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# DIRECT LIMIT DERIVED FROM TWIST PRODUCT ON Γ-SEMIHYPERGROUPS

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Abstract. The aim of this research work is to define a new class of hyperstructure that we call direct system. An important tool in the theory of homological algebra is the direct limit. We will present the construction of the direct limit of a direct system derived from $(\Delta, G)$-set on $\Gamma$-semihypergroups. Also, we prove the direct limit is unique up to isomorphism.

## 1. Introduction

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [9], at the $8^{th}$ Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [2–4].

Recently, the notion of $\Gamma$-hyperstructure introduced and studied by many researcher and represent an intensively was studied field of research, for example, see [1, 5, 6, 8]. The concept of $\Gamma$-semihypergroups was introduced by Davvaz et al. [1, 8] and is a generalization of semigroups, a generalization of semihypergroups and a generalization of $\Gamma$-semigroups.

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In this paper, we define the notion of left(right) $(\Delta, G)$-set, $(G_1, \Delta, G_2)$-biset, twist product, push out systems direct system and direct limits. Also, we prove that direct limit exists and unique.

## 2. Γ-semihypergroup and Twist Product

In this section we present some notion of Γ-semihypergroup and introduce a relation denoted by $\rho^*$ which we shall use in order to define a new derived structure of Γ-semihypergroup that we called twist product. These definitions and results are necessary for the next section.

**Definition 2.1.** [8] Let $G$ and $\Gamma$ be nonempty sets and $\alpha : G \times G \longrightarrow P^*(G)$ be a hyperoperation, where $\alpha \in \Gamma$ and $P^*(G)$ be the set of all nonempty subset of $G$. Then, $G$ is called Γ-*hypergroupoid.*

For any two nonempty subset $G_1$ and $G_2$, we define

$$G_1 \alpha G_2 = \bigcup_{g_1 \in G_1, g_2 \in G_2} g_1 \alpha g_2, \quad G_1 \alpha \{x\} = G_1 \alpha x, \quad \{x\} \alpha G_2 = x \alpha G_2.$$

A Γ-hypergroupiod $G$ is a called Γ-*semihypergroup* if for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$, we have $(x\alpha y)\beta z = x\alpha(y\beta z)$, which means that

$$\bigcup_{u \in x\alpha y} u\beta z = \bigcup_{v \in y\beta z} x\alpha v.$$

*Example* 2.1. Let $\Gamma \subseteq \mathbb{N}$ be a nonempty set. We define

$$x\widehat{\alpha}y = \{z \in \mathbb{N} : z \geq \max\{x, \alpha, y\}\},$$

where $\alpha \in \widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$ and $x, y \in \mathbb{N}$. Then, $\mathbb{N}$ is a $\widehat{\Gamma}$-semihypergroup.

*Example* 2.2. Let $\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then, we define hyperoperations $x\alpha_k y = xyk\mathbb{Z}$. Hence $\mathbb{Z}$ is a Γ-semihypergroup.

*Example* 2.3. Let $G$ be a nonempty set and $\Gamma$ be a nonempty subset of $G$. We define $x\widehat{\alpha}y = \{x, \alpha, y\}$, where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$. Then, $G$ is a $\widehat{\Gamma}$-semihypergroup.

*Example* 2.4. Let $G$ be a group, $H$ be a normal subgroup of $G$ and $\Gamma \subseteq G$ be a nonempty subset. For all $g_1, g_2 \in G$ and $\alpha \in \widehat{\Gamma}$, where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$ we define

$$g_1 \widehat{\alpha} g_2 = g_1 \alpha g_2 H.$$

Then, $G$ is a Γ-semihypergroup.

Let $G$ be a Γ-semihypergroup. Then, an element $e_\alpha \in G$ is called $\alpha$-*identity* if for every $x \in G$, we have $x \in e_\alpha \alpha x \cap x \alpha e_\alpha$ and $e_\alpha$ is called *scaler* $\alpha$-*identity* if $x = e_\alpha \alpha x = x \alpha e_\alpha$. We note that if for every $\alpha \in \Gamma$, $e$ is a scaler $\alpha$-identity, then $x\alpha y = x\beta y$, where $\alpha, \beta \in \Gamma$ and $x, y \in G$. Indeed,

$$x\alpha y = (x\beta e)\alpha y = x\beta(e\alpha y) = x\beta y.$$

Let $G$ be a $\Gamma$-semihypergroup and for every $\alpha \in \Gamma$ has an $\alpha$-identity. Then, $G$ is called a $\Gamma$-semihypergroup with identity.

**Definition 2.2.** Let $G$ be a $\Gamma$-semihypergroup and $\rho$ be an equivalence relation on $G$. Then, $\rho$ is called *right regular* if $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ there is $t_2 \in y\alpha g$ such that $t_1 \rho t_2$ and for every $s_1 \in y\alpha g$ there is $s_1 \in x\alpha g$ such that $s_1 \rho s_2$. In a same way, we can define left regular relations.

**Proposition 2.1.** *Let $G$ be a $\Gamma$-semihypergroup and $\rho$ be a regular relation on $G$. Then, $[G : \rho] = \{\rho(x) : x \in G\}$ is a $\widehat{\Gamma}$-semihypergroup with respect to the following hyperoperation*

$$\rho(x)\widehat{\alpha}\rho(y) = \{\rho(z) : z \in \rho(x)\alpha\rho(y)\},$$

*where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.*

*Proof.* The proof is straightforward. □

Let $G$ be a $\Gamma$-semihypergroup and $\alpha \in \Gamma$. We define $x \circ y = x\alpha y$ for every $x, y \in G$. Hence $(G, \circ)$ becomes a semihypergroup, we denote this semihypergroup by $G[\alpha]$.

