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ON STABILITY AND BOUNDEDNESS PROPERTIES OF SOLUTIONS OF CERTAIN SECOND ORDER NON-AUTONOMOUS NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. In this paper, sufficient criteria for the existence of solutions to uniform asymptotic stability and boundedness problems associated with certain second order nonlinear non autonomous ordinary differential equation are established with the aid of Lyapunov's direct method. Furthermore, the appropriate complete Lyapunov function is given explicitly. Our results complement some well known results on the second order differential equations in the literature.

1. Introduction

Consider the second order non autonomous damped and forced nonlinear ordinary differential equation of the form

$$(a(t)\dot{x})' + b(t)f(x,\dot{x})\dot{x} + c(t)g(x) = p(t;x,\dot{x}), \tag{1.1}$$

where $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$, $p \in C([0,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $a(t)$, $b(t)$ and $c(t)$ are positive functions.

The functions a, b, c, f, g and p depend only on the arguments displayed explicitly and they are such that existence, uniqueness and continuous dependence on the initial condition are guaranteed. The equation (1.1) can be written as

$$a(t)\ddot{x} + [\dot{a}(t) + b(t)f(x,\dot{x})]\dot{x} + c(t)g(x) = p(t;x,\dot{x})$$

with an equivalent system,

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\frac{1}{a(t)}[\dot{a}(t)y + b(t)f(x,y)y + c(t)g(x) - p(t;x,y)].\end{aligned} \tag{1.2}$$

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On substituting $\frac{1}{a(t)} = d(t)$ for all $a(t) \neq 0$ into system (1.2), we have

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -[\dot{a}(t)y + b(t)f(x,y)y + c(t)g(x) - p(t;x,y)]d(t). \end{aligned} \tag{1.3}$$

The dots indicate differentiation with respect to the independent variable t.

For over six decades, numerous works have been done by various authors on autonomous and non autonomous second order nonlinear ordinary differential equations and obtained many interesting results, for examples see [7,10–12,14] and the references cited therein.

In these aforementioned works, the authors made use of Lyapunov direct's method by constructing different Lyapunov functions and obtained criteria which ensure the existence of solutions of some qualitative properties of solutions of the problems considered while some constructed Lyapunov functions, some adopted existing Lyapunov functions to give suitable criteria that guaranteed their results. For more exposition on Lyapunov functions and the qualitative properties of solutions of ordinary differential equations, readers are referred to [5, 13, 17, 24, 25, 35, 36].

Furthermore, many of these existing Lyapunov functions are either incomplete or contain signum functions.

In 1974, Baker [3] established some sufficient conditions that guarantee the uniform stability of the trivial solutions of the following second order nonlinear differential equation

$$\left(r(t)u'\right)' + \phi\left(t, u, u'\right)u' + p(t)f(u) = 0.$$

with the equivalent system

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -\frac{1}{r(t)}\left[r'(t)y + \phi(t,x,y)y + p(t)f(x)\right], \end{aligned} \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$, $\phi : I \times \mathbb{R}^2 \to \mathbb{R}$ and $p : I \to \mathbb{R}$ are continuous, and $r : I \to \mathbb{R}$ is differentiable, and $r(t) > 0$ for all $t \in I$, $I = [0, \infty]$.

He constructed more than one Lyapunov functions for the system (1.4) and obtained conditions to establish his stability results.

In 2011, Tunc [31] studied and extended the results of Baker [3]. He also considered

$$\left(r(t)x'\right)' + \phi\left(t, x, x'\right)x' + p(t)f(x) = p(t, x, x'), \tag{1.5}$$

with an equivalent system given as

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -\frac{1}{r(t)}\left[r'(t)y + \phi(t,x,y)y + p(t)f(x)\right] + \frac{1}{r(t)}q(t,x,y), \end{aligned} \tag{1.6}$$

where $q : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. He constructed more than one Lyapunov functions for the system (1.6) and proved boundedness of the solutions.

Other second order differential equations similar to equations (1.1) and (1.5) have been considered by [1–6, 8, 9, 15, 16, 18–23, 26–30, 32–34].

