HIGH DIMENSIONAL FRACTIONAL COUPLED SYSTEMS: NEW EXISTENCE AND UNIQUENESS RESULTS

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ABSTRACT. In this paper, we study a class of high dimensional coupled fractional differential systems using Caputo approach. We investigate the existence of solutions using Schaefer fixed point theorem. Moreover, new existence and uniqueness results are obtained by using the contraction mapping principle. Finally, Some examples are presented to illustrate our main results.

1. Introduction and Preliminaries

Fractional differential equations have gained a great interest because of their many applications in modeling of physical and chemical processes and in engineering sciences. For the basic theory of fractional differential equations, see [1–10, 12–14, 17, 22]. Moreover, the nonlinear coupled systems involving fractional derivatives are also very important, since they occur in various problems of applied mathematics. In the literature, we can find many papers dealing with the existence and uniqueness of solutions. For more details, we refer the reader to [11, 16, 18–21] and the references therein.

Motivated by cited coupled systems-papers, in this work, we discuss the existence and uniqueness of solutions for the following problem:

$$(1.1) \quad \begin{cases} D^{\alpha_1} u_1(t) = \displaystyle\sum_{i=1}^{l} f_i^1(t, u_1(t), \ldots, u_m(t), D^{\gamma_1} u_1(t), \ldots, D^{\gamma_m} u_m(t)), \quad t \in J, \\[2mm] D^{\alpha_2} u_2(t) = \displaystyle\sum_{i=1}^{l} f_i^2(t, u_1(t), \ldots, u_m(t), D^{\gamma_1} u_1(t), \ldots, D^{\gamma_m} u_m(t)), \quad t \in J, \\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad\vdots \\[2mm] D^{\alpha_m} u_m(t) = \displaystyle\sum_{i=1}^{l} f_i^m(t, u_1(t), \ldots, u_m(t), D^{\gamma_1} u_1(t), \ldots, D^{\gamma_m} u_m(t)), \quad t \in J, \\[2mm] u_k(0) = a_0^k, \\[2mm] u_k^{(j)}(0) = 0, \quad j = 1, 2, \ldots, n-2, \\[2mm] u_k^{(n-1)}(0) = J^{r_k} u_k(\tau_k), \quad r_k > 0, \end{cases}$$

where $k = 1, 2, \ldots, m$. We suppose that $n - 1 < \alpha_k < n$, $\gamma_k \in \,]0, n-1[$, $k = 1, 2, \ldots, m$, $n \in \mathbb{N}^* - \{1\}$, $m, l \in \mathbb{N}^*$, $\tau \in \,]0, 1[$, $J := [0, 1]$. The derivatives D^{α_k}, D^{γ_k}, $k = 1, 2, \ldots, m$, are taken in the sense of Caputo. For the functions $\left(f_i^k\right)_{i=1,\ldots,l}^{k=1,2,\ldots,m}$: $J \times \mathbb{R}^{2m} \to \mathbb{R}$, we will specify them later.

The paper is organised as follows: We begin by introducing some definitions and lemmas that will be used in the proof of the main results. Then, in the Main Results Section, we prove the existence of solutions theorems. At the last section, some illustrative examples are treated. So, let us now present the basic definitions and lemmas [15].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[0, \infty[$ is defined as:

$$(1.2) \qquad J^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s)\, ds, & \alpha > 0, t \geq 0, \\[3mm] f(t), & \alpha = 0, t \geq 0, \end{cases}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx$.

Definition 1.2. The Caputo derivative of order α for a function $u : [0, \infty) \to \mathbb{R}$, which is at least n-times differentiable can be defined as:

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds = J^{n-\alpha} u^{(n)}(t),$$

for $n - 1 < \alpha < n$, $n \in \mathbb{N}^* - \{1\}$.

We recall the following lemmas [7, 19].

Lemma 1.1. *For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha u(t) = 0$, is given by*

$$u(t) = \sum_{j=0}^{n-1} c_j t^j,$$

where $c_j \in \mathbb{R}$, $j = 0, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.2. *Let $\alpha > 0$. Then*

$$J^\alpha D^\alpha u(t) = u(t) + \sum_{j=0}^{n-1} c_j t^j,$$

where $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 1.3. *Let $q > p > 0$, $g \in L^1([a,b])$. Then $D^p J^q f(t) = J^{q-p} f(t)$, $t \in [a,b]$.*

Lemma 1.4 (Schaefer fixed point Theorem). *Let E be Banach space and $T : E \to E$ is a completely continuous operator. If $V = \{u \in E : u = \mu Tu, 0 < \mu < 1\}$ is bounded, then T has a fixed point in E.*

The proof of the following auxiliary lemma is crucial for the problem (1.1).