**Definition 2.3.** Let $G_1$ and $G_2$ be $\Gamma$-semihypergroup with identity. Then, a map $\varphi : G_1 \longrightarrow G_2$ is called *$\alpha$-homomorphism* if $\varphi(x\alpha y) = \varphi(x)\alpha\varphi(y)$ and $\varphi(e_\alpha) = e_\alpha$ for every $x, y \in G_1$. If for every $\alpha \in \Gamma$, $\varphi$ is an $\alpha$-homomorphism, then $\varphi$ is called *homomorphism*.

**Definition 2.4.** Let $G$ be a $\Gamma$-semihypergroup with identity and $X$, $\Delta$ be nonempty sets. We say that $X$ is a *left* $(\Delta, G)$*-set* if for every $\delta \in \Delta$ there is an action $\delta : G \times X \longrightarrow X$ with the following properties:

$$(g_1\alpha g_2)\delta x = g_1\alpha(g_2\delta x),$$
$$e_\alpha \alpha x = x,$$

for every $g_1, g_2 \in G$, $\alpha \in \Gamma$, $x \in X$ and $\delta \in \Delta$.

In a same way, we can define a *right* $(\Delta, G)$*-set*. Let $G_1$ and $G_2$ be $\Gamma$-semihypergroups and $X$ be a nonempty set. Then, we say that $X$ is a $(G_1, \Delta, G_2)$*-bisets* if it is a left $(\Delta, G_1)$-set, right $(\Delta, G_2)$-set and

$$(g_1\delta_1 x)\delta_2 g_2 = g_1\delta_1(x\delta_2 g_2),$$

for every $\delta_1, \delta_2 \in \Delta$, $g_1 \in G_1$, $g_2 \in G_2$ and $x \in X$.

A $\Gamma$-semihypergroup $G$ is called *commutative* when $x\alpha y = y\alpha x$, for every $x, y \in G$ and $\alpha \in \Gamma$. If $G$ is a commutative $\Gamma$-semihypergroup, then there is no distinction between a left and a right $(\Delta, G)$-set. A $(\Delta, G)$-left subset $Y$ of $X$ such that $Y\Delta X \subseteq Y$. A map $\varphi : X \longrightarrow Y$ from a left $(\Delta, G)$-set $X$ into a left $(\Delta, G)$-set $Y$ is called *morphism* if

$$\varphi(g\delta x) = g\delta\varphi(x),$$

for every $x \in X, \delta \in \Delta$ and $g \in G$. In a same way we can define a morphism of right $(\Delta, G)$-sets. An equivalence relation $\rho$ on left $(\Delta, G)$-set $X$ is called *congruence*, if for every $x, y \in X$, $\delta \in \Delta$ and $g \in G$

$$x\rho y \Longrightarrow (g\delta x)\rho(g\delta y).$$

The quotient $[X : \rho]$ is a left $(\widehat{\Delta}, G)$-set by following operation:

$$g\widehat{\delta}(\rho(x)) = \rho(g\delta x),$$

where $\widehat{\Delta} = \{\widehat{\delta} : \delta \in \Delta\}$. The map $\pi : X \longrightarrow [X : \rho]$ defined by $\pi(x) = \rho(x)$, for every $x \in X$ is a morphism.

*Example* 2.5. Let $G$ be a $\Gamma$-semihypergroup and $G_1$ be a $\Gamma$-subsemihypergroup of $G$. Then, $G_1$ is a left $(\Gamma, G_1)$-set in the obvious way.

*Example* 2.6. Let $\rho$ be a left regular relation on $\Gamma$-semihypergroup $G$. Then, there is a well-defined action of $G$ on $[G : \rho]$ given by $g\widehat{\alpha}(\rho(x)) = \rho(g\alpha x)$, where $\widehat{\alpha} \in \widehat{\Gamma}$ and $x \in G$ such that $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$. Hence, with this definition $[G : \rho]$ is a left $(\widehat{\Gamma}, G)$-set.

It is easy to see that the cartesian product $X \times Y$ of a left $(\Delta, G_1)$-set $X$ and a right $(\Delta, G_2)$-set $Y$ becomes $(G_1, \widehat{\Delta}, G_2)$-biset if we make the obvious definition

$$g_1\widehat{\delta}_1(x, y) = (g_1\delta_1 x, y), \quad (x, y)\widehat{\delta}_2 g_2 = (x, y\delta_2 g_2),$$

where $\widehat{\delta}_1, \widehat{\delta}_2 \in \widehat{\Delta}$, $x \in X, y \in Y$ and $g_1 \in G_1$, $g_2 \in G_2$.

Let $X$, $Y$ and $Z$ be $(G_1, \Delta, G_2)$-biset, $(G_2, \Delta, G_3)$-biset, and $(G_1, \Delta, G_3)$-biset (respectively). Then, the cartesian product $X \times Y$ is $(G_1, \Delta, G_3)$-biset. A map $\varphi : X \times Y \longrightarrow Z$ is called $\delta$-*bimap* if

$$\varphi(x\delta g_2, y) = \varphi(x, g_2\delta y),$$

where $x \in X$, $y \in Y$, $z \in Z$, $g_2 \in G_2$ and $\delta \in \Delta$.

