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A REMARK ON THE PAPER "ON WEAKLY SYMMETRIC SPACETIMES" (KRAGUJEVAC J. MATH. 36(2) (2012), 299–308)

KANAK KANTI BAISHYA

Abstract. The object of the present paper is to rectify the example in Section 5 and nullify the Theorem 5.1.

1. Remark

In [1], the authors have stated that:
(a) [1, Theorem 5.1, p. 306] "Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = x^2 \left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \right] - \left(dx^4\right)^2,$$

where $i, j = 1, 2, 3, 4$. Then $(\mathbb{R}^4, g)$ is a weakly symmetric spacetime whose scalar curvature is non-zero and non-constant".

But statement (a) is found to be false. Let us recall the definition of a weakly symmetric manifold.

A non-flat Riemannian manifold is called a weakly symmetric manifold if it realizes the relation (1.2) of [1, p. 300]. The local expression of the relation (1.2) of [1, p. 300], is

$$R_{hijk,l} = A_l R_{hijk} + D_h R_{lijk} + D_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}, \tag{1.1}$$

where A_l and D_l are two non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind is denoted by WS_n.

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The covariant derivative of the Riemann curvature tensor R_{hijk} is defined as [2, p. 85]

$$R_{hijk,l} = \frac{\partial R_{hijk}}{\partial x^l} - \Gamma^a_{hl} R_{aijk} - \Gamma^a_{il} R_{hajk} - \Gamma^a_{jl} R_{hiak} - \Gamma^a_{kl} R_{hija}.$$

The only non-vanishing components of the Christoffel symbols and the Riemann curvature tensor are [1, p. 305–306]

$$\Gamma^1_{12} = \Gamma^3_{23} = \Gamma^2_{22} = -\Gamma^2_{11} = -\Gamma^2_{33} = \frac{1}{2x^2},$$
$$R_{1221} = R_{2332} = -\frac{1}{2x^2}, R_{1331} = \frac{1}{4x^2},$$

obtained by the symmetry and skew-symmetry properties of the above mentioned components. Now making use of the definition of the covariant derivative, we have

$$\begin{aligned} R_{2331,1} &= \frac{\partial R_{2331}}{\partial x^1} - \Gamma^a_{12} R_{a331} - \Gamma^a_{13} R_{2a31} - \Gamma^a_{13} R_{23a1} - \Gamma^a_{11} R_{233a} \\ &= -\Gamma^1_{12} R_{1331} - \Gamma^2_{11} R_{2332} = -\frac{3}{8}.\frac{1}{(x^2)^2}. \end{aligned}$$

In a similar manner, we can find $R_{1221,2} = \frac{3}{2}.\frac{1}{(x^2)^2}$ and $R_{1331,2} = -\frac{3}{4}.\frac{1}{(x^2)^2}$. Now, by virtue of (1.1) and $R_{2331,1}$, we have

$$\begin{aligned} R_{2331,1} &= A_1 R_{2331} + D_2 R_{1331} + D_3 R_{2131} + D_3 R_{2311} + D_1 R_{2331} \\ &\Rightarrow D_2 = -\frac{3}{2}.\frac{1}{x^2}, \text{ as } R_{2331} = 0 = R_{2131} = R_{2311}. \end{aligned}$$

Analogously, from (1.1), $R_{1221,2}$ and $R_{1331,2}$ we can easily bring out $A_2 = -\frac{3}{x^2}$ and $D_2 = \frac{1}{x^2} \neq -\frac{3}{2}.\frac{1}{x^2}$.

Consequently, the spacetime $(\mathbb{R}^4, g)$ under considered metric g can not be a WS_4. This completes the proof that statement (a) is false.

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Department of Mathematics,
Kurseong College, Dowhill Road, Kurseong,
Darjeeling-734203, West Bengal, India
E-mail address: kanakkanti.kc@gmail.com