and G_2, denoted by $G_1 \star G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining every neighbor of the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 for $i = 1, 2, \ldots, n_1$. The neighborhood corona $G_1 \star G_2$ has $n_1(n_2 + 1)$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges.

Many researchers have worked on the corona, edge corona and neighborhood corona of two graphs. Barik et al. [2] provided complete information about the adjacency spectrum of $G_1 \circ G_2$ for a connected graph G_1 and a regular graph G_2 and the Laplacian spectrum of $G_1 \circ G_2$ for arbitrary graphs G_1 and G_2. Hou and Shiu [10] found the adjacency spectrum of $G_1 \diamond G_2$ for a connected regular graph G_1 and a regular graph G_2 and the Laplacian spectrum of the same for a connected regular graph G_1 and a graph G_2. In [14], Wang and Zhou gave complete information about the signless Laplacian spectrum of $G_1 \circ G_2$ for a graph G_1 and a regular graph G_2 and the signless

Laplacian spectrum of $G_1 \diamond G_2$ for a connected regular graph G_1 and a regular graph G_2. In [7], Gopalapillai determined the adjacency spectrum and Laplacian spectrum of $G_1 \star G_2$ for a regular graph G_1 and an arbitrary graph G_2. Here we compute the full normalized Laplacian spectrum of $G_1 \circ G_2$, $G_1 \diamond G_2$ and $G_1 \star G_2$ for two regular graphs G_1 and G_2, with G_1 connected.

To prove our results we need the following matrix products and few results on them. Recall that, the *Kronecker product* of matrices $A = (a_{ij})$ of size $m \times n$ and B of size $p \times q$, denoted by $A \otimes B$, is defined to be the $mp \times nq$ partition matrix $(a_{ij}B)$. It is known [9] that for matrices M, N, P and Q of suitable sizes, $MN \otimes PQ = (M \otimes P)(N \otimes Q)$. This implies that for nonsingular matrices M and N, $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$. It is also known that [9], for square matrices M and N of order k and s respectively, $\det(M \otimes N) = (\det M)^s (\det N)^k$. For two matrices A and B, of same size $m \times n$, the *Hadamard product* $A \bullet B$ of A and B is a matrix of the same size $m \times n$ with entries given by $(A \bullet B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (entrywise multiplication). Hadamard product is commutative, that is $A \bullet B = B \bullet A$.

We also need the lemma below in the proof of our results.

Lemma 1.1 (Schur Complement [6]). *If Q is a non-singular square matrix and the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices, then we have,*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |Q||M - NQ^{-1}P|.$$

2. Our Results

Throughout the paper for any integer k, I_k denotes the identity matrix of size k and J_k denotes the column vector of size k, whose all entries are 1. In the lemma below we represent the normalized Laplacian matrix of corona, edge corona and neighborhood corona of two regular graphs in terms of Kronecker product and Hadamard product of matrices.

Lemma 2.1. *For $i = 1, 2$, let G_i be r_i-regular graph with order n_i and size m_i. Then we have the following:*
(i) $\mathcal{L}(G_1) = I_{n_1} - \frac{1}{r_1}A(G_1)$, $\mathcal{L}(G_2) = I_{n_2} - \frac{1}{r_2}A(G_2)$.
(ii) $\mathcal{L}(G_1 \circ G_2) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -C_{n_2}^T \otimes I_{n_1} \\ -(C_{n_2}^T \otimes I_{n_1})^T & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1} \end{pmatrix}$,
where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(r_1+n_2)(r_2+1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{r_1+n_2}$ and $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+1}$.
(iii) $\mathcal{L}(G_1 \diamond G_2) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -R(G_1) \otimes C_{n_2}^T \\ -(R(G_1) \otimes C_{n_2}^T)^T & I_{m_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)) \end{pmatrix}$,
where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(r_1+r_1n_2)(r_2+2)}}$,

$B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{r_1+r_1 n_2}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+2}$ and $R(G_1)$ is the vertex-edge incidence matrix of G_1 of order $n_1 \times m_1$ with entry $r_{ij} = 1$ if vertex i incident to the edge j and 0 otherwise.

(iv) $\mathcal{L}(G_1 \star G_2) = \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -C_{n_2}^T \otimes A(G_1) \\ -(C_{n_2}^T \otimes A(G_1))^T & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1} \end{pmatrix}$,

where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(r_1+r_1 n_2)(r_2+r_1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{r_1+r_1 n_2}$ and $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+r_1}$.