Lemma 1.5. *Assume that $\left(Q_i^k\right)_{i=1,\ldots,l}^{k=1,\ldots,m} \in C([0,1], \mathbb{R})$, $m, l \in \mathbb{N}^*$. And consider the problem*

$$(1.3) \qquad D^{\alpha_k} u_k(t) = \sum_{i=1}^{l} Q_i^k(t), \quad t \in J, n-1 < \alpha_k < n, n \in \mathbb{N}^* - \{1\},$$

associated with the conditions:

$$(1.4) \qquad \begin{cases} u_k(0) & = a_0^k, \\ u_k^{(j)}(0) & = 0, \quad j = 1, 2, \ldots, n-2, \\ u_k^{(n-1)}(0) & = J^{r_k} u_k(\tau_k), \end{cases}$$

where $k = 1, 2, \ldots, m$. Then for all $k = 1, 2, \ldots, m$, we have

$$u_k(t) = -\sum_{i=1}^{l} \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} Q_i^k(s)\, ds + a_0^k + \frac{\Gamma(r_k + n) t^{n-1}}{(n-1)! \left(\tau_k^{r_k + n - 1} - \Gamma(r_k + n)\right)}$$

$$(1.5) \qquad \times \left(\sum_{i=1}^{l} \int_0^{\tau_k} \frac{(\tau_k - s)^{\alpha_k + r_k - 1}}{\Gamma(\alpha_k + r_k)} Q_i^k(s)\, ds - \frac{a_0^k \tau_k^{r_k}}{\Gamma(r_k + 1)} \right).$$

Proof. Thanks to (1.2) and by (1.3), we have

$$(1.6) \qquad u_k(t) = \sum_{i=1}^{l} \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} Q_i^k(s)\, ds - c_0^k - c_1^k t - c_2^k t^2 - \cdots - c_{n-1}^k t^{n-1},$$

where $c_j^k \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n - 1$, $k = 1, 2, \ldots, m$, $m \in \mathbb{N}^*$, and $n - 1 < \alpha_k < n$, $n \in \mathbb{N}^* - \{1\}$.

For all $k = 1, 2, \ldots, m$, we observe that

$$\begin{cases} u_k(0) & = -c_0^k, \\ u_k^{(j)}(0) & = -j! c_j^k, \quad j = 1, 2, \ldots, n - 2, \\ u_k^{(n-1)}(0) & = -(n-1)! c_{n-1}^k. \end{cases}$$

The conditions (1.4) will allow us to obtain

$$(1.7) \quad \begin{cases} c_0^k & = -a_0^k, \\ c_j^k & = 0, \quad j = 1, 2, \ldots, n - 2, \\ c_{n-1}^k & = \dfrac{\Gamma(r_k + n)}{\Gamma(n)\left(\tau_k^{r_k + n - 1} - \Gamma(r_k + n)\right)} \left(a_0^k \dfrac{\tau_k^{r_k}}{\Gamma(r_k + 1)} - \displaystyle\sum_{i=1}^{l} J^{\alpha_k + r_k} Q_i^k(\tau_k)\right), \end{cases}$$

where $k = 1, 2, \ldots, m$. The values of c_j^k given by (1.7) replaced in (1.6) imply (1.5). This completes the proof of Lemma 1.5. $\square$

Now, we consider the space:

$$S := \{(u_1, u_2, \ldots, u_n) : u_k \in C([0,1], \mathbb{R}), D^{\gamma_k} u_k \in C([0,1], \mathbb{R}), k = 1, 2, \ldots, m\},$$

endowed with the norm

$$\|(u_1, u_2, \ldots, u_m)\|_S = \max_{1 \leq k \leq m} \left(\|u_k\|_\infty, \|D^{\gamma_k} u_k\|_\infty\right).$$

This Banach space will be used in the proof of the theorems.

2. Main Results

We impose the following hypotheses:

(H_1): There exist nonnegative constants $\left(\mu_{i,\ i=1,\ldots,l}^{k,\ k,=1,2,\ldots,m}\right)_{j,\ j=1,2,\ldots,2m}$, such that for all $t \in [0,1]$ and for all $(u_1, u_2, \ldots, u_{2m}), (v_1, v_2, \ldots, v_{2m}) \in \mathbb{R}^{2m}$, we have

$$\left|f_i^k(t, u_1, u_2, \ldots, u_{2m}) - f_i^k(t, v_1, v_2, \ldots, v_{2m})\right| \leq \sum_{j=1}^{2m} \left(\mu_i^k\right)_j |u_j - v_j|.$$

(H_2): The functions $\left(f_i^k\right)_{i=1,2,\ldots,l}^{k=1,2,\ldots,m} : [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}; m, l \in \mathbb{N}^*$ are continuous.

