

PRODUCT CORDIAL LABELING OF LINE GRAPH OF SOME GRAPHS

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ABSTRACT. The line graph $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in G . In this paper we have investigated product cordial labeling of line graph of some product cordial graphs.

1. INTRODUCTION

The graph labeling is one of the concepts in graph theory which has attracted many researchers to work on it because of its diversified and rigorous application in fields such as design and analysis of communication networks, military surveillance, social sciences, optimization and linear algebra. Many types of graph labelings are available in the existing literature. A brief account of comprehensive bibliography of papers on the concept of graph labeling is available in a dynamic survey of graph labeling by Gallian [2].

We begin with finite, undirected and connected graph $G = (V(G), E(G))$ without loops and multiple edges. For the various graph theoretic notations and terminology, we follow West [9].

Definition 1.1. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of mapping is the set of vertices (edges) then the labeling is called a *vertex (an edge) labeling*.

In 1987, Cahit [1] has introduced the concept of cordial labeling which is defined as follows.

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Definition 1.2. For a graph $G = (V(G), E(G))$, a vertex labeling function $f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$. Then f is called a *cordial labeling* of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ is the number of vertices of G having label i under f and $e_f(i)$ is the number of edges of G having label i under f^* for $i = 1, 2$. A graph G is called *cordial graph* if it admits a cordial labeling.

Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are A-Cordial labeling, H-Cordial labeling, Prime cordial labeling, Product cordial labeling, Edge product cordial labeling, etc.

Definition 1.3. For a graph $G = (V(G), E(G))$, a vertex labeling function $f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling function $f^* : E(G) \rightarrow \{0, 1\}$ defined as $f^*(e = uv) = f(u)f(v)$. Then f is called a *product cordial labeling* of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *product cordial graph* if it admits a product cordial labeling.

In 2004, Sundaram et al. [3] have introduced the concept of product cordial labeling. In the same paper they have investigated product cordial labeling for some standard graphs.

Vaidya and Barasara [4–8] have also investigated many results on product cordial labeling.

Definition 1.4. The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G , with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G .

In this paper we have investigated product cordial labeling of line graph of some product cordial graphs.

2. MAIN RESULTS

Definition 2.1. The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Theorem 2.1. *The graph $L(M(P_n))$ is product cordial for odd n and not product cordial for even n .*

Proof. Let e_1, e_2, \dots, e_{n-2} be the edges of path P_{n-1} in $M(P_n)$ and $e'_1, e'_2, \dots, e'_{2n-2}$ are the added edges for the construction of $M(P_n)$. Then $V(L(M(P_n))) = \{e_1, e_2, \dots, e_{n-2}, e'_1, e'_2, \dots, e'_{2n-2}\}$. Hence $|V(L(M(P_n)))| = 3n - 4$ and $|E(L(M(P_n)))| = 7n - 14$.

To define $f : V(L(M(P_n))) \rightarrow \{0, 1\}$, we consider following two cases.

Case 1: When n is odd.

$$\begin{aligned}
 f(e_i) &= 1, & 1 \leq i \leq \frac{n-1}{2}, \\
 f(e_i) &= 0, & \frac{n+1}{2} \leq i \leq n-2, \\
 f(e'_i) &= 1, & 3 \leq i \leq n+1, \\
 f(e'_i) &= 0, & n+2 \leq i \leq 2(n-1) \text{ and } i = 1, 2.
 \end{aligned}$$

In view of above labeling pattern we have

$$\begin{aligned}
 v_f(1) &= v_f(0) + 1 = \frac{3n-3}{2}, \\
 e_f(1) + 1 &= e_f(0) = \frac{7n-13}{2}.
 \end{aligned}$$

Thus we have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

Case 2: When n is even.

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\frac{3n-4}{2}$ vertices out of $3n-4$ vertices. The vertices with label 0 will give rise at least $\frac{7n-12}{2}$ edges with label 0 and at most $\frac{7n-16}{2}$ edges with label 1 out of total $7n-14$ edges. Therefore $|e_f(1) - e_f(0)| \geq 2$. Consequently the graph is not product cordial.

Hence, the graph $L(M(P_n))$ is product cordial for odd n and not product cordial for even n . □

Example 2.1. The line graph of $M(P_5)$ and its product cordial labeling is shown in Figure 1.

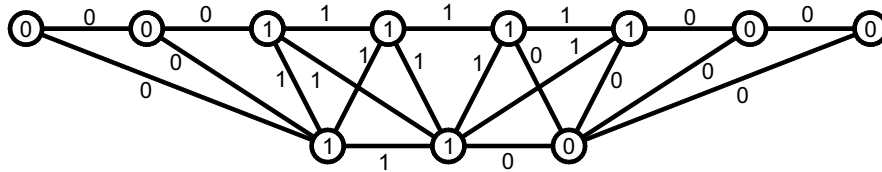


FIGURE 1. $L(M(P_5))$ and its product cordial labeling.