Proof. Since G_1 and G_2 are r_1 and r_2 regular graphs respectively (i) is immediate.
(ii) For any two graphs G_1 and G_2, the adjacency matrix of $G_1 \circ G_2$ is given by [2]:

$$A(G_1 \circ G_2) = \begin{pmatrix} A(G_1) & J_{n_2}^T \otimes I_{n_1} \\ (J_{n_2}^T \otimes I_{n_1})^T & A(G_2) \otimes I_{n_1} \end{pmatrix}.$$

Since G_1 is a r_1 regular graph with n_1 vertices and G_2 is a r_2 regular graph with n_2 vertices, the normalized Laplacian matrix of $G_1 \circ G_2$ is

$$\begin{aligned}
\mathcal{L}(G_1 \circ G_2) = I_{n_1(n_2+1)} &- \begin{pmatrix} \frac{1}{\sqrt{r_1+n_2}} I_{n_1} & O \\ O & \frac{1}{\sqrt{r_2+1}} I_{n_2} \otimes I_{n_1} \end{pmatrix} \\
&\times \begin{pmatrix} A(G_1) & J_{n_2}^T \otimes I_{n_1} \\ (J_{n_2}^T \otimes I_{n_1})^T & A(G_2) \otimes I_{n_1} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{r_1+n_2}} I_{n_1} & O \\ O & \frac{1}{\sqrt{r_2+1}} I_{n_2} \otimes I_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} I_{n_1} - \frac{1}{r_1+n_2} A(G_1) & -C_{n_2}^T \otimes I_{n_1} \\ -(C_{n_2}^T \otimes I_{n_1})^T & \left[I_{n_2} - \frac{1}{r_2+1} A(G_2) \right] \otimes I_{n_1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -C_{n_2}^T \otimes I_{n_1} \\ -(C_{n_2}^T \otimes I_{n_1})^T & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1} \end{pmatrix}.
\end{aligned}$$

(iii) For any two graphs G_1 and G_2, the adjacency matrix of $G_1 \diamond G_2$ is given by [10]:

$$A(G_1 \diamond G_2) = \begin{pmatrix} A(G_1) & R(G_1) \otimes J_{n_2}^T \\ (R(G_1) \otimes J_{n_2}^T)^T & I_{m_1} \otimes A(G_2) \end{pmatrix}.$$

Since G_1 is a r_1 regular graph with n_1 vertices, m_1 edges and G_2 is a r_2 regular graph with n_2 vertices, m_2 edges, the normalized Laplacian matrix of $G_1 \diamond G_2$ is

$$\begin{aligned}
\mathcal{L}(G_1 \diamond G_2) = I_{n_1+m_1 n_2} &- \begin{pmatrix} \frac{1}{\sqrt{r_1+r_1 n_2}} I_{n_1} & O \\ O & I_{m_1} \otimes \frac{1}{\sqrt{r_2+2}} I_{n_2} \end{pmatrix} \\
&\times \begin{pmatrix} A(G_1) & R(G_1) \otimes J_{n_2}^T \\ (R(G_1) \otimes J_{n_2}^T)^T & I_{m_1} \otimes A(G_2) \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{r_1+r_1 n_2}} I_{n_1} & O \\ O & I_{m_1} \otimes \frac{1}{\sqrt{r_2+2}} I_{n_2} \end{pmatrix} \\
&= \begin{pmatrix} I_{n_1} - \frac{1}{r_1+r_1 n_2} A(G_1) & -R(G_1) \otimes C_{n_2}^T \\ -(R(G_1) \otimes C_{n_2}^T)^T & I_{m_1} \otimes \left[I_{n_2} - \frac{1}{r_2+2} A(G_2) \right] \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -R(G_1) \otimes C_{n_2}^T \\ -(R(G_1) \otimes C_{n_2}^T)^T & I_{m_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)) \end{pmatrix}.$$

(iv) For any two graphs G_1 and G_2, the adjacency matrix of $G_1 \star G_2$ is given by [7]:

$$A(G_1 \star G_2) = \begin{pmatrix} A(G_1) & J_{n_2}^T \otimes A(G_1) \\ (J_{n_2}^T \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{pmatrix}.$$

Since G_1 is a r_1 regular graph with n_1 vertices and G_2 is a r_2 regular graph with n_2 vertices, the normalized Laplacian matrix of $G_1 \star G_2$ will be

$$\begin{aligned} \mathcal{L}(G_1 \star G_2) &= I_{n_1(n_2+1)} - \begin{pmatrix} \frac{1}{\sqrt{r_1+r_1n_2}} I_{n_1} & O \\ O & \frac{1}{\sqrt{r_2+r_1}} I_{n_2} \otimes I_{n_1} \end{pmatrix} \\ &\times \begin{pmatrix} A(G_1) & J_{n_2}^T \otimes A(G_1) \\ (J_{n_2}^T \otimes A(G_1))^T & A(G_2) \otimes I_{n_1} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{r_1+r_1n_2}} I_{n_1} & O \\ O & \frac{1}{\sqrt{r_2+r_1}} I_{n_2} \otimes I_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} I_{n_1} - \frac{1}{r_1+r_1n_2} A(G_1) & -C_{n_2}^T \otimes A(G_1) \\ -(C_{n_2}^T \otimes A(G_1))^T & \left[I_{n_2} - \frac{1}{r_2+r_1} A(G_2)\right] \otimes I_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}(G_1) \bullet B(G_1) & -C_{n_2}^T \otimes A(G_1) \\ -(C_{n_2}^T \otimes A(G_1))^T & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1} \end{pmatrix}. \end{aligned}$$