(H_3): There exist nonnegative functions $\left(\omega_i^k\right)_{i=1,\ldots,l}^{k=1,2\ldots,m} \in C([0,1])$, such that: for all $t \in [0,1]$ and for all $(u_1, u_2, \ldots, u_{2m}) \in \mathbb{R}^{2m}$

$$\left|f_i^k(t, u_1, u_2, \ldots, u_{2m})\right| \leq \omega_i^k(t),$$

with

$$\sup_{t \in J} \omega_i^k(t) = C_i^k.$$

Setting the following quantities:

$$\Sigma_k = \sum_{j=1}^{2m} \sum_{i=1}^{l} \left(\mu_i^k\right)_j,$$

$$\digamma_k = \frac{1}{\Gamma\left(\alpha_k+1\right)} + \frac{\Gamma\left(r_k+n\right)\tau_k^{\alpha_k+r_k}}{(n-1)!\left|\tau_k^{r_k+n-1}-\Gamma\left(r_k+n\right)\right|\Gamma\left(\alpha_k+r_k+1\right)},$$

$$\digamma_k^* := \frac{1}{\Gamma\left(\alpha_k-\gamma_k+1\right)} + \frac{\Gamma\left(r_k+n\right)\tau_k^{\alpha_k+r_k}}{\Gamma\left(n-\gamma_k\right)\left|\tau_k^{r_k+n-1}-\Gamma\left(r_k+n\right)\right|\Gamma\left(\alpha_k+r_k+1\right)},$$

and

$$W_k = \left|a_0^k\right| + \frac{\Gamma\left(r_k+n\right)\left|a_0^k\right|\tau_k^{r_k}}{(n-1)!\left|\tau_k^{r_k+n-1}-\Gamma\left(r_k+n\right)\right|\Gamma\left(r_k+1\right)},$$

$$W_k^* = \frac{\Gamma\left(r_k+n\right)\left|a_0^k\right|\tau_k^{r_k}}{\Gamma\left(n-\gamma_k\right)\left|\tau_k^{r_k+n-1}-\Gamma\left(r_k+n\right)\right|\Gamma\left(r_k+1\right)}.$$

We now prove the first main result.

Theorem 2.1. *Assume that* (H_1) *and the following are satisfied*

(2.1) $$\max_{1\leq k\leq m}\Sigma_k\left(\digamma_k,\digamma_k^*\right) < 1.$$

Then, the problem (1.1) *has a unique solution on* J.

Proof. Let us take the operator $P: S \to S$ given by

$$P\left(u_1,u_2,\ldots,u_m\right)(t) := \left(P_1\left(u_1,u_2,\ldots,u_m\right)(t),\ldots,P_m\left(u_1,u_2,\ldots,u_m\right)(t)\right), \quad t \in J,$$

such that

$$P_k\left(u_1,u_2,\ldots,u_m\right)(t)$$

$$:= -\sum_{i=1}^{l}\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)} f_i^k\left(t,u_1\left(t\right),\ldots,u_m\left(s\right),D^{\gamma_1}u_1\left(s\right),\ldots,D^{\gamma_m}u_m\left(s\right)\right)ds + a_0^k$$

$$+ \frac{\Gamma\left(r_k+n\right)t^{n-1}}{(n-1)!\left(\tau_k^{r_k+n-1}-\Gamma\left(r_k+n\right)\right)}$$

$$\times \left(\sum_{i=1}^{l}\int_0^{\tau_k} \frac{(\tau_k-s)^{\alpha_k+r_k-1}}{\Gamma\left(\alpha_k+r_k\right)} f_i^k\left(t,u_1\left(t\right),\ldots,u_m\left(s\right),D^{\gamma_1}u_1\left(s\right),\ldots,D^{\gamma_m}u_m\left(s\right)\right)ds\right.$$

$$\left. - \frac{a_0^k\tau_k^{r_k}}{\Gamma\left(r_k+1\right)}\right).$$

We shall prove that P is contractive: Let $(u_1,u_2,\ldots,u_m)$, $(v_1,v_2,\ldots,v_m) \in S$. Then, for all $k=1,2,\ldots,m$ and $t \in J$, we get:

$$\left|P_k\left(u_1,u_2,\ldots,u_m\right)(t) - P_k\left(v_1,v_2,\ldots,v_m\right)(t)\right|$$

$$\leq \frac{1}{\Gamma(\alpha_k + 1)} \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right.$$

$$\left. - f_i^k \left(s, v_1(s), \ldots, v_m(s), D^{\gamma_1} v_1(s), \ldots, D^{\gamma_m} v_m(s) \right) \right|$$

$$+ \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{(n-1)! \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)}$$

$$\times \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right.$$

$$\left. - f_i^k \left(s, v_1(s), \ldots, v_m(s), D^{\gamma_1} v_1(s), \ldots, D^{\gamma_m} v_m(s) \right) \right|.$$

Using (H_1), yields

$$\| P_k(u_1, u_2, \ldots, u_m) - P_k(v_1, v_2, \ldots, v_m) \|_\infty$$

$$\leq \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{(n-1)! \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)} \right)$$

$$\times \sum_{i=1}^{l} \left((\mu_i^k)_1 + (\mu_i^k)_2 + \cdots + (\mu_i^k)_{2m} \right) \max_{1 \leq k \leq m} \left(\| u_k - v_k \|_\infty, \| D^{\gamma_k}(u_k - v_k) \|_\infty \right).$$