Recently, Alaba and Ogundare [1] discussed the uniform asymptotic stability and boundedness of solutions of the of the equation (1.4) or (1.5) with $r(t) = 1$ and $\phi(t, x, y) \equiv a(t)f(x, y)$ i.e.

$$\ddot{x} + a(t)f(x, \dot{x})\dot{x} + b(t)g(x) = p(t; x, \dot{x}), \tag{1.7}$$

with the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a(t)f(x, y)y - b(t)g(x) + p(t; x, y), \end{aligned} \tag{1.8}$$

where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$, $p \in C([0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $a(t)$ and $b(t)$ are positive functions.

The functions a, b, f, g and p depend only on the arguments displayed explicitly. In the work, they constructed a single complete Lyapunov function and obtained criteria to establish their results.

The motivation for this present work comes from the papers of Baker [3] and C. Tunc [31]. The purpose of this paper is to extend and improve their obtained results by constructing a single unique complete Lyapunov function with less restrictive conditions.

2. The Main Results

2.1. **Basic Assumptions.** The following are the basic assumptions used to formulate the main results of this article.

Let $a(t)$, $b(t)$, $c(t)$, $d(t)$, f, g and p be continuous functions, $I = [0, ab]$ the Routh-Hurwitz interval, also a, b, c and d are positive i.e, $a, b, c, d > 0$ with $0 < a_0 \leq a \leq a(t) \leq a_1$, $0 < b_0 \leq b \leq b(t) \leq b_1$, $0 < c_0 \leq c \leq c(t) \leq c_1$ and $0 < d_0 < d \leq d(t) \leq d_1$, for all $t \in \mathbb{R}^+$; in addition, a, b, c and d are differentiable with $a(t)$, $b(t)$, $c(t)$ and $d(t)$ being decreasing functions i.e, $\dot{a} \leq 0$, $\dot{b} \leq 0$, $\dot{c} \leq 0$ and $\dot{d} \leq 0$, then:

(i) $|f(x, y)| \leq \alpha \in I$;
(ii) $G_0 = \frac{g(x)-g(0)}{x} \leq \beta \in I$, with $x \neq 0$, $g(0) = 0$ and $\alpha, \beta > 0$;
(iii) $c[c\dot{d}\alpha^2 + \beta(2bd\alpha + \frac{\dot{b}}{b}(b + 1))] > \dot{c}\beta(\delta + 1) + b\alpha^2(2b\dot{d} + \dot{b}d)$ for all α, β, δ all positive;
(iv) $p(t; x, y) = p(t)$ and $|p(t)| \leq W$ for all $t \leq 0$, where W is a constant.

These are the main results with respect to the equation (1.1).

Theorem 2.1. *Suppose conditions* (i)–(iii) *in our basic assumptions hold, then the trivial solution of the system* (1.3) *is globally asymptotically stable.*

Theorem 2.2. *Suppose that the conditions of Theorem* 2.1 *hold and in addition condition* (iv) *of the basic assumption also holds, then there exists* σ $(0 < \sigma < \infty)$

depending only on α, β and δ such that every solution of system (1.3) *satisfies*

$$x^2(t) + \dot{x}^2(t) \le e^{-\sigma t}\left\{Q_1 + Q_2 \int_{t_0}^{t} |p(\tau)| e^{(\frac{\sigma\tau}{2})} d\tau\right\}^2,$$

for all $t \ge t_0$, where the constant $Q_1 > 0$ depends on α, β, δ, t_0, $x(t_0)$, $\dot{x}(t)$ and the constant $Q_2 > 0$ depends on α, β and δ only.

Theorem 2.3. *Suppose the conditions of Theorem* 2.2 *hold with condition* (iv) *replaced with* (*v*) *$|p(t;x,y)| \le (|x| + |y|)\phi(t)$, where $\phi(t)$ is a non-negative and continuous function of t and satisfies $\int_0^t \phi(s)ds \le W < \infty$, $W > 0$ is a constant. Then there exists a constant λ_0 which depends on W, λ_1, λ_2 and t_0 such that every solution $x(t)$ of system* (1.3) *satisfies $|x(t)| \le \lambda_0$, $|\dot{x}(t)| \le \lambda_0$, for sufficiently large t.*

3. Preliminary Results

The main tool employed in proving our main results is the scalar function $V(t;x,y)$ defined as

$$V(t;x,y) = \frac{1}{2bd\alpha}\left\{dc\beta(\delta+1)x^2 + (db\alpha)^2x^2 + \delta y^2 + y^2 + 2bd\alpha xy\right\}. \tag{3.1}$$

To establish that (3.1) is indeed a Lyapunov function for the system (1.3), we shall state and prove the following Lemmas.