Definition 2.2. The *triangular snake* T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3 .

Theorem 2.2. *The graph $L(T_n)$ is product cordial for even n and not product cordial for odd n .*

Proof. Let e_1, e_2, \dots, e_{n-1} be the edges of T_n corresponding to path P_n and $e'_1, e'_2, \dots, e'_{2n-2}$ are the added edges to path P_n for the construction of T_n . Then $V(L(T_n)) =$

$\{e_1, e_2, \dots, e_{n-1}, e'_1, e'_2, \dots, e'_{2n-2}\}$. Hence $|V(L(T_n))| = 3n - 3$ and $|E(L(T_n))| = 7n - 11$.

To define $f : V(L(T_n)) \rightarrow \{0, 1\}$, we consider following two cases.

Case 1: When n is even.

$$\begin{aligned} f(e_i) &= 1, & 1 \leq i \leq \frac{n}{2}, \\ f(e_i) &= 0, & \frac{n}{2} + 1 \leq i \leq n - 1, \\ f(e'_1) &= 0, \\ f(e'_i) &= 1, & 2 \leq i \leq n, \\ f(e'_i) &= 0, & n + 1 \leq i \leq 2(n - 1). \end{aligned}$$

In view of above labeling pattern we have

$$\begin{aligned} v_f(1) &= v_f(0) + 1 = \frac{3n - 2}{2}, \\ e_f(1) + 1 &= e_f(0) = \frac{7n - 10}{2}. \end{aligned}$$

Thus we have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

Case 2: When n is odd.

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\frac{3n-3}{2}$ vertices out of $3n - 3$ vertices. The vertices with label 0 will give rise at least $\frac{7n-7}{2}$ edges with label 0 and at most $\frac{7n-15}{2}$ edges with label 1 out of total $7n - 11$ edges. Therefore $|e_f(1) - e_f(0)| \geq 4$. Consequently the graph is not product cordial.

Hence, the graph $L(T_n)$ is product cordial for even n and not product cordial for odd n . □

Example 2.2. The graph $L(T_4)$ and its product cordial labeling is shown in Figure 2.

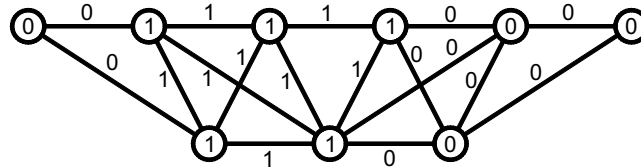


FIGURE 2. $L(T_4)$ and its product cordial labeling.

Definition 2.3. The *armed crown* ACr_n is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge.

Theorem 2.3. *The graph $L(ACr_n)$ is a product cordial graph.*

Proof. For graph ACr_n , let e_1, e_2, \dots, e_n be the edges of C_n and e'_1, e'_2, \dots, e'_n be the edges corresponding to paths and $e''_1, e''_2, \dots, e''_n$ be the edges joining path and cycle. Then $V(L(ACr_n)) = \{e_1, e_2, \dots, e_n, e'_1, e'_2, \dots, e'_n, e''_1, e''_2, \dots, e''_n\}$.

Hence $|V(L(ACr_n))| = 3n$ and $|E(L(ACr_n))| = 4n$.

To define $f : V(L(ACr_n)) \rightarrow \{0, 1\}$ we consider following two cases.

Case 1: When n is odd.

$$\begin{aligned} f(e_i) &= 1, & 1 \leq i \leq n, \\ f(e'_1) &= 1, \\ f(e'_i) &= 0, & 2 \leq i \leq n, \\ f(e''_i) &= 1, & 1 \leq i \leq \frac{n-1}{2}, \\ f(e''_i) &= 0, & \frac{n+1}{2} \leq i \leq n. \end{aligned}$$

In view of above labeling pattern we have

$$\begin{aligned} v_f(1) &= v_f(0) + 1 = \frac{3n+1}{2}, \\ e_f(1) &= e_f(0) = 2n. \end{aligned}$$

Case 2: When n is even.

$$\begin{aligned} f(e_i) &= 1, & 1 \leq i \leq n, \\ f(e'_i) &= 0, & 1 \leq i \leq n, \\ f(e''_i) &= 1, & 1 \leq i \leq \frac{n}{2}, \\ f(e''_i) &= 0, & \frac{n}{2} + 1 \leq i \leq n. \end{aligned}$$

In view of above labeling pattern we have

$$\begin{aligned} v_f(1) &= v_f(0) = \frac{3n}{2}, \\ e_f(1) &= e_f(0) = 2n. \end{aligned}$$

Thus in both the case we have $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. Hence, the graph $L(ACr_n)$ is a product cordial graph. \square

Example 2.3. The line graph of ACr_5 and its product cordial labeling is shown in Figure 3.