**Definition 2.5.** [7] A pair $(P, \psi)$ consisting of $(G_1, \Delta, G_3)$-biset $P$ and a $\delta$-bimap $\psi : X \times Y \longrightarrow P$ will be called a *twist product* of $X$ and $Y$ over $G_2$ if for every $(G_1, \Delta, G_3)$-biset $Z$ and for every bimap $\omega : X \times Y \longrightarrow Z$ there exists a unique bimap $\overline{\omega} : P \longrightarrow Z$ such that $\overline{\omega} \circ \psi = \omega$.

Suppose that $\rho$ is an equivalence relation on $X \times Y$ as follows:

$$\rho = \{((x\delta g, y), (x, g\delta y)), x \in X, y \in Y, g \in G_2\}.$$

Let us define $X \ominus Y$ to be $[X \times Y : \rho^*]$, where $\rho^*$ is a transitive closure of $\rho$. We denote a typical element $\rho^*(x, y)$ by $x \ominus y$. By definition of $\rho^*$, we have $x\delta g \ominus y = x \ominus g\delta y$, where $\delta \in \Delta$.

**Proposition 2.2.** [7] *Let $X$ and $Y$ be $(G_1, \Delta, G_2)$-biset and $(G_2, \Delta, G_3)$-biset, respectively. Then, two elements $x \ominus y$ and $x' \ominus y'$ are equal if and only if $(x, y) = (x', y')$ or* *there exist* $x_1, x_2, \dots, x_{n-1}$ *in* $X$, $y_1, y_2, \dots, y_{n-1} \in Y$, $g_1, g_2, \dots, g_n, h_1, h_2, \dots, h_{n-1} \in G_2$ *and* $\delta \in \Delta$ *such that*

$$\begin{aligned} x &= x_1\delta g_1, & g_1\delta y &= h_1\delta y_1, \\ x_1\delta h_1 &= x_2\delta g_2, & g_2\delta y_1 &= h_2\delta y_2, \\ &\vdots & &\vdots \\ x_i\delta g_i &= x_{i+1}\delta g_{i+1}, & g_{i+1}\delta y_i &= h_{i+1}\delta y_{i+1}, \\ &\vdots & &\vdots \\ x_{n-1}\delta h_{n-1} &= x^{'}\delta g_n, & g_n\delta y_{n-1} &= y^{'}. \end{aligned}$$

**Theorem 2.1.** [7] *Let* $X$ *and* $Y$ *be* $(G_1, \Delta, G_2)$*-biset and* $(G_2, \Delta, G_3)$*-biset. Then,* $(X \ominus Y, \pi)$ *is a twist product of* $X$ *and* $Y$ *over* $G_2$.

*Proof.* It is easy to see that $\pi : X \times Y \longrightarrow X \ominus Y$ is a $\delta$-bimap such that $\pi(x, y) = x \ominus y$. Let $\omega : X \times Y \longrightarrow Z$, where $Z$ is a $(G_1, \Delta, G_3)$-biset and $\omega$ is a $\delta$-bimap. We define $\overline{\omega} : X \ominus Y \longrightarrow Z$ by

$$\overline{\omega}(x \ominus y) = \omega(x, y).$$

Let $x \ominus y = x^{'} \ominus y^{'}$. By 2.2, we have

$$\omega(x, y) = \omega(x_1\delta g_1, y) = \omega(x_1, g_1\delta y) = \omega(x_1, g_1\delta h_1) = \cdots = \omega(x^{'}, y^{'}).$$

Hence $\overline{\omega}(x \ominus y) = \overline{\omega}(x^{'} \ominus y^{'})$. It is easy to see that $\overline{\omega}$ is a $\delta$-bimap, $\overline{\omega} \circ \pi = \omega$ and $\overline{\omega}$ is unique with respect. □

**Theorem 2.2.** *Let* $X$ *and* $Y$ *be* $(G_1, \Delta, G_2)$*-biset and* $(G_2, \Delta, G_3)$*-biset. Then the twist product* $X$ *and* $Y$ *over* $G_2$ *is unique up to isomorphism.*

*Proof.* Suppose that $(P, \psi)$ and $(P^{'}, \psi^{'})$ are twist product of $X$ and $Y$ over $G_2$. By definition we find a unique $\overline{\psi^{'}} : P \longrightarrow P^{'}$ and $\overline{\psi} : P^{'} \longrightarrow P$ such that $\psi \circ \overline{\psi^{'}} = \psi^{'}$ and $\overline{\psi} : P^{'} \longrightarrow P$ such $\psi^{'} \circ \overline{\psi} = \psi$. Since $\psi \circ \overline{\psi^{'}} \circ \overline{\psi} = \psi$, we have $\overline{\psi^{'}} \circ \overline{\psi} = Id_{P^{'}}$. By a similar argument $\overline{\psi} \circ \overline{\psi^{'}} = Id_P$. □