Thus,

$$\| P_k(u_1, u_2, \ldots, u_m) - P_k(v_1, v_2, \ldots, v_m) \|_\infty$$

$$(2.2) \qquad \leq \Sigma_k \mathcal{F}_k \left\| (u_1 - v_1, \ldots, u_m - v_m, D^{\gamma_1}(u_1 - v_1), \ldots, D^{\gamma_m}(u_m - v_m)) \right\|_S.$$

On the other hand, we have

$$\left| D^{\gamma_k} P_k(u_1, u_2, \ldots, u_m)(t) - D^{\gamma_k} P_k(v_1, v_2, \ldots, v_m)(t) \right|$$

$$\leq \frac{t^{\alpha_k - \gamma_k}}{\Gamma(\alpha_k - \gamma_k + 1)} \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right.$$

$$\left. - f_i^k \left(s, v_1(s), \ldots, v_m(s), D^{\gamma_1} v_1(s), \ldots, D^{\gamma_m} v_m(s) \right) \right|$$

$$+ \frac{\Gamma(r_k + n) t^{n - \gamma_k - 1} \tau_k^{\alpha_k + r_k}}{\Gamma(n - \gamma_k) \left| \tau_k^{r_k + n - 1} - \Gamma(r + n) \right| \Gamma(\alpha_k + r_k + 1)}$$

$$\times \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right.$$

$$\left. - f_i^k \left(s, v_1(s), \ldots, v_m(s), D^{\gamma_1} v_1(s), \ldots, D^{\gamma_m} v_m(s) \right) \right|.$$

Then,

$$\| D^{\gamma_k} P_k(u_1, u_2, \ldots, u_m) - D^{\gamma_k} P_k(v_1, v_2, \ldots, v_m) \|_\infty$$

$$\leq \left(\frac{1}{\Gamma(\alpha_k - \gamma_k + 1)} + \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{\Gamma(n - \gamma_k) \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)} \right)$$

$$\times \sum_{i=1}^{2m} \sum_{i=1}^{l} \mu_i^k \left\| (u_1 - v_1, \ldots, u_m - v_m, D^{\gamma_1}(u_1 - v_1), \ldots, D^{\gamma_m}(u_m - v_m)) \right\|_S .$$

So,

$$\left\| D^{\gamma_k} P_k(u_1, u_2, \ldots, u_m) - D^{\gamma_k} P_k(v_1, v_2, \ldots, v_m) \right\|_\infty$$

(2.3) $\quad \leq \Sigma_k F_k^* \left\| (u_1 - v_1, \ldots, u_m - y_m, D^{\gamma_1}(u_1 - v_1), \ldots, D^{\gamma_m}(u_m - v_m)) \right\|_S .$

Thanks to (2.3) and (2.2), we obtain

$$\left\| P(u_1, u_2, \ldots, u_m) - P(v_1, v_2, \ldots, v_m) \right\|_S$$

(2.4) $\quad \leq \max_{1 \leq k \leq n} \Sigma_k(F_k, F_k^*) \left\| (u_1 - v_1, \ldots, u_m - v_m, D^{\gamma_1}(u_1 - v_1), \ldots, D^{\gamma_m}(u_m - v_m)) \right\|_S .$

By (2.1) the operator P is contractive. Hence, P has a unique fixed point which is a solution of the system (1.1). This ends the proof. $\qquad \square$

The following result presents new conditions to the existence of one solution. We prove the result by using Schaefer theorem:

Theorem 2.2. *Let* $\left(f_i^k \right)_{i=1,2,\ldots,l}^{k=1,2,\ldots,m}$ *satisfied* (H_2) *and* (H_3). *Then, the problem* (1.1) *has at least one solution on* J.

Proof. First, we prove that P is completely continuous. We consider the set

$$\Omega_\sigma := \left\{ (u_1, u_2, \ldots, u_m) \in S; \left\| (u_1, u_2, \ldots, u_m) \right\|_S \leq \sigma, \ \sigma > 0 \right\},$$

and we show that P maps bounded sets into bounded sets in S.

Let $(u_1, u_2, \ldots, u_m) \in \Omega_\sigma$. The hypothesis (H_3) implies that

$$\left\| P_k(u_1, u_2, \ldots, u_m) \right\|_\infty$$

$$\leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right|$$

$$+ \left| a_0^k \right| + \frac{\Gamma(r_k + n) t^{n-1} \left| a_0^k \right| \tau_k^{r_k}}{(n-1)! \left| \tau_k^{r_k+n-1} - \Gamma(r_k + n) \right| \Gamma(r_k + 1)}$$

$$+ \frac{\Gamma(r_k + n) t^{n-1} \tau_k^{\alpha_k + r_k}}{(n-1)! \left| \tau_k^{r_k+n-1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)}$$

$$\times \sup_{s \in J} \sum_{i=1}^{l} \left| f_i^k \left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1} u_1(s), \ldots, D^{\gamma_m} u_m(s) \right) \right|$$