Lemma 3.1. *Subject to the assumptions of Theorem* 2.1*, there exist positive constants $\lambda_i = \lambda_i(a,b,c,d,\alpha,\beta,\delta)$ $i = 1,2$ such that*

$$\lambda_1(x^2 + y^2) \le V(t;x,y) \le \lambda_2(x^2 + y^2)$$

Proof. Clearly, $V(t;0,0) = 0$. By rewriting (3.1) we have

$$V = \frac{1}{2bd\alpha}\left\{dc\beta(\delta+1)x^2 + \delta y^2 + (bd\alpha x + y)^2\right\}, \tag{3.2}$$

$$V \ge \frac{1}{2bd\alpha}\left\{dc\beta(\delta+1)x^2 + \delta y^2\right\}, \tag{3.3}$$

and

$$V(t;x,y) \ge \lambda_1(x^2 + y^2), \tag{3.4}$$

where $\lambda_1 = \lambda_0 \times \min\{d_0c_0\beta(\delta+1), \delta\}$ and $\lambda_0 = \frac{1}{2db\alpha}$.

After applying the inequality $|xy| \le \frac{1}{2}|x^2 + y^2|$ on equation (3.1), we have

$$V \le \frac{1}{2bd\alpha}\left\{\left[dc\beta(\delta+1) + (bd\alpha)^2\right]x^2 + [\delta+1]\,y^2 + bd\alpha x^2 + bd\alpha y^2\right\}, \tag{3.5}$$

$$V \le \frac{1}{2bd\alpha}\left\{\left[dc\beta(\delta+1) + (bd\alpha)^2 + bd\alpha\right]x^2 + [bd\alpha + \delta + 1]\,y^2\right\}, \tag{3.6}$$

and

$$V(t;x,y) \le \lambda_2(x^2+y^2), \tag{3.7}$$

where $\lambda_2 = \lambda_0 \times \max\{d_1c_1\beta(\delta+1)+(d_1b_1\alpha)^2+b_1d_1\alpha, b_1d_1\alpha+\delta+1\}$ and $\lambda_0 = \frac{1}{2db\alpha}$. From equations (3.4) and (3.7), we have

$$\lambda_1(x^2+y^2) \le V(t;x,y) \le \lambda_2(x^2+y^2) \tag{3.8}$$

which completes the proof. □

Lemma 3.2. *Subject to the assumptions of Theorem* 2.1, *there exist positive constants* $\lambda_j = \lambda_j(a,b,c,d,\alpha,\beta,\delta)$ *with* $j = 3,4$ *such that for any solution* (x,y) *of system* (1.3)

$$\dot V(t;x,y)|_{(1.4)} = \left.\frac{dV(t;x,y)}{dt}\right|_{(1.4)} \le -\lambda_3\left(x^2+y^2\right) + \lambda_4(|x|+|y|)|p(t;x,y)|.$$

Proof. After differentiating the equation (3.1) with respect to t, we have

$$\left.\frac{dV}{dt}\right|_{(1.4)} = \dot V|_{(1.4)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot x + \frac{\partial V}{\partial y}\dot y \tag{3.9}$$

by using (1.3) in (3.9). Therefore, differentiation of (3.1) partially, with respect to t, gives

$$\begin{aligned}\frac{\partial V}{\partial t} =& \frac{2bd\alpha}{4b^2d^2\alpha^2}\Bigg\{\left[d\dot c\beta(\delta+1)+\dot dc\beta(\delta+1)+2\dot d db^2\alpha^2+2\dot b bd^2\alpha^2\right]x^2+2(\dot bd+b\dot d)\alpha xy\\ &-\frac{2\alpha(\dot bd+b\dot d)}{4b^2d^2\alpha^2}\left(\left[dc\beta(\delta+1)+(dc\alpha)^2\right]x^2+[\delta+1]y^2+2bd\alpha xy\right)\Bigg\}\\ =&\frac{1}{2bd\alpha}\Bigg\{\left(\left[d\dot c\beta(\delta+1)+\dot dc\beta(\delta+1)+2\dot d db^2\alpha^2+2\dot b bd^2\alpha^2\right]x^2+2(\dot bd+b\dot d)\alpha xy\right)\\ &-\frac{1}{2b^2d^2\alpha}\Bigg(\left[\dot b d^2c\beta(\delta+1)+\dot bd(db\alpha)^2+b\dot ddc\beta(\delta+1)+b\dot dd^2c^2\alpha^2\right]x^2\\ &+\left[\dot bd(\delta+1)+b\dot d(\delta+1)\right]y^2+2\dot bbd^2\alpha xy+2b^2\dot dd\alpha xy\Bigg)\Bigg\}.\end{aligned}$$