We can generalize the notion of twist product three bisets. Let $X$, $Y$, $Z$ and $W$ be $(G_1, \Delta, G_2)$-biset, $(G_2, \Delta, G_3)$-biset, $(G_3, \Delta, G_4)$-biset and $(G_1, \Delta, G_4)$-biset. Then, a map $\varphi : X \times Y \times Z \longrightarrow Z$ is called $\delta$*-trimap* if for $x \in X, y \in Y$ and $z \in Z$ and $g_2 \in G_2$, $g_3 \in G_3$ and $\delta \in \Delta$

$$\varphi(x\delta g_2, y, z) = \varphi(x, g_2\delta y, z), \quad \varphi(x, y\delta g_3, z) = \varphi(x, y, g_3\delta z).$$

A pair $(P, \psi)$, where $P$ is a $(G_1, \Delta, G_4)$-biset and $\psi : X \times Y \times Z \longrightarrow P$ is a $\delta$-trimap is said to be twist if for every $(G_1, \Delta, G_4)$-biset $W$ and every $\delta$-trimap $\phi : X \times Y \times Z \longrightarrow W$ there is a unique $\overline{\phi} : P \longrightarrow W$ such that $\overline{\phi} \circ \psi = \phi$. A similar argument shows that $X \ominus (Y \ominus Y)$, together with the obvious trimap $(x, y, z) \longrightarrow x \ominus (y \ominus z)$ is also a twist product of $X$, $Y$ and $Z$.

**Proposition 2.3.** [7] *Let $X$, $Y$, $Z$ be $(G_1,\Delta,G_2)$-biset, $(G_2,\Delta,G_3)$-biset, $(G_3,\Delta,G_4)$-biset, respectively. Then, $X\ominus(Y\ominus Z)\cong(X\ominus Y)\ominus Z$.*

Suppose that $\varphi: X_1\longrightarrow X_2$ is a morphism and

$$\operatorname{kerl}\varphi:\{(a,b)\in X_1\times X_1:\varphi(a)=\varphi(b)\}.$$

This relation on $X_1$ is an equivalence relation and is called *kernel of* $\varphi$.

**Theorem 2.3.** *Let $G$ be a $\Gamma$-semihypergroup, $X_1$, $X_2$ be left $(\Delta,G)$-sets, $\varphi:X_1\longrightarrow X_2$ be a morphism and $\rho\subseteq\operatorname{kerl}\varphi$ be a congruence relation on $X_1$. Then, $[X_1:\rho]$ is a $(\widehat{\Delta},G)$-set, where $\widehat{\Delta}=\{\widehat{\delta}:\delta\in\Delta\}$ and there is a monomorphism $\widehat{\varphi}:[X_1:\rho]\longrightarrow\operatorname{Im}\varphi$.*

*Proof.* It is easy to see that $[X_1:\rho]$ is a $(\widehat{\Delta},G)$-set. We Define $\widehat{\varphi}:[X_1:\rho]\longrightarrow X_2$ by

$$\widehat{\varphi}(\rho(x))=\varphi(x).$$

Let $\rho(a)=\rho(b)$. Then,

$$(a,b)\in\rho\Longrightarrow(a,b)\in\operatorname{kerl}\varphi\Longrightarrow\varphi(a)=\varphi(b).$$

This implies that $\widehat{\varphi}$ is well-defined. If $g\in G$ and $\rho(a)\in[X_1:\rho]$, then

$$\widehat{\varphi}(g\widehat{\delta}\rho(a))=\widehat{\varphi}(\rho(g\delta a))=\varphi(g\delta a)=g\delta\varphi(a)=g\delta\widehat{\varphi}(\rho(a)).$$

Hence $\widehat{\varphi}$ is a morphism. □

**Proposition 2.4.** *Let $\rho_1$ and $\rho_2$ be congruence relations on $(\Delta,G)$-set $X$ such that $\rho_1\subseteq\rho_2$. Then,*

$$[\rho_2:\rho_1]=\{(\rho_1(a),\rho_1(b))\in[X:\rho_1]\times[X:\rho_2]:(a,b)\in\rho_2\},$$

*is a congruence relation on $[X:\rho_1]$ and*

$$[[X:\rho_1]:[\rho_2:\rho_1]]\simeq[X:\rho_2].$$

*Proof.* The proof is straightforward. □

## 3. Direct Limit

In this section we introduce a non additive version of direct limit that is important in homological algebra. We prove that the direct limit exists and is unique.

Let $(J,\leq)$ be a partially ordered set and $\{X_j\}_{j\in J}$ be a collection of $(G_1,\Delta,G_2)$-bisets and for all $i,j\in J$ such that $i\leq j$, there is a morphism $\omega_{ij}:X_i\longrightarrow X_j$ with the following properties:

1) $\omega_{ii}=I_{X_i}$,
2) $\omega_{ij}\circ\omega_{jk}=\omega_{ik}$.

Then, we say that $(X_i,\omega_{ij})$ is a *direct system* of $(G_1,\Delta,G_2)$-bisets.

We say that a $(G_1,\Delta,G_2)$-biset $X$ is called *direct limit* of this direct system if there exist morphisms $\omega_i:X_i\longrightarrow X$ such that $\omega_j\circ\omega_{ij}=\omega_i$ and if there exists a $(G_1,\Delta,G_2)$- biset $Y$ has the property that there exist a morphism $\lambda_i:X_i\longrightarrow Y$ such that $\lambda_j\circ\alpha_{ij}=\lambda_i$ with $i\leq j$ and $\alpha_{ij}:X_i\longrightarrow X_j$ is a morphism, then there is a unique morphism $\lambda:X\longrightarrow Y$ such that $\lambda\circ\omega_i=\lambda_i$.