(2.5)

$$\leq \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{(n-1)! \left| \tau_k^{r_k+n-1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)} \right) \sup_{s \in J} \sum_{i=1}^{l} \omega_i^k(s)$$

$$+ \left| a_0^k \right| + \frac{\Gamma(r_k + n) \left| a_0^k \right| \tau_k^{r_k}}{(n-1)! \left| \tau_k^{r_k+n-1} - \Gamma(r_k + n) \right| \Gamma(r_k + 1)} .$$

Therefore,

$$\|P_k(u_1, u_2, \ldots, u_m)\|_\infty$$

$$\leq \left(\frac{1}{\Gamma(\alpha_k + 1)} + \frac{\Gamma(r_k + n)\tau_k^{\alpha_k + r_k}}{(n-1)!\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(\alpha_k + r_k + 1)}\right)$$

$$\times \sum_{i=1}^{l} C_i^k + \left|a_0^k\right| + \frac{\Gamma(r_k + n)\left|a_0^k\right|\tau_k^{r_k}}{(n-1)!\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(r_k + 1)}$$

$$(2.6) \qquad \leq F_k \sum_{i=1}^{l} C_i^k + W_k.$$

Furthermore, we get

$$\|D^{\gamma_k} P_k(u_1, u_2, \ldots, u_m)\|_\infty$$

$$\leq \frac{t^{\alpha_k - \gamma_k}}{\Gamma(\alpha_k - \gamma_k + 1)}\sup_{s \in J}\sum_{i=1}^{l}\left|f_i^k\left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1}u_1(s), \ldots, D^{\gamma_m}u_m(s)\right)\right|$$

$$+ \frac{\Gamma(r_k + n) t^{n-\gamma_k-1}\left|a_0^k\right|\tau_k^{r_k}}{\Gamma(n - \gamma_k)\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(r_k + 1)}$$

$$+ \frac{\Gamma(r_k + n) t^{n-\gamma_k-1}\tau_k^{\alpha_k + r_k}}{\Gamma(n - \gamma_k)\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(\alpha_k + r_k + 1)}$$

$$\times \sup_{s \in J}\sum_{i=1}^{l}\left|f_i^k\left(s, u_1(s), \ldots, u_m(s), D^{\gamma_1}u_1(s), \ldots, D^{\gamma_m}u_m(s)\right)\right|$$

$$\leq \left(\frac{1}{\Gamma(\alpha_k - \gamma_k + 1)} + \frac{\Gamma(r_k + n)\tau_k^{\alpha_k + r_k}}{\Gamma(n - \gamma_k)\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(\alpha_k + r_k + 1)}\right)$$

$$\times \sum_{i=1}^{l}\omega_i^k(t) + \frac{\Gamma(r_k + n)\left|a_0^k\right|\tau_k^{r_k}}{\Gamma(n - \gamma_k)\left|\tau_k^{r_k+n-1} - \Gamma(r_k+n)\right|\Gamma(r_k + 1)}.$$

Hence,

$$(2.7) \qquad \|D^{\gamma_k} P_k(u_1, u_2, \ldots, u_m)\|_\infty \leq F_k^* \sum_{i=1}^{l} C_i^k + W_k^*.$$

From (2.6) and (2.7), we get

$$(2.8) \quad \|P(u_1, u_2, \ldots, u_m)\|_S \leq \max_{1 \leq k \leq m}\left(F_k\sum_{i=1}^{l}C_i^k + W_k, \ F_k^*\sum_{i=1}^{l}C_i^k + W_k^*\right) < \infty.$$

So, P maps bounded sets into bounded sets in S.

The hypothesis (H_2) implies that P is continuous on S. Now, let $t_1, t_2 \in [0, 1]$; $t_1 < t_2$, and $(u_1, u_2, \ldots, u_m) \in S$, we have:

$$\left\| P_k (u_1, u_2, \ldots, u_m)(t_2) - P_k (u_1, u_2, \ldots, u_m)(t_1) \right\|_\infty$$

(2.9)
$$\leq \frac{1}{\Gamma(\alpha_k + 1)} \left(2(t_2 - t_1)^{\alpha_k} + \left(t_2^{\alpha_k} - t_1^{\alpha_k} \right) \right) \sum_{i=1}^{l} C_i^k$$

$$+ \frac{\Gamma(r_k + n) \left| a_0^k \right| \tau_k^{r_k}}{(n-1)! \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(r_k + 1)} \left(t_2^{n-1} - t_1^{n-1} \right)$$

$$+ \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{(n-1)! \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)} \left(t_2^{n-1} - t_1^{n-1} \right) \sum_{i=1}^{m} C_i^k,$$

and

$$\left\| D^{\gamma_k} P_k (u_1, u_2, \ldots, u_m)(t_2) - D^{\gamma_k} P_k (u_1, u_2, \ldots, u_m)(t_1) \right\|_\infty$$

$$\leq \frac{2(t_2 - t_1)^{\alpha_k - \gamma_k} + \left(t_2^{\alpha_k - \gamma_k} - t_1^{\alpha_k - \gamma_k} \right)}{\Gamma(\alpha_k - \gamma_k + 1)} \sum_{i=1}^{l} C_i^k$$

$$+ \frac{\Gamma(r_k + n) \left| a_0^k \right| \tau_k^{r_k}}{\Gamma(n - \gamma_k) \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(r_k + 1)} \left(t_2^{n - \gamma_k - 1} - t_1^{n - \gamma_k - 1} \right)$$