□

Notation 2.1. Let G be a graph on n vertices, B and C be matrices of size $n \times n$ and $n \times 1$ respectively. For any parameter λ, we have the notation

$$\chi_G(B, C, \lambda) = C^T(\lambda I_n - (\mathcal{L}(G) \bullet B))^{-1} C.$$

We note that the notation is similar to the notion 'coronal' which was introduced by McLeman[13].

Theorem 2.1. *For $i = 1, 2$, let G_i be r_i-regular graph on n_i vertices with G_1 connected. Then the normalized Laplacian spectrum of $G_1 \circ G_2$ consists of*
(i) *the eigenvalue $\frac{1+r_2\delta_j}{r_2+1}$ with multiplicity n_1 for every eigenvalue δ_j, $j = 2, \ldots, n_2$, of $\mathcal{L}(G_2)$;*
(ii) *two simple eigenvalues*

$$\frac{(2n_2+n_2r_2+r_1+r_1\mu_i+r_1r_2\mu_i) \pm \sqrt{(2n_2+n_2r_2+r_1+r_1\mu_i+r_1r_2\mu_i)^2 - 4r_1\mu_i(r_2+1)(r_1+n_2)}}{2(r_2+1)(r_1+n_2)},$$

for each eigenvalue μ_i, $i = 1, 2, \ldots, n_1$, of $\mathcal{L}(G_1)$.

Proof. The normalized Laplacian characteristic polynomial of $G_1 \circ G_2$ is

$$\begin{aligned} &f_{G_1 \circ G_2}(\lambda) \\ &= \det(\lambda I_{n_1(n_2+1)} - \mathcal{L}(G_1 \circ G_2)) \\ &= \det \begin{pmatrix} \lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) & C_{n_2}^T \otimes I_{n_1} \\ (C_{n_2}^T \otimes I_{n_1})^T & \lambda I_{n_1 n_2} - ((\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1}) \end{pmatrix} \\ &= \det \begin{pmatrix} \lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) & C_{n_2}^T \otimes I_{n_1} \\ (C_{n_2}^T \otimes I_{n_1})^T & (\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1} \end{pmatrix} \end{aligned}$$

$$= \det((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1})$$
$$\times \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) - (C_{n_2}^T \otimes I_{n_1})$$
$$\times ((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1})^{-1}(C_{n_2}^T \otimes I_{n_1})^T) \quad \text{[by Lemma 1.1]}$$
$$= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{n_1}$$
$$\times \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) - (C_{n_2}^T(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1}C_{n_2}) \otimes I_{n_1})$$
$$= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{n_1}$$
$$\times \det((\lambda - \chi_{G_2}(B(G_2), C_{n_2}, \lambda))I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1))) \quad \text{[from Notation 2.1]}.$$

Since $\mathcal{L}(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+1}A(G_2)$, one gets $\mathcal{L}(G_2) \bullet B(G_2) = \frac{1}{r_2+1}(I_{n_2} + r_2\mathcal{L}(G_2))$. As G_2 is regular, the sum of all entries on every row of $\mathcal{L}(G_2)$ is zero. That means $\mathcal{L}(G_2)C_{n_2} = \left(1 - \frac{r_2}{r_2}\right) C_{n_2} = 0C_{n_2}$. Then

$$(\mathcal{L}(G_2) \bullet B(G_2))C_{n_2} = \left(1 - \frac{r_2}{r_2+1}\right) C_{n_2} = \frac{1}{r_2+1}C_{n_2}$$

and

$$(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))C_{n_2} = \left(\lambda - \frac{1}{r_2+1}\right) C_{n_2}.$$

Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+n_2)(r_2+1)}$. Therefore,

$$\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = C_{n_2}^T(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1}C_{n_2}$$
$$= \frac{C_{n_2}^T C_{n_2}}{\left(\lambda - \frac{1}{r_2+1}\right)} = \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda - \frac{1}{r_2+1}\right)}.$$

Again, since $\mathcal{L}(G_1) \bullet B(G_1) = I_{n_1} - \frac{1}{r_1+n_2}A(G_1)$, then $\mathcal{L}(G_1) \bullet B(G_1) = \frac{1}{r_1+n_2}(n_2 I_{n_2} + r_1\mathcal{L}(G_1))$.