**Theorem 3.1.** *Let $(X_i, \omega_{ij})$ be a direct system. Then, the direct limit exist.*

*Proof.* Suppose that $(X_i, \omega_{ij})$ is a direct system. Without loses of generality we suppose that the sets $X_i$ are pairwise disjoint. Let $D = \bigcup_{i\in I} X_i$ and $\theta^*$ be equivalence relation generated by the following relation:

$$d_1\theta d_2 \Longleftrightarrow \text{there exists } i \leq j, \quad d_1 \in X_i, \quad \omega_{ij}(d_1) = d_2,$$

where $d_1, d_2 \in D, i, j \in I$. We prove that $[D : \theta^*]$ is a direct limit. To see this, we define a morphism $\omega_i : X_i \longrightarrow [D : \theta^*]$, for each $i \in I$ by

$$\omega_i(x_i) = \theta^*(x_i),$$

where $x_i \in X_i$. We have

$$\omega_j \circ \omega_{ij}(x_i) = \theta^*(\omega_{ij}(x_i)) = \theta^*(x_i),$$

for every $x_i \in X_i$. Let $Y$ be a $(G_1, \Delta, G_2)$-biset and $\lambda_i : X_i \longrightarrow Y$ be a morphism such that $\lambda_j \circ \alpha_{ij} = \lambda_i$. We define a morphism $\varphi : D \longrightarrow Y$ by

$$\varphi(x_i) = \lambda_i(x_i), \quad x_i \in X_i, \quad i \in I.$$

Let $d_1\theta d_2$. Then, there are $i, j \in I$ and $\omega_{ij}$ such that $\omega_{ij}(d_1) = d_2$. This implies that

$$\varphi(d_2) = \varphi(\omega_{ij}(d_1)) = \lambda_j(\omega_{ij}(d_1)) = \lambda_i(d_1) = \varphi(d_1).$$

Hence $(d_1, d_2) \in ker\varphi$ and by Proposition 2.4, there exists morphism $\widehat{\varphi} : [D : \theta^*] \longrightarrow Y$ defined by

$$\widehat{\varphi}(\theta^*(x_i)) = \varphi(x_i),$$

where $x_i \in X_i$, $i \in I$. Also, $\widehat{\varphi} \circ \omega_i(x_i) = \widehat{\varphi}(\theta^*(x_i)) = \lambda_i(x_i)$. This implies that $\widehat{\varphi} \circ \omega_i = \lambda_i$. If $\psi$ is a morphism with the same properties, then for every $x_i \in X_i$ and $i \in I$,

$$\psi(\theta^*(x_i)) = \psi(\omega_i(x_i)) = \lambda_i(x_i) = \widehat{\varphi}(\theta^*(x_i)).$$

Therefore, $\widehat{\varphi} = \psi$. This completes the proof. □

**Proposition 3.1.** *The direct limit of direct system $(X_i, \omega_{ij})_{i,j\in I}$ is unique up to isomorphism.*

*Proof.* Suppose that $X$ and $Y$ are direct limits of direct system $(X_i, \omega_{ij})_{i,j\in I}$. By definition we have a unique $\lambda : X \longrightarrow Y$ and $\lambda' : Y \longrightarrow X$ such that $\omega_i \circ \lambda = \lambda_i$ and $\lambda_i \circ \lambda' = \omega_i$. Hence

$$\omega_i \circ (\lambda \circ \lambda') = (\omega_i \circ \lambda) \circ \lambda' = \lambda_i \circ \lambda' = \omega_i,$$
$$\lambda_i \circ (\lambda' \circ \lambda) = \lambda_i.$$

Therefore, $\lambda \circ \lambda' = I_X$ and $\lambda' \circ \lambda = I_Y$ and so $X \cong Y$. □

**Proposition 3.2.** *Let $(X_n, \omega_n)$ be a direct system and $[D : \theta^*]$ be the direct limit of the this direct system. Then, the map $\beta_n : X_n \longrightarrow [D : \theta^*]$ is one to one if and only if the maps $\omega_n$ are one to one.*

*Proof.* Suppose that all maps $\omega_n : X_n \longrightarrow X_{n+1}$ are one to one and $\beta_m(a_m) = \beta_m(b_m)$. This implies that $\theta^*(a_m) = \beta^*(b_m)$. Hence

$$\text{there exists } x_1, x_2, \ldots, x_n \in D : a_m = x_1,\ b_m = x_n \text{ and } x_i\theta x_{i+1}.$$

By the definition of $\theta$, since every $\omega_n$ is one to one it follows that $a_m = b_m$. This implies that $\beta_n : X_n \longrightarrow X_{n+1}$ is one to one. Conversely, suppose that for some $n$, $\omega_n$ is not one to one and $\beta_n$ is one to one. Hence for some $a_n \neq b_n$ we have $\omega_n(a_n) = \omega_n(b_n)$. By definition $a_n\theta b_n$ and $\theta^*(a_n) = \theta^*(b_n)$ and so $\beta^*(a_n) = \beta^*(b_n)$. That is contradiction. □