(2.10)

$$+ \frac{\Gamma(r_k + n) \tau_k^{\alpha_k + r_k}}{\Gamma(n - \gamma_k) \left| \tau_k^{r_k + n - 1} - \Gamma(r_k + n) \right| \Gamma(\alpha_k + r_k + 1)} \left(t_2^{n - \gamma_k - 1} - t_1^{n - \gamma_k - 1} \right) \sum_{i=1}^{l} C_i^k.$$

The right-hand sides of the above inequalities (2.9) and (2.10) are independent of $(u_1, u_2, \ldots, u_m)$ and tend to zero as $t_2 - t_1 \to 0$. Then the operator P, is equicontinuous. Hence, the operator P is a completely continuous.

Finally we show that

$$\lambda := \left\{ (u_1, u_2, \ldots, u_m) \in S, \ (u_1, u_2, \ldots, u_m) = \eta P (u_1, u_2, \ldots, u_m) \ 0 < \eta < 1 \right\}$$

is bounded.

For $(u_1, u_2, \ldots, u_m) \in \lambda$, we have $(u_1, u_2, \ldots, u_m)(t) = \eta P (u_1, u_2, \ldots, u_m)(t)$, for all $t \in [0, 1]$. Using (2.8), we get

$$\left\| (u_1, u_2, \ldots, u_m) \right\|_S \leq \eta \max_{1 \leq k \leq m} \left(F_k \sum_{i=1}^{l} C_i^k + W_k, \ F_k^* \sum_{i=1}^{l} C_i^k + W_k^* \right) < \infty.$$

Thus, λ is a bounded. By Lemma 1.4, the operator P has a fixed point which is a solution of (1.1). Theorem 2.2 is thus proved. $\square$

3. Examples

In this section, we give some examples to illustrate our theorems.

Example 3.1. Consider the following system:

(3.1)

$$
\begin{cases}
D^{\frac{11}{3}} u_1(t) = \dfrac{1}{12\pi^3(t+1)}\left(\cos(u_1(t)) + \cos(u_2(t)) + \dfrac{\left|D^{\frac{8}{3}}u_1(t) + D^{\frac{5}{2}}u_2(t)\right|}{\left(1 + \left|D^{\frac{4}{3}}u_1(t) + D^{\frac{3}{2}}u_2(t)\right|\right)}\right) \\
\qquad + \dfrac{1}{64\pi(t+1)}\left(\dfrac{|u_1(t)+u_2(t)|}{2\pi(1+|u_1(t)+u_2(t)|)} + \sin D^{\frac{8}{3}}u_1(t) + \sin D^{\frac{5}{2}}u_2(t)\right), \\
\hfill t \in [0,1], \\
D^{\frac{19}{5}} u_2(t) = \dfrac{1}{14\pi^3 e^t}\left(\dfrac{|u_1(t)+u_2(t)|}{1+|u_1(t)+u_2(t)|} + \sin\left(D^{\frac{8}{3}}u_1(t)\right) + \sin\left(D^{\frac{5}{2}}u_2(t)\right)\right) \\
\qquad + \dfrac{t^2}{6\pi^3}\left(\dfrac{\left|\sin(u_1(t)) + \cos(u_2(t)) + \cos\left(D^{\frac{8}{3}}u_1(t)\right) + \sin\left(D^{\frac{5}{2}}u_2(t)\right)\right|}{1 + \left|\sin(u_1(t)) + \cos(u_2(t)) - \cos\left(D^{\frac{8}{3}}u_1(t)\right) + \sin\left(D^{\frac{5}{2}}u_2(t)\right)\right|}\right), \\
\hfill t \in [0,1], \\
u_1(0) = \sqrt{3}, \quad u_1^{(1)}(0) = u_1^{(2)}(0) = 0, \quad u_1^{(3)}(0) = J^{\frac{1}{2}}\left(\dfrac{3}{7}\right), \\
u_2(0) = \sqrt{5}, \quad u_2^{(1)}(0) = u_2^{(2)}(0) = 0, \quad u_2^{(3)}(0) = J^{\frac{1}{3}}\left(\dfrac{6}{7}\right).
\end{cases}
$$