Further simplification yields to

$$\begin{aligned}\frac{\partial V}{\partial t} =&\frac{1}{bd\alpha}\Bigg\{\frac{1}{2}\left[d\dot c\beta(\delta+1)+2\dot d db^2\alpha^2+\dot bbd^2\alpha^2-\frac{\dot b}{b}(\delta+1)-\dot ddc^2\alpha^2\right]x^2\\ &-\frac{1}{2}\left[\dot bb(\delta+1)+\frac{\dot d}{d}(\delta+1)\right]y^2\Bigg\}.\end{aligned} \tag{3.10}$$

Also differentiating the equation (3.1) partially with respect to x gives

$$\frac{\partial V}{\partial x} = \frac{1}{db\alpha}\left\{ \left[dc\beta(\delta+1) + (db\alpha)^2\right] x + db\alpha y\right\},$$

$$\frac{\partial V}{\partial x}\dot{x} = \frac{1}{db\alpha}\left\{ \left[dc\beta(\delta+1) + (db\alpha)^2\right] xy + db\alpha y^2\right\}. \tag{3.11}$$

Finally differentiating the equation (3.1) partially with respect to y gives

$$\begin{aligned}\frac{\partial V}{\partial y} =& \frac{1}{db\alpha}\left\{(\delta+1)y + db\alpha x\right\},\\ \frac{\partial V}{\partial y}\dot{y} =& \frac{1}{db\alpha}\left\{((\delta+1)y + db\alpha x)(-\dot{a}dy - bdf(x,y) - cdg(x) + dp(t;x,y))\right\}\\ \leq& \frac{1}{bd\alpha}\left\{ -bcd^2\alpha\beta x^2 - \dot{a}d(\delta+1)y^2 - bd\alpha(\delta+1)y^2 - b^2d^2\alpha^2 xy - cd\beta(\delta+1)xy \right.\\ &\left. + p(t;x,y)d^2b\alpha x + p(t;x,y)(\delta+1)dy\right\},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial V}{\partial y}\dot{y} =& \frac{1}{bd\alpha}\left\{ \left(-bcd^2\alpha\beta x^2 - \left[\dot{a}d(\delta+1) + bd\alpha(\delta+1)\right]\right) y^2\right.\\ &\left. - \left[b^2d^2\alpha^2 + cd\beta(\delta+1)\right] xy + p(t;x,y)d^2b\alpha x + p(t;x,y)(\delta+1)dy\right\}.\end{aligned} \tag{3.12}$$

If we substitute equations (3.10), (3.11) and (3.12) in (3.9) then it yields to

$$\begin{aligned}\frac{dV}{dt} =& -\frac{1}{bd\alpha}\left\{\left(bcd^2\alpha\beta + \frac{1}{2}\left[\dot{d}dc^2\alpha^2 + \frac{\dot{b}}{b}dc\beta(\delta+1)\right]\right.\right.\\ &\left. - \frac{1}{2}\left[d\dot{c}\beta(\delta+1) + 2\dot{d}db^2\alpha^2 + \dot{b}bd^2\alpha^2\right]\right)x^2\\ &+ \left(\dot{a}d(\delta+1) + bd\alpha\delta + \frac{1}{2}\left(\frac{\dot{b}}{b}(\delta+1) + \frac{\dot{d}}{d}(\delta+1)\right)\right) y^2\\ &\left. + p(t;x,y)\left(d^2b\alpha x + (\delta+1)dy\right)\right\}\end{aligned} \tag{3.13}$$

and

$$\frac{dV}{dt} = \dot{V}(t;x,y) \leq -\lambda_3\left(x^2+y^2\right) + \lambda_*(|x|+|y|)p(t;x,y), \tag{3.14}$$

with $\lambda_0 = \frac{1}{db\alpha}$, $\lambda_* = \{d^2b\alpha, d(\delta+1)\}$ and