Now, if δ_j is an eigenvalue of $\mathcal{L}(G_2)$ and μ_i is an eigenvalue of $\mathcal{L}(G_1)$ then

$$f_{G_1 \circ G_2}(\lambda) = \prod_{j=1}^{n_2}\left(\lambda - \frac{1 + r_2\delta_j}{r_2+1}\right)^{n_1}$$
$$\times \prod_{i=1}^{n_1}\left(\lambda - \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda - \frac{1}{r_2+1}\right)} - \frac{n_2 + r_1\mu_i}{r_1+n_2}\right).$$

(i) Since the only pole of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda - \frac{1}{r_2+1}\right)}$ is $\lambda = \frac{1}{r_2+1}$ and 0 is an eigenvalue of $\mathcal{L}(G_2)$, then $\frac{1+r_2\delta_j}{r_2+1}$ is an eigenvalue of $\mathcal{L}(G_1 \circ G_2)$ with multiplicity n_1 for $j = 2, \ldots, n_2$.
(ii) The remaining $2n_1$ eigenvalues are obtained by solving the equation

$$\lambda - \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda - \frac{1}{r_2+1}\right)} - \frac{n_2 + r_1\mu_i}{r_1+n_2} = 0$$

or

$$(r_2+1)(r_1+n_2)\lambda^2-(2n_2+n_2r_2+r_1+r_1\mu_i+r_1r_2\mu_i)\lambda+r_1\mu_i=0.$$

So the eigenvalues are,

$$\lambda_i=\frac{(2n_2+n_2r_2+r_1+r_1\mu_i+r_1r_2\mu_i)\pm\sqrt{(2n_2+n_2r_2+r_1+r_1\mu_i+r_1r_2\mu_i)^2-4r_1\mu_i(r_2+1)(r_1+n_2)}}{2(r_2+1)(r_1+n_2)},$$

for $i=1,2,\ldots,n_1$. □

Theorem 2.2. *For $i=1,2$, let G_i be r_i-regular graph on n_i vertices and m_i edges with G_1 connected. Then the normalized Laplacian spectrum of $G_1\diamond G_2$ consists of*
(i) *the eigenvalue $\frac{2+r_2\delta_j}{r_2+2}$ with multiplicity m_1 for every eigenvalue δ_j, $j=2,\ldots,n_2$, of $\mathcal{L}(G_2)$;*
(ii) *two simple eigenvalues*

$$\frac{(2+4n_2+r_2n_2+r_2\mu_i+2\mu_i)\pm\sqrt{(2+4n_2+r_2n_2+r_2\mu_i+2\mu_i)^2-4\mu_i(n_2+2)(r_2+2)(n_2+1)}}{2(r_2+2)(n_2+1)},$$

for each eigenvalue μ_i, $i=1,2,\ldots,n_1$, of $\mathcal{L}(G_1)$;
(iii) *the eigenvalue $\frac{2}{r_2+2}$ with multiplicity (m_1-n_1) if $n_1<m_1$.*

Proof. The normalized Laplacian characteristic polynomial of $G_1\diamond G_2$ is

$$\begin{aligned}
&f_{G_1\diamond G_2}(\lambda)\\
&=\det(\lambda I_{n_1+m_1n_2}-\mathcal{L}(G_1\diamond G_2))\\
&=\det\begin{pmatrix}\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1)) & R(G_1)\otimes C_{n_2}^T\\ (R(G_1)\otimes C_{n_2}^T)^T & \lambda I_{m_1n_2}-I_{m_1}\otimes((\mathcal{L}(G_2)\bullet B(G_2)))\end{pmatrix}\\
&=\det\begin{pmatrix}\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1)) & R(G_1)\otimes C_{n_2}^T\\ (R(G_1)\otimes C_{n_2}^T)^T & I_{m_1}\otimes(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))\end{pmatrix}\\
&=\det((\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))\otimes I_{m_1})\det(\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1))-(R(G_1)\otimes C_{n_2}^T)\\
&\qquad\times(I_{m_1}\otimes(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2))))^{-1}(R(G_1)\otimes C_{n_2}^T)^T)\qquad\text{[by Lemma 1.1]}\\
&=\det(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))^{m_1}\times\det(\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1))\\
&\qquad-(R(G_1)\otimes C_{n_2}^T)(I_{m_1}\otimes(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2))))^{-1}(R(G_1)\otimes C_{n_2}^T)^T)\\
&=\det(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))^{m_1}\det(\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1))\\
&\qquad-(R(G_1)I_{m_1}R(G_1)^T)\otimes(C_{n_2}^T(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))^{-1}C_{n_2}))
\end{aligned}$$