For this example, we have: $n = 4$, $l = 2$, $\alpha_1 = 11/3$, $\alpha_2 = 19/5$, $\gamma_1 = 8/3$, $\gamma_2 = 5/2$, $r_1 = 1/2$, $r_2 = 1/3$, $\tau_1 = 3/7$, $\tau_2 = 6/7$, $J = [0,1]$, and

$$
f_1^1(t, u_1, u_2, u_3, u_4) = \frac{1}{12\pi^3(t+1)}\left(\cos u_1 + \cos u_2 + \frac{|u_3 + u_4|}{(1 + |u_3 + u_4|)}\right),
$$

$$
f_2^1(t, u_1, u_2, u_3, u_4) = \frac{1}{64\pi(t+1)}\left(\frac{|u_1 + u_2|}{2\pi(1 + |u_1 + u_2|)} + \sin u_3 + \sin u_4\right),
$$

$$
f_1^2(t, u_1, u_2, u_3, u_4) = \frac{1}{14\pi^3 e^t}\left(\frac{|u_1 + u_2|}{1 + |u_1 + u_2|} + \sin u_3 + \sin u_4\right),
$$

$$
f_2^2(t, u_1, u_2, u_3, u_4) = \frac{t^2}{6\pi^3}\left(\frac{|\sin u_1 + \cos u_2 + \cos u_3 + \sin u_4|}{1 + |\sin u_1 + \cos u_2 - \cos u_3 + \sin u_4|}\right).
$$

For all $t \in J$ and $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, we get:

$$
\left|f_1^1(t, u_1, u_2, u_3, u_4) - f_1^1(t, v_1, v_2, v_3, v_4)\right|
$$
$$
\leq \frac{1}{12\pi^3}|u_1 - v_1| + \frac{1}{12\pi^3}|u_2 - v_2| + \frac{1}{12\pi^3}|u_3 - v_3| + \frac{1}{12\pi^3}|u_4 - v_4|,
$$
$$
\left|f_2^1(t, u_1, u_2, u_3, u_4) - f_2^1(t, v_1, v_2, v_3, v_4)\right|
$$
$$
\leq \frac{1}{128\pi^2}|u_1 - v_1| + \frac{1}{128\pi^2}|u_2 - v_2| + \frac{1}{64\pi}|u_3 - v_3| + \frac{1}{64\pi}|u_4 - v_4|,
$$
$$
\left|f_1^2(t, u_1, u_2, u_3, u_4) - f_1^2(t, v_1, v_2, v_3, v_4)\right|
$$
$$
\leq \frac{1}{14\pi^3}|u_1 - v_1| + \frac{1}{14\pi^3}|u_2 - v_2| + \frac{1}{14\pi^3}|u_3 - v_3| + \frac{1}{14\pi^3}|u_4 - v_4|,
$$

$$\left| f_2^2 \left(t, u_1, u_2, u_3, u_4 \right) - f_2^2 \left(t, v_1, v_2, v_3, v_4 \right) \right|$$

$$\leq \frac{1}{6\pi^3} \left| u_1 - v_1 \right| + \frac{1}{6\pi^3} \left| u_2 - v_2 \right| + \frac{1}{6\pi^3} \left| u_3 - v_3 \right| + \frac{1}{6\pi^3} \left| u_4 - v_4 \right|.$$

We can take:

$$\left(\mu_1^1 \right)_1 = \left(\mu_1^1 \right)_2 = \left(\mu_1^1 \right)_3 = \left(\mu_1^1 \right)_4 = \frac{1}{12\pi^3},$$

$$\left(\mu_2^1 \right)_1 = \left(\mu_2^1 \right)_2 = \frac{1}{128\pi^2}, \quad \left(\mu_2^1 \right)_3 = \left(\mu_2^1 \right)_4 = \frac{1}{64\pi},$$

$$\left(\mu_1^2 \right)_1 = \left(\mu_1^2 \right)_2 = \left(\mu_1^2 \right)_3 = \left(\mu_1^2 \right)_4 = \frac{1}{14\pi^3},$$

$$\left(\mu_2^2 \right)_1 = \left(\mu_2^2 \right)_2 = \left(\mu_2^2 \right)_3 = \left(\mu_2^2 \right)_4 = \frac{1}{6\pi^3}.$$

Indeed,

$$\Sigma_1 F_1 = 0.001517, \Sigma_2 F_2 = 0.001820,$$
$$\Sigma_1 F_1^* = 0.022304, \Sigma_2 F_2^* = 0.027009.$$

Thus,

$$\max \left(\Sigma_1 F_1, \Sigma_2 F_2, \Sigma_1 F_1^*, \Sigma_2 F_2^* \right) < 1.$$

Using Theorem 2.1, system (3.1) has a unique solution on J.