$$\lambda_3 = \lambda_0 \times \max\left\{bcd^2\alpha\beta + \frac{1}{2}\left[\dot{d}dc^2\alpha^2 + \frac{\dot{b}}{b}dc\beta(\delta+1)\right] - \frac{1}{2}\left[d\dot{c}\beta(\delta+1) + 2\dot{d}db^2\alpha^2 \right.\right.$$
$$\left.\left. + \dot{b}bd^2\alpha^2\right], \left[\dot{a}d(\delta+1) + bd\alpha\delta + \frac{1}{2}\left(\frac{\dot{b}}{b}(\delta+1) + \frac{\dot{d}}{d}(\delta+1)\right)\right]\right\}.$$

Inequality (3.14) can also be simplified and given as

$$\frac{dV}{dt} = \dot{V}(t;x,y) \le -\lambda_3\left(x^2+y^2\right) + \lambda_4\left(x^2+y^2\right)^{\frac{1}{2}}|p(t;x,y)| \tag{3.15}$$

with $\lambda_4 = \sqrt{2}\lambda_*$ which completes the proof. □

Remark 3.1. When $p(t;x,y) = 0$, (3.15) becomes

$$\frac{dV}{dt} = \dot{V}(t;x,y) \le -\lambda_3\left(x^2+y^2\right). \tag{3.16}$$

Lemma 3.1 established that V is a positive definite function while Lemma 3.2 showed that $\dot{V}$ is negative definite function; hence V is a Lyapunov function for the system (1.3).

4. Proof of the Main Results

Now, we shall give the proofs of our results stated in section 4 as follows.

Proof of Theorem 2.1. From Lemmas 3.1 and 3.2 with $p(t;x,\dot{x}) \equiv 0$, it had been established that function $V(t;x,y)$ is a Lyapunov function for the system (1.3). Hence, the trivial solution of system (1.3) is globally asymptotically stable (g.a.s), that is, every solution $x(t)$, $\dot{x}(t)$ of the system (1.3) satisfies $x^2(t) + \dot{x}^2(t) \to 0$ as $t \to \infty$. □

Proof of Theorem 2.2. From (3.15) by replacing $p(t;x,\dot{x})$ with $p(t)$ we obtain

$$\dot{V}(t;x,y) \le -\lambda_3\left(x^2+y^2\right) + \lambda_4\left(x^2+y^2\right)^{\frac{1}{2}} p(t). \tag{4.1}$$

From (3.4), we have

$$x^2+y^2 \le \frac{1}{\lambda_1}V. \tag{4.2}$$

Hence

$$\lambda_3\left(x^2+y^2\right) \le \lambda_3\frac{V}{\lambda_1}. \tag{4.3}$$

Using inequalities (4.2) and (4.3), inequality (4.1) becomes

$$\frac{dV}{dt} \le -\lambda_6 V + \lambda_5 V^{\frac{1}{2}}|p(t)|, \tag{4.4}$$

where $\lambda_6 = \frac{\lambda_3}{\lambda_1}$ and $\lambda_5 = \frac{\lambda_4}{\lambda_1^{\frac{1}{2}}}$. The inequality (4.4) can also be expressed as

$$\frac{dV}{dt} \leq -2\lambda_7 V + \lambda_5 V^{\frac{1}{2}}|p(t)|,$$

with $\lambda_7 = \frac{\lambda_6}{2}$. Therefore,

$$\dot{V} + \lambda_7 V \leq -\lambda_7 V + \lambda_5 V^{\frac{1}{2}}|p(t)|,$$

$$\dot{V} + \lambda_7 V \leq \lambda_5 V^{\frac{1}{2}} \left\{ |p(t)| - \lambda_8 V^{\frac{1}{2}} \right\}, \tag{4.5}$$

where $\lambda_8 = \frac{\lambda_7}{\lambda_5}$. Thus (4.5) becomes $\dot{V} + \lambda_7 V \leq \lambda_5 V^{\frac{1}{2}} V^*$, where

$$V^* = |p(t)| - \lambda_8 V^{\frac{1}{2}}. \tag{4.6}$$

When $|p(t)| \leq \lambda_8 V^{\frac{1}{2}}$, then (4.6) becomes

$$V^* \leq 0 \tag{4.7}$$

and when $|p(t)| \geq \lambda_8 V^{\frac{1}{2}}$, then (4.6) becomes $V^* \geq 0$. By substituting (4.7) into (4.5), we have $\dot{V} + \lambda_7 V \leq \lambda_9 V^{\frac{1}{2}}|p(t)|$, where $\lambda_9 = \frac{\lambda_5}{\lambda_8}$. This implies that

$$\dot{V} V^{-\frac{1}{2}} + \lambda_7 V^{\frac{1}{2}} \leq \lambda_9 |p(t)|. \tag{4.8}$$