[It is well known [6] that $R(G_1)R(G_1)^T=A(G_1)+r_1I_{n_1}$ and $A(G_1)=r_1(I_{n_1}-\mathcal{L}(G_1))$, so one gets $R(G_1)R(G_1)^T=r_1(2I_{n_1}-\mathcal{L}(G_1))$]

$$\begin{aligned}
&=\det(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))^{m_1}\det(\lambda I_{n_1}-(\mathcal{L}(G_1)\bullet B(G_1))\\
&\qquad-r_1(2I_{n_1}-\mathcal{L}(G_1))\otimes\chi_{G_2}(B(G_2),C_{n_2},\lambda))\qquad\text{[from Notation 2.1]}\\
&=\det(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))^{m_1}\det((\lambda-2r_1\chi_{G_2}(B(G_2),C_{n_2},\lambda))I_{n_1}\\
&\qquad-(\mathcal{L}(G_1)\bullet B(G_1))+r_1\chi_{G_2}(B(G_2),C_{n_2},\lambda)\mathcal{L}(G_1)).
\end{aligned}$$

Since $\mathcal{L}(G_2)\bullet B(G_2)=I_{n_2}-\frac{1}{r_2+2}A(G_2)$, we get $\mathcal{L}(G_2)\bullet B(G_2)=\frac{1}{r_2+2}(2I_{n_2}+r_2\mathcal{L}(G_2))$. The sum of all entries on every row of $\mathcal{L}(G_2)$ is zero because G_2 is regular. That means $\mathcal{L}(G_2)C_{n_2} = (1 - \frac{r_2}{r_2})C_{n_2} = 0C_{n_2}$. Then

$$(\mathcal{L}(G_2) \bullet B(G_2))C_{n_2} = \left(1 - \frac{r_2}{r_2+2}\right) C_{n_2} = \frac{2}{r_2+2} C_{n_2}$$

and

$$(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))C_{n_2} = \left(\lambda - \frac{2}{r_2+2}\right) C_{n_2}.$$

Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+r_1n_2)(r_2+2)}$. Hence,

$$\begin{aligned}\chi_{G_2}(B(G_2), C_{n_2}, \lambda) &= C_{n_2}^T (\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1} C_{n_2} \\ &= \frac{n_2}{(r_1 + r_1n_2)(r_2 + 2)\left(\lambda - \frac{2}{r_2+2}\right)}.\end{aligned}$$

As $\mathcal{L}(G_1) \bullet B(G_1) = I_{n_1} - \frac{1}{r_1+r_1n_2} A(G_1)$, we have

$$\mathcal{L}(G_1) \bullet B(G_1) = \frac{1}{r_1 + r_1n_2}(r_1n_2 I_{n_2} + r_1\mathcal{L}(G_1)) = \frac{1}{n_2+1}(n_2 I_{n_2} + \mathcal{L}(G_1)).$$

Now, if δ_j is an eigenvalue of $\mathcal{L}(G_2)$ and μ_i is an eigenvalue of $\mathcal{L}(G_1)$ then,

$$\begin{aligned}f_{G_1 \diamond G_2}(\lambda) = \prod_{j=1}^{n_2} \left(\lambda - \frac{2 + r_2\delta_j}{r_2+2}\right)^{m_1} \prod_{j=1}^{n_1} \Bigg(\lambda &- \frac{2n_2}{(n_2+1)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)} \\ &- \frac{n_2 + \mu_i}{n_2+1} + \frac{n_2\mu_i}{(n_2+1)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)}\Bigg).\end{aligned}$$

(i) Since the only pole of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+r_1n_2)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)}$ is $\lambda = \frac{2}{r_2+2}$ and 0 is an eigenvalue of $\mathcal{L}(G_2)$, one gets that $\frac{2+r_2\delta_j}{r_2+2}$ is an eigenvalue of $\mathcal{L}(G_1 \diamond G_2)$ with multiplicity m_1 for $j = 2, \ldots, n_2$.
(ii) The $2n_1$ eigenvalues are obtained by solving the equation

$$\lambda - \frac{2n_2}{(n_2+1)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)} - \frac{n_2+\mu_i}{n_2+1} + \frac{n_2\mu_i}{(n_2+1)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)} = 0.$$

or

$$(n_2+1)(r_2+2)\lambda^2 - (2 + 4n_2 + r_2n_2 + r_2\mu_i + 2\mu_i)\lambda + \mu_i(n_2+2) = 0.$$

So the eigenvalues are,

$$\lambda_i = \frac{(2+4n_2+r_2n_2+r_2\mu_i+2\mu_i) \pm \sqrt{(2+4n_2+r_2n_2+r_2\mu_i+2\mu_i)^2 - 4\mu_i(n_2+2)(r_2+2)(n_2+1)}}{2(r_2+2)(n_2+1)}$$

for $i = 1, 2, \ldots, n_1$.