**Proposition 3.3.** *Let $(X_i, \omega_{ij})$ be a direct system $(G_2, \Delta, G_3)$-biset, $D$ be the direct limit of this direct system and $H_1$, $H_2$ be $(G_1, \Delta, G_2)$, $(G_3, \Delta, G_4)$-biset, respectively. Then, $H_1 \ominus D \ominus H_2$ is the direct limit of direct system $(H_1 \ominus X_i \ominus H_2, I_{H_1} \ominus \omega_{ij} \ominus I_{H_2})$.*

*Proof.* Suppose that

$$I_{H_1} \ominus \omega_{ij} \ominus I_{H_2} : H_1 \ominus X_i \ominus H_2 \longrightarrow H_1 \ominus X_j \ominus H_2,$$

$$I_{H_1} \ominus \omega_i \ominus I_{H_2} : H_1 \ominus X_i \ominus H_2 \longrightarrow H_1 \ominus D \ominus H_2.$$

Obviously,

$$(I_{H_1} \ominus \omega_i \ominus I_{H_2}) \circ (I_{H_1} \ominus \omega_{ij} \ominus I_{H_2}) = I_{H_1} \ominus \omega_i \ominus I_{H_2},$$

for $i \leq j$. Let $Q$ be a $(G_1, \Delta, G_4)$-biset and $\sigma_i : H_1 \ominus X_i \ominus H_2 \longrightarrow Q$ such that $(I_{H_1} \ominus \omega_{ij} \ominus I_{H_2}) \circ \sigma_j = \sigma_i$, for all $i \leq j$ and $T$ is the disjoint union $X_i$ and $\theta^*$ is the equivalence relation generated by the following relation:

$$d_1\theta d_2 \Longleftrightarrow \text{there exists } i, j \in I\ \omega_{ij}(d_2) = d_1.$$

We know that $\omega_i(x_i) = \theta^*(x_i)$ for all $x_i \in X_i$. We define $\mu : H_1 \times T \times H_2 \longrightarrow Q$ by

$$\mu(h_1, t_i, h_2) = \sigma_i(h_1 \ominus t_i \ominus h_2).$$

We have

$$\begin{aligned}
\mu(h_1, \omega_{ij}(t_i), h_2) &= \sigma_i(h_1 \ominus \omega_{ij}(t_i) \ominus h_2) \\
&= \sigma_j(I_{H_2} \ominus \omega_{ij} \ominus I_{H_2})(h_1 \ominus t_i \ominus h_2) \\
&= \sigma_i(h_1 \ominus t_i \ominus h_2) \\
&= \mu(h_1, t_i, h_2).
\end{aligned}$$

Hence $\mu$ induces a map $\widehat{\mu} : H_1 \times T \times H_2 \longrightarrow Q$ defined by

$$\widehat{\mu}(h_1, \theta^*(t_i), h_2) = \sigma_i(h_1 \ominus t_i \ominus h_2).$$

For all $g_2 \in G_2$ and $t_i \in T$ we have

$$\widehat{\mu}(h_1\delta g_2, \theta^*(t_i), h_2) = \sigma_i(h_1\delta g_2 \ominus t_i \ominus h_2) = \sigma_i(h_1 \ominus g_2\delta x_i \ominus h_2) = \widehat{\mu}(h_1, g_2\theta^*(t_i), h_2),$$

and similarly, for every $g_3 \in G_3$ we have

$$\widehat{\mu}(h_1, \theta^*(t_i), g_3\delta h_2) = \widehat{\mu}(h_1, \theta^*(t_i)\delta g_3, h_2).$$

Hence $\widehat{\mu}$ induces a map $\xi : H_1 \ominus T \ominus H_2 \longrightarrow Q$ given by

$$\xi(h_1 \ominus \theta^*(t_i) \ominus h_2) = \sigma_i(h_1 \ominus t_i \ominus h_2).$$

It is easy to see that $\xi$ is morphism and $(I_{H_1}\omega_i \ominus I_{H_2}) \circ \xi = \sigma_i$ and $\xi$ is unique. This completes the proof. $\square$

Suppose that $X_1$, $X_2$, $X_3$, $X_4$ are $(\Delta, G)$-sets and $\varphi_i : X_1 \longrightarrow X_i$, $\psi_j : X_j \longrightarrow X_4$ are morphisms for $2 \leq i \leq 3$ and $2 \leq j \leq 3$ such that $\psi_2 \circ \varphi_1 = \psi_3 \circ \varphi_2$. If there are $\psi_2^{'} : X_2 \longrightarrow X_4^{'}$ and $\psi_3^{'} : X_3 \longrightarrow X_4^{'}$, such that $\psi_3^{'} \circ \varphi_2 = \psi_2^{'} \circ \varphi_1$, then there is a morphism $\lambda : X_4 \longrightarrow X_4^{'}$ such that $\lambda \circ \psi_2 = \psi_2^{'}$ and $\lambda \circ \psi_3 = \psi_3^{'}$. A system $[X_i, \varphi_j, \psi_k]$ for $1 \leq i \leq 4$, $1 \leq j \leq 2$, $1 \leq k \leq 2$ is called *push out.*

**Proposition 3.4.** *Let $X_1$, $X_2$, $X_3$ be $(\Delta, G)$-sets, $\varphi_1 : X_1 \longrightarrow X_2$ and $\varphi_2 : X_1 \longrightarrow X_3$ be morphisms. Then, there is a push out system and $x_2 \in X_2$, $x_3 \in X_3$, $\psi_2(x_2) = \psi_3(x_3)$ implies that $x_2 \in \operatorname{Im} \varphi_1$.*