Multiplying both sides of (4.8) by $e^{\frac{\lambda_7 t}{2}}$, we have $e^{\frac{\lambda_7 t}{2}} \left\{ \dot{V} V^{-\frac{1}{2}} + \lambda_7 V^{\frac{1}{2}} \right\} \leq e^{\frac{\lambda_7 t}{2}} \lambda_9 |p(t)|$, that is,

$$2\frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{(\frac{\lambda_7 t}{2})} \right\} \leq e^{\frac{\lambda_7 t}{2}} \lambda_9 |p(t)|. \tag{4.9}$$

Integrating both sides of (4.9) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{\lambda_7 \xi}{2}} \right\}_{t_0}^{t} \leq \int_{t_0}^{t} \frac{1}{2} e^{\frac{\lambda_7 \tau}{2}} \lambda_9 |p(\tau)| d\tau,$$

which implies that

$$\left\{ V^{\frac{1}{2}}(t) \right\} e^{(\frac{\lambda_7 t}{2})} \leq V^{\frac{1}{2}}(t_0) e^{\frac{\lambda_7 t_0}{2}} + \frac{\lambda_9}{2} \int_{t_0}^{t} |p(\tau)| e^{\frac{\lambda_7 \tau}{2}} d\tau$$

or

$$V^{\frac{1}{2}}(t) \leq e^{-(\frac{\lambda_7 t}{2})} \left\{ V^{\frac{1}{2}}(t_0) e^{\frac{\lambda_7 t_0}{2}} + \frac{\lambda_9}{2} \int_{t_0}^{t} |p(\tau)| e^{\frac{\lambda_7 \tau}{2}} d\tau \right\}. \tag{4.10}$$

Using (3.4) and (4.10), we have

$$\lambda_1 (x^2(t) + \dot{x}^2(t)) \leq e^{-(\lambda_7 t)} \left\{ \lambda_2 \left(x(t)^2 + \dot{x}^2(t) \right)^{\frac{1}{2}} e^{\frac{\lambda_7 t_0}{2}} + \frac{\lambda_9}{2} \int_{t_0}^{t} |p(\tau)| e^{\frac{\lambda_7 \tau}{2}} d\tau \right\}^2,$$

for all $t \geq 0$. Thus

$$x^2(t) + \dot{x}^2(t) \leq \frac{1}{\lambda_1}\left\{e^{-\lambda_7 t}\left\{\lambda_2\left(x^2(t)+\dot{x}^2(t)\right)^{\frac{1}{2}} e^{\frac{\lambda_7 t_0}{2}} + \frac{\lambda_9}{2}\int_{t_0}^{t}|p(\tau)|e^{\frac{\lambda_7\tau}{2}}d\tau\right\}^2\right\}$$

$$\leq e^{-\lambda_7 t}\left\{Q_1 + Q_2\int_{t_0}^{t}|p(\tau)|e^{\frac{\lambda_7\tau}{2}}d\tau\right\}^2, \tag{4.11}$$

where $Q_1 = \lambda_2(x(t)^2 + \dot{x}^2(t))^{\frac{1}{2}}e^{\frac{\lambda_7 t_0}{2}}$ and $Q_2 = \frac{\lambda_9}{2}$.

By putting $\lambda_7 = \sigma$ in the inequality (4.11) and gives

$$x^2(t) + \dot{x}^2(t) \leq e^{-\sigma t}\left\{Q_1 + Q_2\int_{t_0}^{t}|p(\tau)|e^{(\frac{\sigma\tau}{2})}d\tau\right\}^2. \tag{4.12}$$

Hence, this completes the proof. □

Remark 4.1. From the proof of the Theorem 2.2, the following can be pointed out as direct consequence of the Theorem. If $p(t,x,y) \equiv 0$, then the trivial solution of (1.3) is uniformly asymptotically stable.