(iii) Since G_1 is connected regular graph, then $n_1 \leq m_1$. If $n_1 = m_1$ then all eigenvalues are obtained by (i) and (ii). If $n_1 < m_1$ then the remaining $n_1 + m_1 n_2 - m_1(n_2 - 1) - 2n_1 = m_1 - n_1$ normalized Laplacian eigenvalues of G must come from the only pole $\lambda = \frac{2}{r_2+2}$ of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+r_1n_2)(r_2+2)\left(\lambda - \frac{2}{r_2+2}\right)}$. □

Theorem 2.3. *For $i = 1, 2$, let G_i be r_i-regular graph on n_i vertices with G_1 connected. Then the normalized Laplacian spectrum of $G_1 \star G_2$ consists of*
(i) *the eigenvalue $\frac{r_1+r_2\delta_j}{r_2+r_1}$ with multiplicity n_1 for every eigenvalue δ_j, $j = 2, \ldots, n_2$, of $\mathcal{L}(G_2)$;*
(ii) *two simple eigenvalues*

$$\frac{(r_1+2r_1n_2+r_2n_2+r_1\mu_i+r_2\mu_i) \pm \sqrt{(r_1+2r_1n_2+r_2n_2+r_1\mu_i+r_2\mu_i)^2 - 4r_1\mu_i(r_2+r_1)(n_2+1)(1-n_2\mu_i+2n_2)}}{2(r_2+r_1)(n_2+1)}$$

for each eigenvalue μ_i, $i = 1, 2, \ldots, n_1$, of $\mathcal{L}(G_1)$.

Proof. The normalized Laplacian characteristic polynomial of $G_1 \star G_2$ is

$$\begin{aligned}
f_{G_1\star G_2}(\lambda) &= \det(\lambda I_{n_1(n_2+1)} - \mathcal{L}(G_1 \star G_2)) \\
&= \det\begin{pmatrix} \lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) & C_{n_2}^T \otimes A(G_1) \\ (C_{n_2}^T \otimes A(G_1))^T & \lambda I_{n_1 n_2} - ((\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1}) \end{pmatrix} \\
&= \det\begin{pmatrix} \lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) & C_{n_2}^T \otimes A(G_1) \\ (C_{n_2}^T \otimes A(G_1))^T & (\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1} \end{pmatrix} \\
&= \det((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1}) \\
&\quad \times \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) - (C_{n_2}^T \otimes A(G_1)) \\
&\quad \times ((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1})^{-1} (C_{n_2}^T \otimes A(G_1))^T) \quad \text{[by Lemma 1.1]} \\
&= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{n_1} \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) \\
&\quad - (C_{n_2}^T(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1} C_{n_2}) \otimes A(G_1)^2) \\
&= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{n_1} \det(\lambda I_{n_1} \\
&\quad - (\mathcal{L}(G_1) \bullet B(G_1)) - \chi_{G_2}(B(G_2), C_{n_2}, \lambda) A(G_1)^2) \qquad \text{[from Notation 2.1]} \\
&= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{n_1} \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) \\
&\quad - \chi_{G_2}(B(G_2), C_{n_2}, \lambda) r_1^2 (I_{n_1} - \mathcal{L}(G_1))^2) \qquad \text{[as } A(G_1) = r_1(I_{n_1} - \mathcal{L}(G_1))].
\end{aligned}$$

Since $\mathcal{L}(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+r_1} A(G_2)$, we have $\mathcal{L}(G_2) \bullet B(G_2) = \frac{1}{r_2+r_1}(r_1 I_{n_2} + r_2 \mathcal{L}(G_2))$. As G_2 is regular, the sum of all entries on every row of $\mathcal{L}(G_2)$ is zero. That means, $\mathcal{L}(G_2) C_{n_2} = (1 - \frac{r_2}{r_2}) C_{n_2} = 0 C_{n_2}$. Then

$$(\mathcal{L}(G_2) \bullet B(G_2)) C_{n_2} = \left(1 - \frac{r_2}{r_2 + r_1}\right) C_{n_2} = \frac{r_1}{r_2 + r_1} C_{n_2}$$

and

$$(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) C_{n_2} = \left(\lambda - \frac{r_1}{r_2 + r_1}\right) C_{n_2}.$$

Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+r_1n_2)(r_2+r_1)}$. Hence,

$$\begin{aligned}\chi_{G_2}(B(G_2), C_{n_2}, \lambda) &= C_{n_2}^T(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1} C_{n_2} \\ &= \frac{n_2}{(r_1 + r_1n_2)(r_2 + r_1)\left(\lambda - \frac{r_1}{r_2+r_1}\right)}.\end{aligned}$$

Again, since $\mathcal{L}(G_1) \bullet B(G_1) = I_{n_1} - \frac{1}{r_1+r_1n_2} A(G_1)$, we get

$$\begin{aligned}\mathcal{L}(G_1) \bullet B(G_1) &= \frac{1}{r_1 + r_1n_2}(r_1n_2 I_{n_2} + r_1\mathcal{L}(G_1)) \\ &= \frac{1}{n_2+1}(n_2 I_{n_2} + \mathcal{L}(G_1)).\end{aligned}$$

Now, if δ_j is an eigenvalue of $\mathcal{L}(G_2)$ and μ_i is an eigenvalue of $\mathcal{L}(G_1)$ then,

$$\begin{aligned}f_{G_1 \star G_2}(\lambda) = &\prod_{j=1}^{n_2}\left(\lambda - \frac{1+r_2\delta_j}{r_2+1}\right)^{n_1} \\ &\times \prod_{j=1}^{n_2}\left(\lambda - \frac{n_2+\mu_i}{n_2+1} - \frac{n_2r_1(\mu_i^2 - 2\mu_i + 1)}{(n_2+1)(r_2+r_1)\left(\lambda - \frac{r_1}{r_2+r_1}\right)}\right).\end{aligned}$$

(i) Since the only pole of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+r_1n_2)(r_2+r_1)\left(\lambda - \frac{r_1}{r_2+r_1}\right)}$ is $\lambda = \frac{r_1}{r_2+r_1}$ and 0 is an eigenvalue of $\mathcal{L}(G_2)$, then $\frac{r_1+r_2\delta_j}{r_2+r_1}$ is an eigenvalue of $\mathcal{L}(G_1 \star G_2)$ with multiplicity n_1 for $j = 2, \ldots, n_2$.

(ii) The remaining $2n_1$ eigenvalues are obtained by solving the equation

$$\lambda - \frac{n_2+\mu_i}{n_2+1} - \frac{n_2r_1(\mu_i^2 - 2\mu_i + 1)}{(n_2+1)(r_2+r_1)\left(\lambda - \frac{r_1}{r_2+r_1}\right)} = 0.$$

or

$$(n_2+1)(r_2+r_1)\lambda^2 - (r_1 + 2r_1n_2 + r_2n_2 + r_1\mu_i + r_2\mu_i)\lambda + r_1\mu_i(1 - n_2\mu_i + 2n_2) = 0.$$

So the eigenvalues are,

$$\lambda_i = \frac{(r_1+2r_1n_2+r_2n_2+r_1\mu_i+r_2\mu_i) \pm \sqrt{(r_1+2r_1n_2+r_2n_2+r_1\mu_i+r_2\mu_i)^2 - 4r_1\mu_i(r_2+r_1)(n_2+1)(1-n_2\mu_i+2n_2)}}{2(r_2+r_1)(n_2+1)}$$

for $i = 1, 2, \ldots, n_1$. □

Example 2.1. Let us consider $G_1 = C_4$ and $G_2 = K_3$. Then the normalized Laplacian eigenvalues of G_1 are $1 - \cos\frac{2\Pi k}{4}$ for $k = 0, 1, 2, 3$ and the normalized Laplacian eigenvalues of G_2 are 0 and $\frac{3}{2}$ with multiplicity 2.

Now using the result of Theorem 2.1, we get the normalized Laplacian spectrum of $G_1 \circ G_2$, as

$$\left\{\frac{4}{3}\text{ (multiplicity 8)}, 0, \frac{14}{15}, \frac{10 \pm \sqrt{70}}{15}\text{ (multiplicity 2)}, \frac{13 \pm \sqrt{109}}{15}\right\}.$$

From Theorem 2.2, we get the normalized Laplacian spectrum of $G_1 \diamond G_2$, as

$$\left\{\frac{5}{4} \text{ (multiplicity 8)}, 0, \frac{5}{4}, \frac{3 \pm \sqrt{2}}{4} \text{ (multiplicity 2)}, \frac{7 \pm \sqrt{3}}{8}\right\}.$$

Finally by Theorem 2.3, we get the normalized Laplacian spectrum of $G_1 \star G_2$, as

$$\left\{\frac{5}{4} \text{ (multiplicity 8)}, 0, \frac{5}{4}, \frac{1}{2} \text{ (multiplicity 2)}, 1 \text{ (multiplicity 2)}, \frac{7 \pm \sqrt{33}}{8}\right\}.$$