Example 3.2. We consider the following system:

$$(3.2) \begin{cases} D^{\frac{19}{4}} u_1 \left(t \right) = \dfrac{t \sin \left(D^{\frac{7}{2}} u_1 \left(t \right) + D^{\frac{4}{3}} u_2 \left(t \right) + D^{\frac{15}{4}} u_3 \left(t \right) \right)}{1 + \left| \sin \left(u_1 \left(t \right) + u_2 \left(t \right) + u_3 \left(t \right) \right) \right|} \\ \qquad + \dfrac{e^t \sin \left(u_1 \left(t \right) D^{\frac{7}{2}} u_1 \left(t \right) \right)}{1 + \left| \cos \left(u_2 \left(t \right) + D^{\frac{4}{3}} u_2 \left(t \right) + u_3 \left(t \right) + D^{\frac{15}{4}} u_3 \left(t \right) \right) \right|}, \quad t \in [0,1], \\ D^{\frac{17}{4}} u_2 \left(t \right) = \dfrac{e^t \sin \left(u_1 \left(t \right) + u_2 \left(t \right) + u_3 \left(t \right) \right)}{2 - \sin \left(D^{\frac{7}{2}} u_1 \left(t \right) + D^{\frac{4}{3}} u_2 \left(t \right) + D^{\frac{15}{4}} u_3 \left(t \right) \right)} \\ \qquad + \dfrac{\sin \left(u_2 \left(t \right) u_3 \left(t \right) \right)}{\pi \left(t + 1 \right) + \sin \left(u_1 \left(t \right) D^{\frac{7}{2}} u_1 \left(t \right) D^{\frac{4}{3}} u_2 \left(t \right) + D^{\frac{15}{4}} u_3 \left(t \right) \right)}, \quad t \in [0,1], \\ D^{\frac{25}{6}} u_3 \left(t \right) = \sin \left(u_1 \left(t \right) u_2 \left(t \right) u_3 \left(t \right) \right) e^{t \cos \left(D^{\frac{7}{2}} u_1(t) + D^{\frac{4}{3}} u_2(t) + D^{\frac{15}{4}} u_3(t) \right)} \\ \qquad + \dfrac{t \sin \left(u_1 \left(t \right) u_2 \left(t \right) u_3 \left(t \right) \right)}{e^{D^{\frac{7}{2}} u_1(t) + D^{\frac{4}{3}} u_2(t) + D^{\frac{15}{4}} u_3(t)}}, \quad t \in [0,1], \\ u_1 \left(0 \right) = 3\sqrt{2}, \quad u_1^{(1)} \left(0 \right) = u_1^{(2)} \left(0 \right) = u_1^{(3)} \left(0 \right) = 0, \quad u_1^{(4)} \left(0 \right) = J^{\frac{1}{8}} \left(\frac{1}{2} \right), \\ u_2 \left(0 \right) = \sqrt{7}, \quad u_2^{(1)} \left(0 \right) = u_2^{(2)} \left(0 \right) = u_2^{(3)} \left(0 \right) = 0, \quad u_2^{(4)} \left(0 \right) = J^{\frac{1}{2}} \left(\frac{2}{3} \right), \\ u_3 \left(0 \right) = \sqrt{5}, \quad u_3^{(1)} \left(0 \right) = u_3^{(2)} \left(0 \right) = u_3^{(3)} \left(0 \right) = 0, \quad u_3^{(4)} \left(0 \right) = J^{\frac{5}{8}} \left(\frac{4}{5} \right). \end{cases}$$

We have: $n = 5$, $l = 2$, $\alpha_1 = 19/4$, $\gamma_1 = 7/2$, $\alpha_2 = 17/4$, $\gamma_2 = 4/3$, $\alpha_3 = 25/6$, $\gamma_3 = 15/4$, $r_1 = 1/8$, $r_2 = 1/2$, $r_3 = 5/8$, $\tau_1 = 1/2$, $\tau_2 = 2/3$, $\tau_3 = 4/5$, $J = [0, 1]$. And, for all $(u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{R}^6$, for all $t \in J$, we get

$$\left| f_1^1 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_1^1 (t) = \frac{t \left| \sin u_4 + u_5 + u_6 \right|}{1 + \left| \sin (u_1 + u_2 + u_3) \right|} \leq C_1^1 = 1,$$

$$\left| f_2^1 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_2^1 (t) = \frac{e^t \left| \sin (u_1 u_4) \right|}{1 + \left| \cos (u_2 + u_5 + u_3 + u_6) \right|} \leq C_2^1 = e,$$

$$\left| f_1^2 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_1^2 (t) = \frac{e^t \left| \sin (u_1 + u_2 + u_3) \right|}{2 - \sin (u_4 + u_5 + u_6)} \leq C_1^2 = e,$$

$$\left| f_2^2 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_2^2 (t) = \frac{\left| \sin (u_2 u_3) \right|}{\pi (t + 1) + \sin (u_1 u_4 u_5 + u_6)} \leq C_2^2 = \frac{1}{\pi - 1},$$

$$\left| f_1^3 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_1^3 (t) = \left| \sin (u_1 u_2 u_3) \right| e^{t \cos(u_4 + u_5 + u_6)} \leq C_1^3 = e,$$

$$\left| f_2^3 (t, u_1, u_2, u_3, u_4, u_5, u_6) \right| \leq \omega_2^3 (t) = \frac{t \left| \sin (u_1 (t) u_2 (t) u_3 (t)) \right|}{e^{u_4 t + u_5 + u_6}} \leq C_2^3 = 1.$$

The functions f_i^k are continuous and bounded on $J \times \mathbb{R}^6$. Using Theorem 2.2, the system (3.2) has at least one solution on J.

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