Remark 4.2. If $p(t;x,y) \equiv 0$, then (4.12) reduces to $x^2(t)+\dot{x}^2(t) \leq e^{-\sigma t}Q_1$, as $t \to \infty$, $x^2(t)+\dot{x}^2(t) \to 0$ which implies that the trivial solution of the system (1.3) is globally asymptotically stable.

Proof of Theorem 2.3. Indeed from the inequality (3.15), we have

$$\dot{V} \leq \lambda_*(|x|+|y|)^2\phi(t). \tag{4.13}$$

By using the Schwartz inequalities on (4.13) we obtain

$$\dot{V} \leq \lambda_{10}\left(x^2+y^2\right)\phi(t), \tag{4.14}$$

where $\lambda_{10} = 2\lambda_*$.

From the inequalities (3.4) and (4.10), we have $\dot{V} \leq \lambda_{10}V\phi(t)$, by integrating equation (4.14) from 0 to t_0, we obtain

$$V(t) - V(0) \leq \lambda_{11}\int_{t_0}^{t}V(s)\phi(s)ds, \tag{4.15}$$

where $\lambda_{11} = \frac{\lambda_{10}}{\lambda_1} = \frac{3\lambda_*}{\lambda_1}$. The inequality (4.15) now becomes

$$V(t) \leq V(0) + \lambda_{11}\int_{t_0}^{t}V(s)\phi(s)ds. \tag{4.16}$$

By Gronwall-Bellman inequality (4.16) we obtain

$$V(t) \leq V(0)\exp(\lambda_{11}\int_{t_0}^{t}\phi(s)ds).$$

This completes the proof. □

5. Example

Consider second order non autonomous nonlinear differential equation

$$\left(\frac{2+3t^2}{1+t^2}x'\right)' + \frac{t}{(1+t^2)^2}x' + \frac{4+3t^2}{1+t^2}\left(2x+4x^3\right) = \frac{1}{1+t^2+x^2+x'^2}, \tag{5.1}$$

which can be written as

$$\left(3-\frac{1}{1+t^2}\right)x'' + \frac{3t}{(1+t^2)^2}x' + \frac{4+3t^2}{1+t^2}\left(2x+4x^3\right) = \frac{1}{1+t^2+x^2+x'^2}. \tag{5.2}$$

We state (5.2) as the system form

$$\begin{aligned} x' &= y, \\ y' &= -\frac{3t}{(2+3t^2)(1+t^2)}y - \frac{4+3t^2}{3t^2+2}\left(2x+4x^3\right) + \frac{1+t^2}{(1+t^2+x^2+y^2)(2+3t^2)}. \end{aligned} \tag{5.3}$$

Comparing (5.1) with (1.1), it is clearly seen that

$$\begin{aligned} & a(t) = 3-\frac{1}{1+t^2}, \quad t \geq 0, \quad 2 \leq 3-\frac{1}{1+t^2} \leq 3, \\ & a_0 = 2, \quad a_1 = 3, \\ & b(t) = \frac{t}{(1+t^2)^2}, \quad t \geq 0, \quad \frac{1}{2} \leq \frac{t}{(1+t^2)^2} \leq 1, \\ & b_0 = \frac{1}{4}, \quad b_1 = \frac{2}{5}, \\ & c(t) = 3+\frac{1}{1+t^2}, \quad t > 0, \quad 3 \leq 3+\frac{1}{1+t^2} \leq 4, \\ & c_0 = 3, \quad c_1 = 4, \\ & d_0 = \frac{1}{2}, \quad d_1 = \frac{1}{3}, \\ & \lambda_0 = \frac{1}{\alpha}, \quad \lambda_3 = 12\beta. \end{aligned}$$

The corresponding Lyapunov function to the system (5.3) is given as

$$V = \frac{1}{\alpha}\left\{0.5\beta(\delta+1)x^2 + \delta y^2 + (0.5\alpha x + y)^2\right\} > 0,$$

where α, β, and δ are positive constants and whose derivative is given as

$$\frac{dV}{dt} = \dot{V}(t; x, y) \leq -12\beta\left(x^2+y^2\right),$$

where $\beta > 0$. All conditions stated in Theorem 2.1 are satisfied therefore the zero solution of system (5.3) is globally asymptotic stable.

We have for $p \equiv 0$ that the solutions of (5.1) are globally asymptotic stable.

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