*Proof.* Suppose that $\rho$ is a following relation on $X = X_1 \cup X_2 \cup X_3$ of disjoint $(\Delta, G)$-sets.

$$x\rho y \Longleftrightarrow x \in X_1 \text{ and } y = \varphi_1(x) \text{ or } x \in X_1 \text{ and } \varphi_2(x) = y.$$

Let $\rho^*$ be equivalence relation generated by $\rho$ and $[X : \rho^*]$ be the quotient set on $X$ by $\rho^*$. We define $\psi_2 : X_2 \longrightarrow [X : \rho^*]$ and $\psi_3 : X_3 \longrightarrow [X : \rho^*]$ by

$$\psi_2(x_2) = \rho^*(x_2),\ \psi_3(x_3) = \rho^*(x_3).$$

It is easy to see that $[X, X_i, \varphi_j, \varphi_k]$ is a push out system.

Let $\psi_2(x_2) = \psi_3(x_3)$, then $\rho^*(x_2) = \rho^*(x_3)$. This implies that there are $a_1, a_2, \ldots, a_n \in X$ such that $a_1 = x_2$, $a_n = x_3$ and $a_i \rho a_{i+1}$. Hence $a_i \in X_1$ and $\varphi_1(a_i) = a_{i+1}$ or $\varphi_2(a_i) = a_{i+1}$. This implies that $x_2 \in \operatorname{Im} \varphi_1$. $\square$

Let $G_1$ be a Γ-subsemihypergroup of $G$, $X_1$ and $X_2$ be a $(\Delta, G_1)$-set and a $(\Delta, G)$-set, respectively and $X = X_2 \ominus G$, and $\varphi : X_1 \longrightarrow X_2$ be a morphism on $G_1$. We define the following relation on $X$ as follows:

$$(d_1 \ominus g_1)\xi(d_2 \ominus g_2) \Longleftrightarrow \text{there exists } t_1, t_2 \in X_1,\ \varphi(t_1) = d_1,\ \varphi(t_2) = d_2,\ t_1\delta g_1 = t_2\delta g_2.$$

Suppose that $\xi^*$ is an equivalence relation generated by $\xi$. Hence $[X \ominus G : \xi^*]$ is a $(\Delta, G)$-set and is called *extension* of $X$ by $G$.

**Proposition 3.5.** *Let $G_1$ be a Γ-subsemihypergroup of $G$, $X_1$ be a left $(\Delta, G)$-set, $X_2$ be a left $(\Delta, G_1)$-set and $\varphi : X_1 \longrightarrow X_2$ be a morphism and $T = [X_2 \ominus G : \xi^*]$, where $\Delta \subseteq \Gamma$. Then, $X_1 \ominus G, X_2 \ominus G, X_1, T$, where $\varphi \ominus I : X_1 \ominus G \longrightarrow X_2 \ominus G, \psi : X_1 \ominus G \longrightarrow X_1, \beta : X_1 \longrightarrow T$ and $\pi : X_2 \ominus G \longrightarrow T$ defined as follows:*

$$\varphi \ominus I(x_1 \ominus g) = \varphi(x_1) \ominus g,\ \psi(x_1 \ominus g) = x_1\delta g,$$

$$\beta(x_1) = \xi^*(\varphi(x_1) \ominus e_\delta),\ \pi(x_2 \ominus g) = \xi^*(x_2 \ominus g),$$

*is a push out system.*

*Proof.* Suppose that $x_1 \ominus g \in X_1 \ominus G$. Hence

$$\pi \circ (\varphi \ominus I)(x_1 \ominus g) = \pi(\varphi(x_1) \ominus g) = \xi^*(\varphi(x_1) \ominus g),$$

and

$$\begin{aligned}\beta \circ \psi(x_1 \ominus g) = \beta(x_1 \delta g) &= \xi^*(\varphi(x_1 \delta g) \ominus e_\delta)\\ &= \xi^*(\varphi(x_1)\delta e_\delta \ominus g)\\ &= \xi^*(\varphi(x_1) \ominus e_\delta \delta g)\\ &= \xi^*(\varphi(x_1) \ominus g).\end{aligned}$$

This implies that $\beta \circ \psi = \pi \circ (\varphi \ominus I)$. Let $T^{'}$ be a $(\Delta, G)$-set, $\beta^{'} : X_1 \longrightarrow T^{'}$ and $\sigma : X_2 \ominus G \longrightarrow T^{'}$ such that $\sigma \circ (\varphi \ominus I) = \beta^{'} \circ \psi$. For every $x_1, x_1^{'} \in X_1$ and $g, g^{'} \in G$ such that $\xi^*(\varphi(x_1) \ominus g) = \xi^*(\varphi(x_1^{'}) \ominus g^{'})$. We have

$$\begin{aligned}\pi(\varphi(x_1) \ominus g) = \pi \circ (\varphi \ominus I)(x_1 \ominus g) &= \beta^{'} \circ \psi(x_1 \ominus g)\\ &= \beta^{'}(x_1 \delta g)\\ &= \beta^{'}(x_1^{'} \delta g^{'})\\ &= \psi(\varphi(x_1^{'}) \ominus g^{'}).\end{aligned}$$