Remark 2.1. If G_1 and G_2 are two regular graphs then we find from Theorems 2.1, 2.2 and 2.3, that the normalized Laplacian spectrum of all the coronas depend only on the degrees of regularities, number of vertices, number of edges, and normalized Laplacian eigenvalues of G_1 and G_2. Thus for $i = 1, 2$, if G_i and H_i are $\mathcal{L}$-cospectral regular graphs then $G_1 \circ G_2$ (resp. $G_1 \diamond G_2$ and $G_1 \star G_2$) is $\mathcal{L}$-cospectral with $H_1 \circ H_2$ (resp. $H_1 \diamond H_2$ and $H_1 \star H_2$).

Now we apply the results of the paper and determine some normalized Laplacian cospectral graphs. Since for an r-regular graph G we have $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$, the Lemma below is immediate.

Lemma 2.2. *Two regular graphs are $\mathcal{L}$-cospectral if and only if they are cospectral.*

In the literature there are several regular cospectral graphs, for example see [15]. In Theorem 2.4 below we construct non-regular $\mathcal{L}$-cospectral graphs using coronas. Proof of this theorem follows from Remark 2.1 and Lemma 2.2.

Theorem 2.4. *If G_1 and H_1(not necessarily distinct) are $\mathcal{L}$-cospectral regular graphs, and G_2 and H_2(not necessarily distinct) and not necessarily different from G_1 and H_1, are $\mathcal{L}$-cospectral regular graphs, then $G_1 \circ G_2$ (resp. $G_1 \diamond G_2$ and $G_1 \star G_2$) and $H_1 \circ H_2$ (resp. $H_1 \diamond H_2$ and $H_1 \star H_2$) are $\mathcal{L}$-cospectral graphs.*

Example 2.2. Applying Theorem 2.4 here we construct $\mathcal{L}$-cospectral graphs. We consider regular cospectral graphs G_1 and H_1 [15] as given in Figure 1.

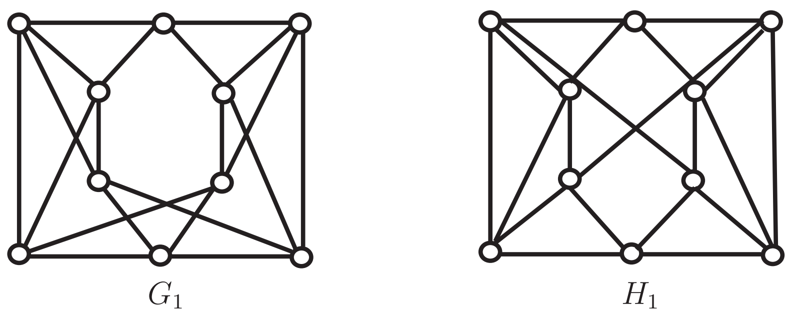

FIGURE 1. Two cospectral regular graphs.

We also consider graphs G_2 and H_2 both of which are copies of K_2. Now by Theorem 2.4 graphs $G_1 \circ K_2$ and $H_1 \circ K_2$ given in Figure 2a and Figure 2b respectively are

$\mathcal{L}$-cospectral. Similarly graphs $G_1 \diamond K_2$ (resp. $G_1 \star K_2$) and $H_1 \diamond K_2$ (resp. $H_1 \star K_2$) are also $\mathcal{L}$-cospectral graphs.

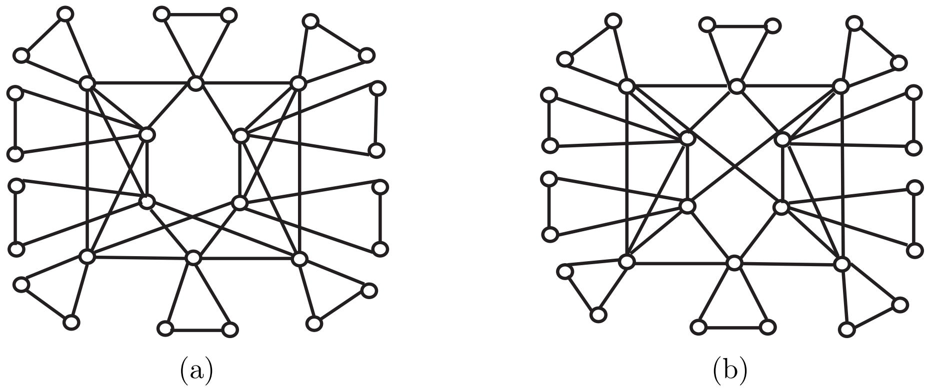

FIGURE 2. Non-regular nonisomorphic $\mathcal{L}$-cospectral graphs.

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