Definition 2.4. The *square of graph* G is denoted by G^2 and defined as the graph with distance 1 or 2 apart in G .

Theorem 2.4. *The graph $L(P_n^2)$ is not product cordial for odd $n > 3$.*

Proof. For the graph $L(P_n^2)$ for odd $n > 3$, $|V(L(P_n^2))| = 2n - 3$ and $|E(L(P_n^2))| = 6n - 16$. In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\lfloor \frac{2n-3}{2} \rfloor$ vertices out of $2n - 3$ vertices. The vertices with label 0 will give rise at least $\frac{7n-19}{2}$ edges with label 0 and at most $\frac{5n-13}{2}$ edges with label 1

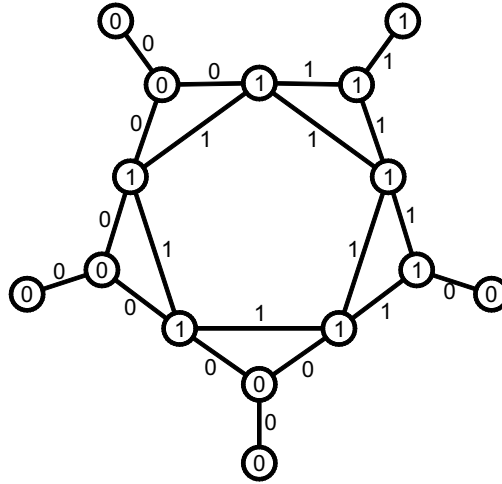


FIGURE 3. $L(ACr_5)$ and its product cordial labeling.

out of total $6n - 16$ edges. Therefore $|e_f(1) - e_f(0)| \geq n - 3 > 1$. Consequently the graph is not product cordial.

Hence, the graph $L(P_n^2)$ is not product cordial for odd $n > 3$. □

Remark 2.1. The graph P_3^2 is same as C_3 and hence $L(P_3^2) = C_3$, which is product cordial as proved in [3].

Definition 2.5. The *splitting graph* $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G .

Theorem 2.5. *The graph $L(S'(P_n))$ is not product cordial for even $n > 2$.*

Proof. For the graph $L(S'(P_n))$ for even $n > 2$, $|V(L(S'(P_n)))| = 3n - 3$ and $|E(L(S'(P_n)))| = 7n - 12$. In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\lfloor \frac{3n-3}{2} \rfloor$ vertices out of $3n - 3$ vertices. The vertices with label 0 will give rise at least $\frac{7n-10}{2}$ edges with label 0 and at most $\frac{7n-14}{2}$ edges with label 1 out of total $7n - 12$ edges. Therefore $|e_f(1) - e_f(0)| \geq 2$. Consequently the graph is not product cordial.

Hence, the graph $L(S'(P_n))$ is not product cordial for even $n > 2$. □

Remark 2.2. The graph $S'(P_2)$ is same as P_4 and hence $L(S'(P_2)) = P_3$, which is product cordial as proved in [3].

Definition 2.6. The *total graph* $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in $T(G)$ whenever they are either adjacent or incident in G .

Theorem 2.6. *The graph $L(T(P_n))$ is not product cordial graph.*

Proof. For the graph $L(T(P_n))$, $|V(L(T(P_n)))| = 4n - 5$ and $|E(L(T(P_n)))| = 12n - 22$. We will consider following two cases.

Case 1: When n is odd. In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\frac{4n-5}{2}$ vertices out of $4n - 5$ vertices. The vertices with label 0 will give rise at least $7n - 13$ edges with label 0 and at most $5n - 9$ edges with label 1 out of total $12n - 22$ edges. Therefore $|e_f(1) - e_f(0)| \geq 2n - 4 > 1$. Consequently the graph is not product cordial.

Case 2: When n is even.

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\lfloor \frac{4n-5}{2} \rfloor$ vertices out of $4n - 5$ vertices. The vertices with label 0 will give rise at least $7n - 12$ edges with label 0 and at most $5n - 10$ edges with label 1 out of total $12n - 22$ edges. Therefore $|e_f(1) - e_f(0)| \geq 2n - 2 > 1$. Consequently the graph is not product cordial.

Hence, the graph $L(T(P_n))$ is not product cordial graph. \square

3. CONCLUDING REMARKS

The graphs T_n , P_n^2 and $T(P_n)$ are proved to be product cordial graphs by Sundaram et al. [3] and $M(P_n)$, AC_n and $S'(P_n)$ are proved to be product cordial graphs by Vaidya and Barasara [4, 7, 8] while here we prove that the line graph of $M(P_n)$, AC_n and T_n are product cordial graphs and the line graph of $S'(P_n)$, P_n^2 and $T(P_n)$ are not product cordial graphs. Thus line graphs are not invariant for product cordial labeling in general.

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