It follows that $\psi$ induces a unique morphism $\omega : T \longrightarrow T^{'}$ by following definition:

$$\omega(\xi^*(x_2 \ominus g)) = \pi(x_2 \ominus g).$$

On the others hand

$$\begin{aligned}\omega \circ \beta(x_1) = \omega(\xi^*(\varphi(x_1) \ominus e_\delta)) &= \pi \circ (\varphi(x_1) \ominus e_\delta)\\ &= \pi \circ (\varphi \ominus I)(x_1 \ominus e_\delta)\\ &= \beta^{'} \circ \psi(x_1 \ominus e_\delta)\\ &= \beta^{'}(x_1).\end{aligned}$$

This completes the proof. □

**Lemma 3.1.** *Let $G_1$ be a $\Gamma$-subsemihypergroup of $G$ and $G_1$ has the extension property in $G$ and $\varphi : X_1 \longrightarrow X_2$ be a morphism on $G_1$ and $x_2 \ominus e_\alpha = \varphi(x_1) \ominus g$. Then, $x_2 \in \operatorname{Im} \varphi$.*

*Proof.* Suppose that $X_1$, $X_2$, $T$ are $(\Delta, G)$-sets and $\varphi : X_1 \longrightarrow X_2$, $\delta_2 : X_2 \longrightarrow T$ and $\lambda : X_2 \longrightarrow T$ are a push out system. Hence $X_1 \ominus G$, $X_2 \ominus G$, $T \ominus G$, $\varphi \ominus I_G : X_1 \ominus G \longrightarrow X_2 \ominus G$, $\lambda \ominus I_G : X_2 \ominus G \longrightarrow T \ominus G$ and $\delta_2 \ominus I_G : X_2 \ominus G \longrightarrow T \ominus G$ is also push out system. Since $x_2 \ominus e_\alpha = \varphi(x_1) \ominus g$ in $X_2 \ominus G$, we have

$$\begin{aligned}
\delta_2(x_2) \ominus e_\alpha = (\delta_2 \ominus I_G)(x_2 \ominus e_\alpha) &= (\delta_2 \ominus I_G)(\varphi(x_1) \ominus g) \\
&= \delta_2(\varphi(x_1)) \ominus g \\
&= \lambda(\varphi(x_1)) \ominus g \\
&= (\lambda \ominus I_G)(\varphi(x_1) \ominus g) \\
&= (\lambda \ominus I_G)(x_2 \ominus e_\alpha) \\
&= \lambda(x_2) \ominus e_\alpha.
\end{aligned}$$

Since $G_1$ has the extension property in $G$ the map $X_2 \longrightarrow X_2 \ominus e_\alpha$ from $X_2$ to $X_2 \ominus G$ is one to one. This implies that $\lambda(x_2) = \delta_2(x_2)$ and by Proposition 3.4, $x_2 \in \operatorname{Im}\varphi$. □

**Theorem 3.2.** *Let $G_1$ be a $\Gamma$-subsemihypergroup of $G$ and $G_1$ has the extension property in $G$ and $\varphi : X_1 \longrightarrow X_2$ be a morphism on $G_1$ such that $\varphi \ominus I : X_1 \ominus X \longrightarrow X_2 \ominus X$ be a morphism, where $X$ be a right $(\Delta, G)$-set. Then, $x_2 \ominus e_\alpha \ominus x = \varphi(x) \ominus g \ominus x'$, implies that there exist $t_1 \in X_1, t_2 \in X$ such that $x_2 \ominus e_\alpha \ominus x = \varphi(t_1) \ominus e_\alpha \ominus t_2$.*

*Proof.* Suppose that $x_2 \ominus e_\alpha \ominus x = \varphi(x) \ominus g \ominus x'$. Let $X_1, X_2, T$ be push out system, where $\varphi : X_1 \longrightarrow X_2, \delta_1 : X_2 \longrightarrow T$ and $\delta_2 : X_2 \longrightarrow T$. Hence $X_1 \ominus X$, $X_2 \ominus X$, and $T \ominus X$ is a push out system, where $\varphi \ominus I : X_1 \ominus X \longrightarrow X_2 \ominus X$, $\delta_1 \ominus I : X_2 \ominus X \longrightarrow T \ominus X$ and $\delta_2 \ominus I : X_2 \ominus X \longrightarrow T \ominus X$ is also push out system. On the other hand on $T \ominus G \ominus X$,

$$\begin{aligned}
\delta_2(x_2) \ominus e_\alpha \ominus x = (\delta_2 \ominus I_G \ominus I_X)(x_2 \ominus e_\alpha \ominus x)) &= (\delta_2 \ominus I_G \ominus I_X)(\varphi(x) \ominus g \ominus x') \\
&= \delta_2\varphi(x) \ominus g \ominus x' \\
&= \delta_2\varphi(x) \ominus g \ominus x' \\
&= \delta_1\varphi(x) \ominus g \ominus x' \\
&= \delta_1(x_2) \ominus e_\alpha \ominus x.
\end{aligned}$$

By extension property we have $\delta_2(x_2) \ominus x = \delta_1(x_2) \ominus x$. Hence by Proposition 3.4 there exist $t_1 \in X_1$ and $t_2 \in X$ such that

$$x_2 \ominus x = (\varphi \ominus I_X)(t_1 \oslash t_2) = \varphi(t_1) \oslash t_2.$$

This implies that

$$x_2 \ominus e_\alpha x = \varphi(t_1) \ominus e_\alpha \ominus t_2.$$

□

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